Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture-16: Exponential of a Linear Operator

Welcome students, in this video we are going to talk about exponential, matrix exponential, but we will start with something called exponential of operator, okay. So basically given an operator we want to talk about the exponential of something like this. So up till now we know that you can put you know given a real number you can talk about the exponential of such a number. But the thing is here we are going to talk about a linear operator. So basically consider a linear operator T from Rn to Rn.

Rn is linear. Now, so I will write it here as TLRN. So, it means RN to RN. Now, we have already, we already know that the operator norm, we can define the operator norm of T is defined as norm of T is nothing but the maximum of mod x less than equal 1 t x. So, t acting on x that will be a real number you see sorry that will be a element of random vector.

So, I am taking the you know So I am computing the norm of each vector and then we are taking the maximum. So first of all why is this valid? Because you see you remember that norm function we actually talked about it. It is a Lipschitz continuous function.

So basically it is continuous function. And t is a linear map from a finite dimension to finite dimension. You can actually show that such a thing is actually going to be continuous. Okay, so norm of tx is nothing but a composition of two continuous function, hence it is continuous and norm x less than equal 1, this is a closed and bounded set, so compact set, right, in RN. So, you see continuous function and compact set, at least it is maxima, right, so that is the object which you are defining as a norm, clear?

Okay, now we also know that there are some properties of norm. So, let us just write down and it satisfies the usual property. So, what are the properties? Norm of t, we know that this is always going to be greater than or equal to 0 and norm of t will be 0 if and only if t equals to 0.

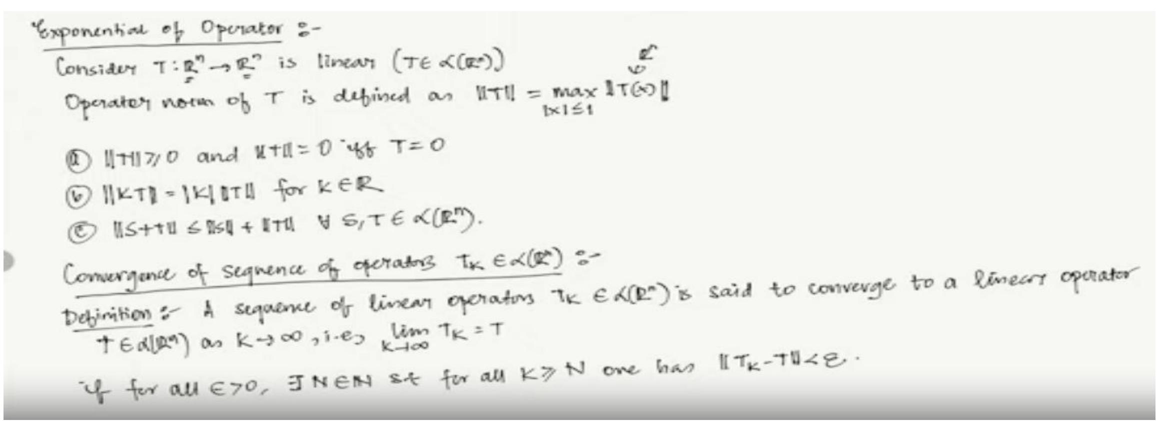
So, basically it satisfies all the norm properties, okay. And b, if you have a constant times a norm t, sorry, a linear operator t, then that is nothing but the modulus of k, okay, and the norm of t, okay. This holds for k in R, okay and we also have that the triangle inequality s plus t is dominated by norm of s plus norm of t this holds for all s and t linear map from rn to rn clear okay so now you see we will define something so now this is fine right we have to define some kind of calculus operators, okay?

And to do that, what we are going to do is we are going to first of all define something called a convergence, okay? Convergence of sequence of operators, sequence of operators, operators, okay? And these sequence of operators, we will write it as a linear map from Rn to Rn, And we want to define a convergence and in which thing this in operator norm. So, let us write down the definition here.

Definition. So, sequence of a sequence of linear operators, a sequence of linear operators, linear operator Tk, clear? In LRNRN is said to converge, is said to converge, converge to a linear operator, okay? So, basically you have a linear operator given to let us say T, I am saying that it will, there is a sequence Tk which converges to T and when is that happening?

That is the question, okay? Linear operator, t which is again in l rn rn, rn to rn, okay, as k tends to infinity, as k tends to infinity. So, basically what we will do is this, we will write it like this, okay, limit k tends to infinity tk is t, yeah, and this convergence is like this. If for all epsilon greater than 0,

Okay, so for any epsilon greater than 0, there exists a natural number n, natural number, okay, such that after some point, such that for all k greater than equal capital N, one has, you can make it as small as possible. Okay, so how in terms of norm, so tk minus t should be less than epsilon. So this is what it means for a convergence. So whenever, please remember this thing, whenever we are saying a sequence of linear operators converges to another linear operator, what does it mean?



It converges, so basically tk minus t, the distance between tk minus t, think of it like that. That is very small. So and what is in that sense, which sense is the distance? So basically it is the norm. And what is this norm? This is nothing but the operator norm.

operator norm. So, basically if you take the operator norm of tk minus t that should be made can be made arbitrarily small. So, the tail should be substantially smaller that is. So, there are some properties which is to satisfy. So, let me put it as a theorem.

So, for st which is a linear map from Rn to Rn and x in Rn. The first property is that T of x, you see, what is T of x? This is the element of Rn. Please remember this thing.

So, that is your norm. That can always be dominated by norm of T. This is operator norm. Please remember operator norm. Norm of T is operator norm. okay and the norm of x so this is just an rn norm yeah okay so how do you prove something like this let's just look at the proof okay so proof proof okay so see so this this should hold for all x in rn right so if x equals to 0 of course equality holds there is nothing to prove here equality holds why there is nothing to prove here because you see t is a linear map right so t of 0 has to be equal to 0 there is no other way

Yes. So, 0 and this side also will be 0. So, basically 0 is equal to 0. Now, if x is not a 0 vector, other than 0, then we define y to be x by norm x. okay why am i defining this because if you do something like this see since x is non-zero norm of x is not zero right so if you define a new vector y which is given by x divided by the length then what happens to y the norm of y is basically 1 okay so then

You can use the operator norm definition. What is norm of t? So norm of t is always dominates norm of t acting at y. See, norm of t is t in operator norm. That is nothing but the maximum of t on the unit circle, right?

So, y by, equals to x by mod x is on the unit circle. So, basically, see, norm of y is nothing but 1, right? Okay. So, and this, since this is nothing but the maximum of t on norm x less than equal 1, okay? This is the definition, right?

Okay. So, this is always dominates t of y, yeah? Okay. Since if it is now, it is always dominant t of y. Now, that is nothing but, you see, this is nothing but t of, what is y? y is nothing but x by norm of x. Okay.

Now, see, you can think of this as a constant, right? So, t of constant times x is c times tx. That's linear map, right? So, you can write it as norm of 1 by norm of x. T of x. I can do that.

Now, see what happens is this norm of x is basically a real number. So, I can take it out. So, this becomes 1 by norm of x norm of Tx. Now, if you put it together, what happens? Then you have norm of Tx is dominated by, if you take this norm here, norm of T times norm of x. Again, I am telling you all the time, please remember which is which.

This T of x is an element of Rn. So the norm here is in Rn. Clear? This is an operator norm. Let's just call it like this operator norm.

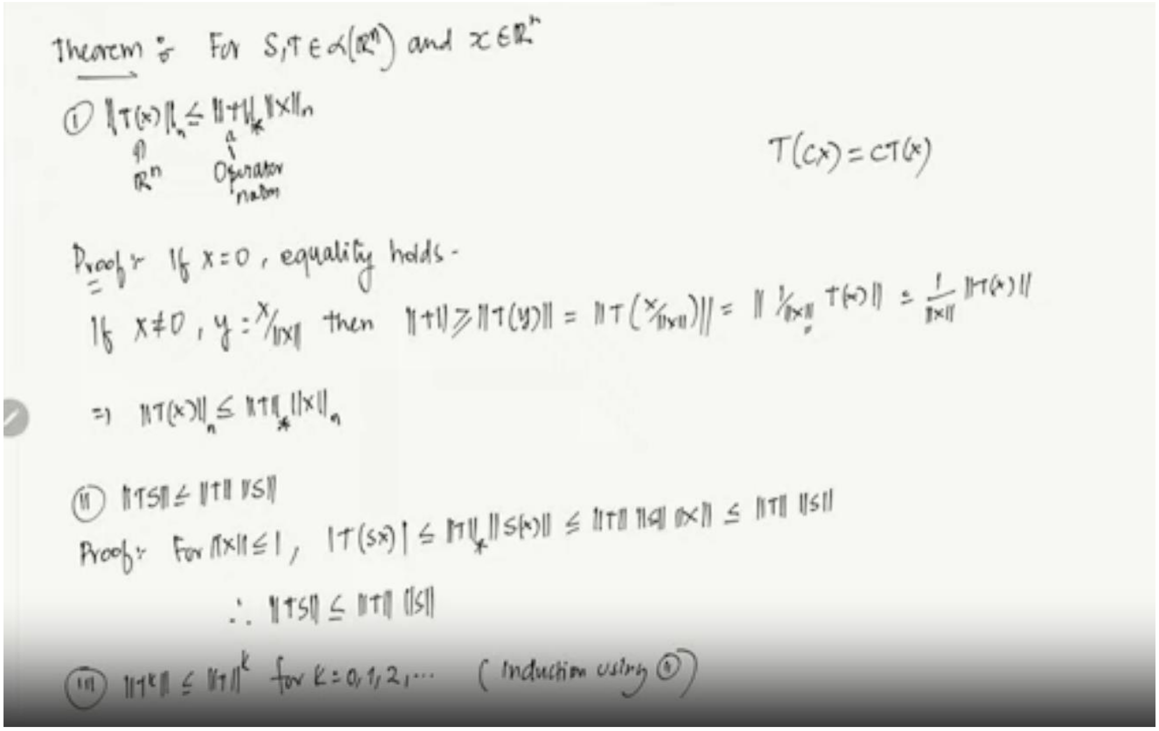
And this is, so let's just call the operator norm as star norm. That will be more suitable. Star norm. Right? Okay?

So star norm and this is n norm. Clear? So please keep that in mind. Don't get confused here. okay so what are the second property which you can actually say that norm of ts so basically the composition of two linear uh this thing operator right t or t and s composition is ts that is dominated by norm of t times norm of s right again very very important but i mean relatively easy okay so let's look at quick look at the proof of this what happens the proof the proof is so for

norm x less than equal 1, okay? You see, I have to look at, we have to look at this, T of Sx, right? So, that is always dominated by norm of T, okay? Norm of Sx, clear? This is star norm, okay?

And norm of Sx is by the 1, it is always dominated by norm of T, norm of S, norm of x, right. So, and since norm x is always less than 1, this is dominated by norm t norm s. See, this I am doing extra, this is already done in week 1, but still we will, this kind of a revision you can think of. Therefore, norm of t s is always dominated by norm of t times norm of s. Yeah, and the third property which we want because we are going to use this thing norm of t power k is always dominated by norm of t to power k. Okay, this holds for k equals to 0, 1, 2.

So, I think you can handle this part. So, please do this thing. So, this is by induction. Induction using 1, using 2. using 2, right? So, this is always true. Why I did it again here?



Because I want these properties, okay? I want to exploit these properties to define a new thing. So, basically, we have a new theorem here.

Theorem, okay? So, you see, given a linear map, T, linear operator, sorry, T, which is linear operator from Rn to Rn, and a point T0, Okay. Positive. Okay.

The series. So, we are basically looking at such series. Okay. Summation. P power K. K to the power K. Pi.

P factorial. K equals to 0 to infinity. Yes. Is absolutely. Absolutely.

And uniformly. And uniformly. convergent convergent for all mod t less than equal T 0, okay. So, you understood what I am saying.

See, for an operator T, we can define T power K, right. So, this is the composition of T with itself for K times. So, here also what we are doing is and we are using the same idea here and we are defining a new series here, okay. And this series is like a series of linear operator, okay. So, basically and what we are saying is this, if you define something like this, you can actually show that series is absolutely and uniformly convergent, okay.

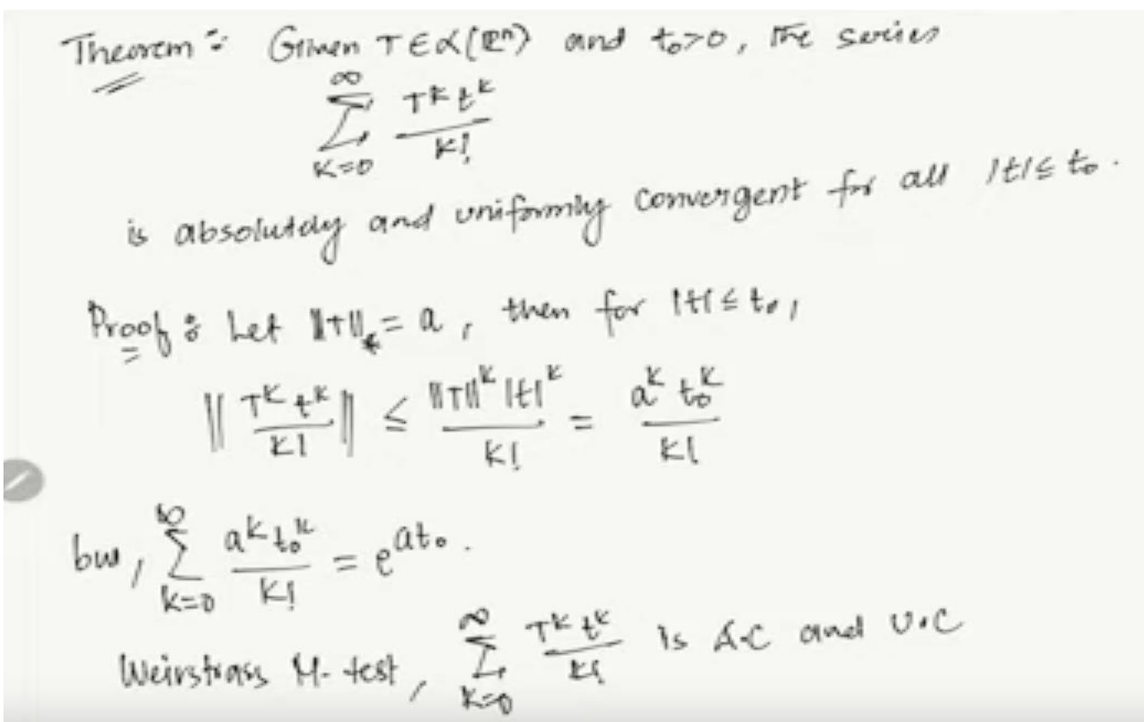
Very, very important theorem, we are going to use this theorem, okay. So, this is one of the fundamental theorems which we are going to use, okay, to define what comes next, which is the most important thing. So, the proof is very easy. So, you see, let, given a T, right, so the norm of T is fixed, let us say that is A, okay, then, then for A, t less than, let us say t 0, one has, you have to have uniform convergence, right?

So, you look at the norm of t power k, t to the power k by k factorial, okay? This is always dominated by norm of t power k, mod of t power k by k factorial. How is this true? You see, we are using 1 and 3, okay? We are using 1 and 3.

one second sorry sorry we are not using 1 and 3 we are using this thing b and 3 okay so this this property b property of norm and this 3 okay this this is 2 this 2 we are using and we got this clear okay so now this norm of t is nothing but a so this is nothing but a power k here and t is dominated by t 0 so it is t 0 power k by k factorial Okay, now what is this? You see, if you take the sum, so, but sum k equals to 0 to infinity, a power k, t 0 power k by k factorial is nothing but e power a t 0, right? So, we can use Weierstrass M test, Weierstrass M test. Okay.

This series is dominated by a convergent series. Okay. Absolutely convergent series. So, essentially what happens is you can actually say that see this series is absolutely and uniformly convergent. That will imply that the series k equals to 0 to infinity.

Okay. T power k by k factorial. Okay. Is absolutely convergent and uniformly convergent. So, that is there.



So, basically we saw that if you write a series like this, given a linear operator, then of course, we can actually say that it is absolutely an uniformly convergent. What we are going to do is this. So, this is very, very important. We are going to define the exponential of a linear operator. So, let T from L Rn.

Then T we define e to the power t, the exponential of t. You see, looks very much like the exponential, right? So, we will define this exponential of t is nothing but summation k equals to 0 to infinity t to the power k by k factorial. So, you see what we are doing is this, that using this series, we know that this series is absolutely and uniformly convergent.

right. So, we are using that part, ok, the absolutely convergence part and we are actually writing a series which looks like this and we are saying that it actually gives you a new type of operator which we are going to call e power t and that is nothing but the sum from k equals to 0 to infinity 3 divided by k factorial, ok. How do you know something like this can happen? Because of the first theorem, because this is absolutely convergence, right, ok. Now,

you see of course the you can use the properties of limit to show it is easy to see it is easy to see to see that e power t okay is a linear operator on rn okay so this is just a property of limit you can use the limit and you can say and of course this is also true okay Norm of e power t is dominated by e power norm of t. Okay. You see. This is what we proved here. Norm of e power t is dominated by e power norm of t. Clear.

So that is what I wrote. Clear. Okay. Fine. Now I will explain to you why we are suddenly interested in exponential of an operator.

But this is very very important. Okay. That is what I can say right now. Okay. So we will write down a definition here.

Definition. What we are going to do is this. Let A be a n cross n matrix. We know that any linear operator is nothing but a n cross n matrix. So, I am starting out with A, which is a n cross n matrix.

And for t in R, we define e power a t. This is the definition which we are using. It is nothing but summation k equals 0 to infinity a power k. T power k by k factorial. Clear? Okay.

So, this is what we are calling as an exponential of a matrix. So, this is called, it is called the exponential of a matrix. Of a matrix. Okay, so if you think of it properly, you see, let us say A is a one cross n matrix, which basically you can think of it as a scalar that this actually matches with your, you know, e power some constant times t, right? That is what the definition is.

Okay, so in that sense, it is true. Okay, and there are some properties of this linear transformation. Okay, what are the properties? So the first property, let us write it as a proposition. Proposition.

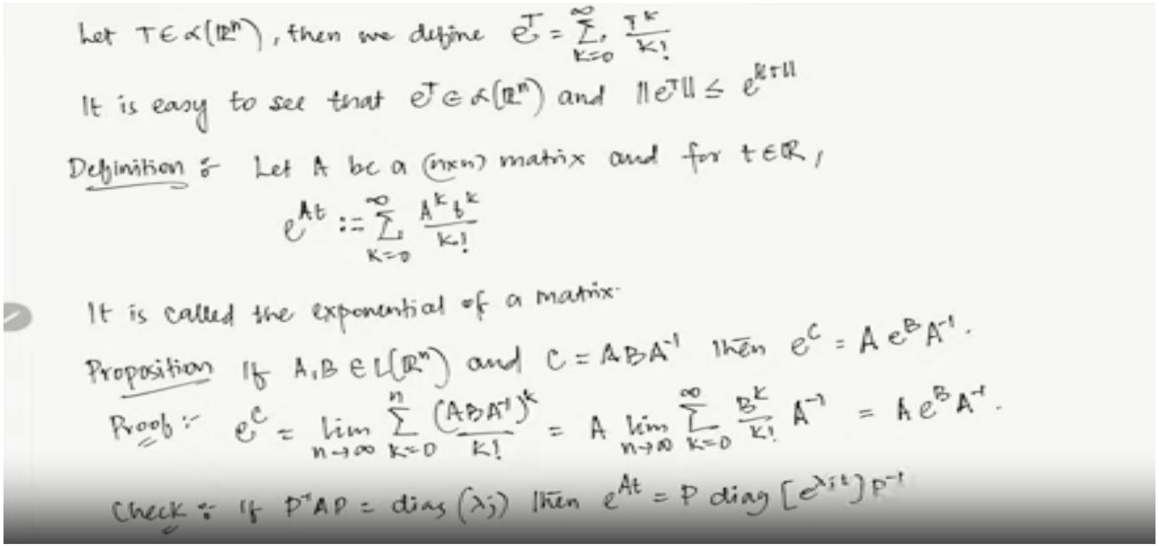
Okay. If a and b are linear transformations, if a and b, so let me put it this way, if a and b are linear transformations in Rn, and you defining a new function c, clear, which is a, b, A inverse. Then, it is very easy to see that E power C, nothing but A, E power B, A inverse. Let us quickly do this thing.

How do we prove something like this? Let us look at the proof. Yes, these propositions are very, very important. We are going to use it a lot of time. Yes.

So, E power, let us say C. What is it? It is limit n tends to infinity okay summation k equals to 0 to n okay a b a inverse whole power k by k factorial right so this is nothing but you see now if you just write it you can break it up using matrices right you can write it as a limit n tends to infinity, summation k equals to 0 to infinity, t power k, sorry, b power k, b power k by k factorial times a inverse, right? This is just, you see, a b a inverse whole power k is nothing but a b power k a inverse, right?

That is what we are using here. Okay, so if you have this and then it is nothing but A e to the power b. A inverse, clear? So, that is the difference. And also please check this part, you have to check it yourself, check it that if P inverse A P is the diagonal matrix, diagonal matrix given by lambda j. So, basically diagonal for lambda 1, lambda 2, lambda and everything else is 0, this is the matrix.

Then e to the power at we are going to use this property again any lot of time. So please remember P diagonal matrix whose diagonal is e power lambda jt P inverse. Very very simple. So you see you do not have to calculate everything.



So basically let us say if you know that a matrix is a diagonal matrix then it is very easy to calculate what exponential is.

Exponential of such a matrix is. Now there is another property proposition which we need to prove. So, what is it? So, let us say that S and T are two linear operators on Rn such that they are commutative S T equals to T S. Yeah, then what happens is e power s plus t equals to e power s times e power t. So, please remember this.

See, if a is a matrix, n cross n matrix, e power at is nothing but basically a matrix, right? Yes. Now, the thing is, If you have, but in real number what happens is e to the power a plus b, e to the power a plus b is nothing but e to the power a times e to the power b. We can write it like this, right, for real numbers. But since these are matrices which you are dealing with, so basically the commutative property does not work, right.

And then what happens is when is e to the power s plus t is e to the power s times e to the power t, that only works if s and t are commutative, okay. So, let us look at the proof. proof see if s t equals to t s okay so one can write t plus s whole power n is nothing but n factorial okay summation j plus k equals to n this is just a binomial theorem nothing special s power j t power k by j factorial k factorial okay this is just binomial theorem nothing special Binomial theorem. Okay?

Right. Therefore, P power S plus T, if you just write it down, what will happen? This will be nothing but summation, summation J plus K equals to N. Right? Equals to N. N equals to 0 to infinity. Clear?

S power J, T power K by J factorial, K factorial. Okay? Clear? This is nothing but summation j equals to 0 to infinity. I can write it like this, right?

s power j by j factorial summation k equals to 0 to infinity t power k by k factorial. Clear? We can write it like this. If you just break it up, you can just see that this is always true. Clear?

Okay. So, Now, you see this is nothing but e power s times e power s. Okay. So, we have this.

Fine. So, this is I mean fine. Okay. So, you see this is actually something what we used here is something called a Cauchy product. If you if you are not familiar with this please check out I mean any mathematical analysis book you can just find it.

Okay. So, also check this part. Check this part that e power t whole inverse, yes. So, exponential t, t is a matrix, right.

So, exponential of a matrix, the inverse of that is nothing but e power minus t, okay, right. There are other some, okay, so that is fine, yes. So, now, the thing is, let us look at some examples and how to calculate the Calculate exponential of a given matrix. Clear?

How to calculate this? So, let us look at, you see, the thing is, we have to go here step by step. So, let us check some simple examples. Yes? And then, we can use that, you know, idea and we can go for a, you know, more complicated examples.

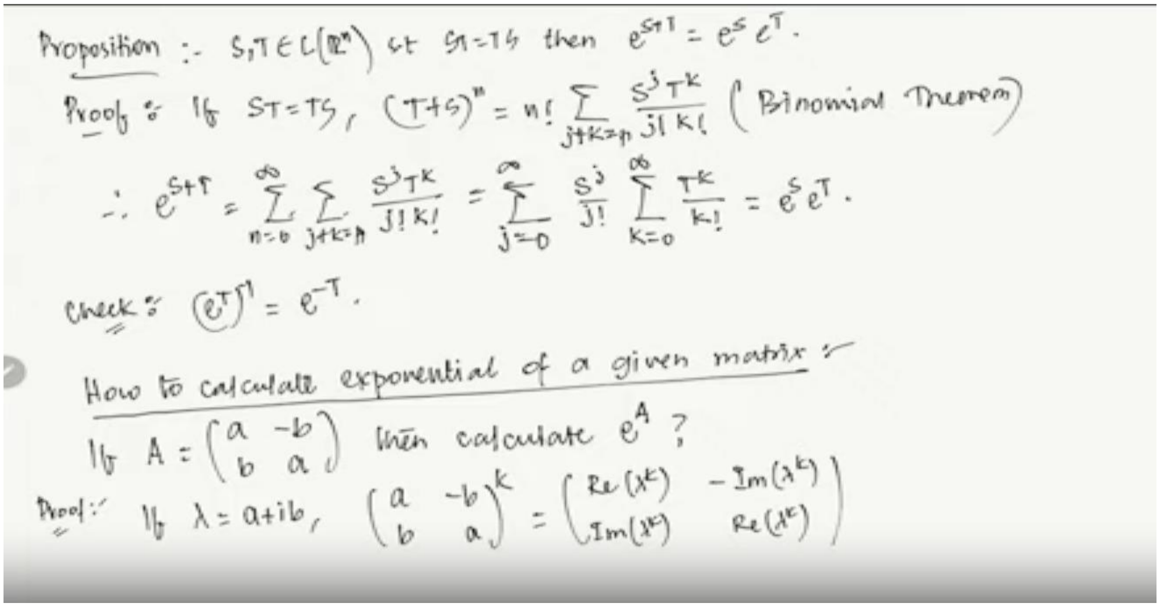
Okay? Right. So, first of all, let us say, if A is a 2 cross 2, let us check with 2 cross 2, A minus B, B, A. Okay? Then, We have to know what is e power a. Okay.

We need to know what is e power a. See, the thing is, then calculate e power a. Calculate e power a. Yes. That is the question. Okay. So, now, you see, if one writes, so this is the proof. Proof.

If we write lambda to be nothing but a plus ib. Sorry. If we write lambda to be a plus ib. Okay. So, basically this is a complex number.

Okay. Then you see a minus b, b a power k. So, this is the small prick which we are using. This is nothing but real part of lambda power k. Real minus imaginary part of lambda power k. Okay. Imaginary part of lambda power k and real part of lambda power k. Yes.

Okay. If lambda is a plus ib, you can of course calculate what is lambda power k, right? And you take the real part, primary part, that is given by this, right? So, we are just writing it down. That is the only thing.



Yes. Therefore, if you want to write e power a now, now I want to calculate e power a, right? Therefore, e power a, what is it? This is nothing but summation k equals to 0 to infinity, right? this this thing right okay by k by k factorial of course okay so this is nothing but real part of lambda power k by k factorial minus imaginary part of lambda power k by k factorial

Okay. Again imaginary part of lambda power k by k factorial and real part of lambda power k by k factorial. Okay. See this is there and there is 1 by k factorial. Right.

For the e power this thing that is the definition which we use. Right. So I am just writing it like that. I hope this is clear. Now you see what happens is this thing is nothing but you can write it as real part of

e power lambda minus imaginary part of e power lambda. And then you have imaginary part of e power lambda and real part of e power lambda. Clear? You see, this is the sum. You can just put everything together and you can just write it like this, right?

So, that is there. Now, what is this? This is nothing but, if you break it up, this becomes e power a. So, everywhere, e power a, I will just write it like this, e power a cosine b and this is minus sine b and this is sine b And this is nothing but cosine. So, what is exponential a?

This is given by this. Now, again, this is one way of doing some, see, this is not like a straightforward way. We will look at many other how to calculate exponential, but this is just an example of some easy calculations. Now, another example, let us just look at another example. If let us say A is another 2 cross 2 matrix, A, B, 0, A. Right?

Okay. Then we need to calculate what is E power A. Let us just calculate that. Okay. See, we can write, one may write, one may write A to be A0, 0A. plus 0B, 0, 0.

Right? So, which is nothing but A times I plus a matrix B. Clear? Now, check this part. Check that A times I and B commutes. There is nothing to check actually.

But anyways. See it is an identity matrix, right? It is constant time identity. Another one is B. So, it comes. So, therefore, what happens is e to the power a is nothing but e to the power ai plus B, right?

Which is nothing but e to the power B is nothing but e to the power this matrix, right? This matrix. Clear? Okay. So, you see, e to the power, let us just calculate this thing.

e to the power B is Is nothing but e to the power 0 b 0 0. This is a matrix. Okay. So this is.

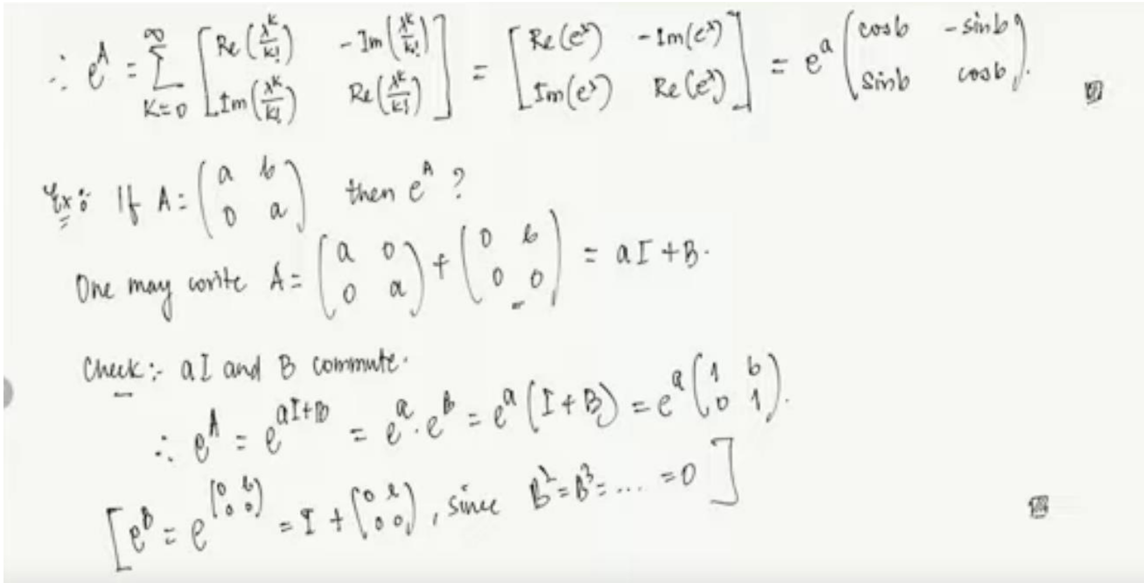
Now the problem is this. See. It is identity. Okay. Plus the matrix 0 b 0 0.

Okay. Plus. Now b squared. The term b squared. That is going to be 0.

Right. b cube is going to be 0. So this will actually be up till here. This will be up till here, right? Since, since, okay, let me write it down.

Since b square equals to b cube equals to, these are all going to be the zero matrix, okay? So, this is identity plus the matrix B. Okay. Now, what is e power ai? This is nothing but e power a. This is easily checked, right?

You can just write it down. I mean, identity matrix, nothing is going to happen. It is basically the same thing, e power a, right? And then it is e power b. So, this is e power a times i plus the matrix b. Okay. Now, if you want to prove it properly, it is e to the power a, 1, b, 0, 1.



Okay. So, I hope this is clear. This is how.