Ordinary Differential Equations (noc 24 ma 78)

Dr Kaushik Bal

Department of Mathematics and Statistics

Indian Institute of Technology, Kanpur

Week-02

Fundamental Matrix - W4P3

So welcome students and in this video we are going to talk about something called a fundamental matrix okay and by the name of it you do realize this it is a very important concept so what is it so essentially let us consider this system so consider the system so a variable coefficient system n cross n system which is given by x prime equals to 80 times x yes where Where A of t, this is a variable coefficient of course, is a continuous n cross n matrix function. Continuous n cross n matrix function. Let me put it this way. So, basically what I mean is if all the coefficients of a and they are going to be continuous on whatever interval we are going to define this system.

Yes. Now, the thing is this. See, we know that if we choose, see, if one choose f of t y to be 80 times y, sorry, f of t x to be 80 times x. tx to be 80 times x yes and since a is given to be continuous c a uh here with respect to t a is continuous right because a only depends on t and uh f of tx with respect to just the second variable x it is c1 right so basically we can use the picard's theorem which we talked about in the earlier video so if we choose this then then by picard's existence and uniqueness picards Existence and uniqueness and uniqueness.

One can find a unique solution, right? One can find an unique solution. unique solution and what is the solution given by let us just call it x of t okay and the solution is in of course is in c1 of i yeah c1 of i to rn satisfying what that some initial data right satisfying x at the point let us say 0 is x naught, okay? So, whenever you are given this initial data, you have a solution which passes through that initial data, that is via, because there is just an unit test, okay?

Now, what happens is this, see, if for this system, let us just call this system 1, okay? Now, see, this x naught is in Rn, right? Yes, and you see that now, if 1 choose x naught okay the vector x naught okay to be let's say e i what is e i which is nothing but the unit vector okay so 0 0 and 1 in the ith variable and then 0 0 0 0 yeah let's say this is the element in rl so basically this is the ith entry the unit ith unit vector if you choose that then look at this system see consider this system now so consider the following following system.

See, what is the system? It is given by f prime equals to 80 times x, right? 80 times x and x at the point 0 is ei, okay? Given this system, let us just call this system as 2. okay, 2 and i, let us just call it 2i, okay, because this depends on i, right, then there exists, we know that a unique solution x, let us just call this solution as xi of t, which is, of course, is in c1 of i to rn, right, satisfying, satisfying,

to i this equation for what i for i between 1 and n so what i am trying to say is this see the thing is this x naught so basically what we are saying is this this equation 1 okay That is the trajectory which a particle is following. Yes, that is what given. And what is x0 at the point x0? That is basically saying that at the point t equals to 0.

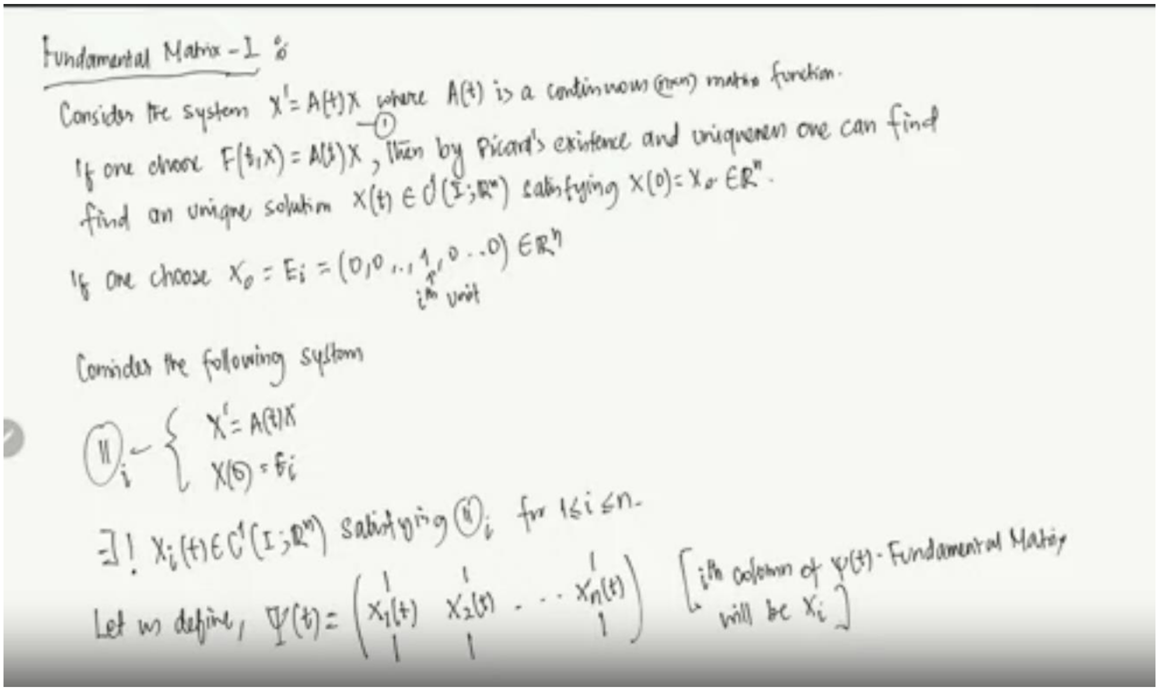
So, when time is 0, it has to pass through the vector x0, right? So, we can choose the vector to be e1, e2, en, right? So, ei is at the unit vector. We can choose that. And for each such choice, I will get a xi, right?

I will get a xi. Now, the thing is this, you see, Let us define this. Let us define this. phi of t. Let us define this phi of t. What is phi of t?

Phi of t, this is what we are going to call a fundamental matrix, which is nothing but x1 of t. So, if you understand x1 of t is basically a vector in Rn, right? So, you just write down as a first column. So, what is the first column of phi, which is x1 of t? What is the second column of t? It is x2 of t. So, what we are going to do is we are going to define this new matrix.

You understand? What is the definition? That the ith column of the fundamental. So, basically the ith column of phi t. Phi t is what? We will call it a fundamental matrix.

Fundamental matrix will be. the xi's is it okay so basically the xi of t will actually form the ith column is it okay so that is your phi now you do realize that phi is a n cross n matrix right and since x1 x2 x3 they are all c1 so basically phi is also c1 so you do not really have to worry about it yes now okay fine now the thing is there are some few properties which we really need to understand properties First of all, if you look at it, you see phi, where is it defined? Phi is defined on i. You see, for every t from the interval i, you are going to get a matrix, n cross n matrix, right? So, phi as a map, yeah, it is mapped from i to where it is a matrix to n cross n, okay, over r. So, that is phi of phi is c1, okay?



It is C1. So, at least continuously differentiable. Why it is C1? Because you do realize that every component of it is continuously differentiable. So, basically, phi is continuously differentiable.

Is it okay? Right. Now, the thing is, this is the first property, which we know. Now, the second property, this property, I am going to write it down. You have to verify it.

It is very easy. Yes, it is actually trivial. So, you please verify it. You see, phi, since it is C1, what happens to phi prime of t? You can actually show that phi prime of t is nothing but a t times phi of t.

Is it okay? Yeah, this is what phi does. Yes. And if you check this part, you see also, do you realize what phi of 0 is, right? Phi of 0 is nothing but, you see, x1 at the point 0, you see, this is x1, the vector x1 at the point 0.

x2 at the point 0 right and xn at the point 0 right that's your phi of 0 so what is that vector what is phi 1 at the point 0 you see phi 1 at the point 0 is nothing but e1 right so the first vector so it is 1 0 0 0 and the second one is 0 1 0 0 you understand so the last one is all 0 0s everything is 0 and it is 1 so basically it is nothing but the identity matrix is it okay so phi hence Hence, let me put it this way. Hence, phi satisfies the matrix phi, okay? That satisfies the matrix differential equation, okay? So, this is a different kind of differential equation.

It is called a matrix differential equation. Differential equation. And what is it given by? Given by phi prime of t. equals to A t times phi t, right?

And phi at the point 0 is identity, yes? Now, I really do not have to tell you that since phi at the point 0 is identity, you can use Abbas theorem, right? Yes, you can use Abbas theorem and you can say that the determinant of phi of t is never going to be 0 for all t in I. Yes, see the thing is initially what is happening is this phi at the point 0 is x1 at the point 0, x2 at the point 0, xn at the point 0, right? See x1 at the point 0, x2 at the point 0, which is nothing but the identity matrix.

So phi at the point 0 is identity matrix. Is it okay? Now, what happens is, of course, I mean, in the last video, we talked about this Abel's formula, right? What is Abel's formula? It says that the Wronskian of x is nothing but Wronskian at some point 0 times the exponential of 80, right?

Trans of 50. Now, the thing is this. See here, what is happening is... So, you take that arbitrary point x naught to be 0 here. So, if you take that, then you do realize that since that matrix, if you take in the Wronskian, that Wronskian of the identity matrix is 1.

So, it is basically never going to be 0. So, in that case, determinant of phi of t is non-zero, which means, therefore, phi is non-singular. Is it okay? So, it is basically a non-singular matrix. Matrix.

Okay. What are we using here? We are using Abel's formula. Formula. Okay.

So, you guys have to check this part out. Please do that. Yes. Okay. So, now, let us look at some examples.

Okay. What are the fundamental matrices? So, let us look at the examples. See, the thing is, you can easily check that if you define, define phi of, sorry t 0 let's define phi of t 0 which is given by e to the power a t if you define this thing yes you can see that this is the fundamental matrix or let me call it principal fundamental matrix i will explain to you what is principle why we are calling it principle fundamental matrix again this matrix also will call it a principal fundamental matrix okay principle

I will explain to you why we call it principle. Principle fundamental matrix. Matrix of what? Of the system. Of the system.

I think you can guess this part. It is given by x prime equals to the constant coefficient system. x prime equals to a times x. Is it okay? a is constant. a is constant.

Of course, n cross n matrix is a constant. So, you can check this part. So, if you are defining phi to be this, then phi will be a fundamental, reasonable fundamental matrix of this system. And you do realize that any solution of this system in that case will look like phi of t0 times x0. So, that is just any solution would look like this.

So, now one thing I want to, yes, principal fundamental matrix. What is the principal fundamental? You do realize that you see here, This x1, x2, x0 equals to EI we chose, right? EIs are what?

EIs are basically the basis of Rn. Is it okay? Now the thing is this, if we change the basis of Rn from EI to some other basis, right? So you choose any n linearly independent vectors and you have another basis. So let us say, you know, a1, a2.

a2 is another basis of Rn, right? It can have many different bases, right? Let us say this is a basis of Rn, okay? Now, if you take this equation, x prime equals to at times x and x at the point 0 is ai, let us say, yeah? You will get a different set of x i's, different set of solutions, right?

Those solutions also you can put it like this, like phi, You can define a different phi and that will also give you a fundamental matrix. Again the same sort of thing, same properties hold. All these same properties hold. Of course phi of 0 will be something else but the thing is doesn't really matter.

Exactly the same. So phi will divide by phi prime t equals to 80 times phi and all other properties will be same except that phi of 0 will be a little bit different. It will be a different matrix which is by the way non-singular. Is it okay? And again, by Abbas formula, you can prove that it is a non-singular matrix.

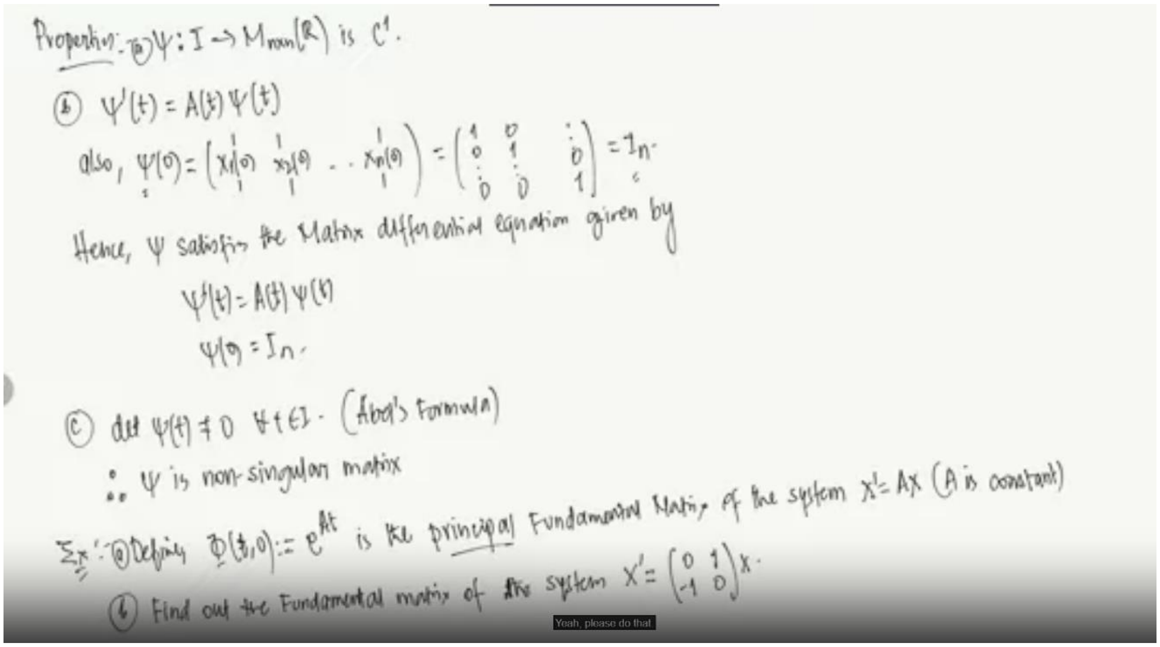
So, that is not a point. The thing is this, you see, there can be many different fundamental matrices for a given system. You understand? Based on the basis which you are choosing. Generally speaking, if you are choosing the standard basis, which is EI's, what I chose here, then that fundamental matrix, we will call it a principal fundamental matrix.

I hope this is clear. Okay? Right. So this is the thing. Now the thing is you can also find I mean if you want.

Yes. So let me call this thing as A and let me write down B. What I want you to do is please check try to find out the fundamental matrix for them. So find the find out the fundamental

fundamental matrix. mental matrix of A, the system, of the system, the system, x prime equals to 0, 1, minus 1, 0 times x. Is it okay? Try to find it out.

See what, if you can, if you can do that part. Yeah, please do that. So, that is it. Now, the thing is this, we are going to prove a very important theorem now. The theorem says this.



So, what is the relation between these fundamental matrices? So, theorem. So, it says that if phi of t phi of t is a, I am writing it like this, fundamental matrix fm of the system of the system x prime equals to A t times x. So, if you have a homogeneous system and phi of t is given to be the fundamental matrix of this homogeneous system. Then, any constant non-singular n cross n matrix matrix

Then for any constant non-singular matrix, n plus n matrix, c, phi t times c, phi t times c is also a fundamental matrix. of the same system, okay? So, let us say what I am saying is this. If you have one fundamental matrix, you can of course produce another one just by multiplying with a constant. And please do not change the order.

So, it is phi t times c, yeah? A t times x of the same system, clear? Also, also we can say that every, okay, fundamental matrix, fundamental matrix is Matrix of the system x prime equals to 80 times x. Okay. Of this system is of the form is of the form.

So basically you can't find any other form of fundamental matrix except this of the form 50 times c. Is it okay? Where c is again where c is as above. So basically a non-singular n cross n matrix. Constraint matrix of course. So what is the theorem?

It is very simple. So essentially what I told you is this. You see there is a principal fundamental matrix and there are other fundamental matrix depending on the basis which you choose. So given a system you can have like infinitely many fundamental matrices depending on what basis you are choosing. What is the principal fundamental matrix?

It is nothing but the matrix which you get with respect to basically the solution which you are going to get if you are choosing the standard unit basis elements. So E1, E2, Ea. Now the thing is this, what this theorem says that what about the other fundamental matrix? Is there any relation between any two fundamental matrices? What it is saying is this, it does not really matter what you are going to choose your basis.

The fundamental matrix essentially will look like if one is given to be phi, another one will look like phi t times the constant c. And it is also true that every fundamental matrix will look like this. So you cannot go on showing something else is a fundamental matrix. So let us see. So, first of all, let us say if phi t is a fundamental matrix, we have to show that phi t times c is also a fundamental matrix. It is very easy.

Now, since phi t is a fundamental matrix, what did I tell you? This is true, right? phi prime of t equals to a t times phi of t. Now, this is quite easy to check as I told you, of course. Yes, please do that. So, this is one.

Let us just call that one. Now, you see, I have to show phi t times c is also a fundamental matrix. So, let us look. Now, You see, phi t times c. Let us just look at the derivative of this.

What is it? Constant c is a constant matrix, right? So, the derivative of that is basically nothing. So, it is essentially phi prime of t acting at c, right? Again, phi prime of t, we know it is a t times phi t, right?

Phi t acting at c, yes? So, you see, by matrix, we know that associativity property holds, right? So, you can write it as a t times phi t times c. Now, phi t times c, see phi t times c prime is nothing but 80 times phi t times c. Therefore, phi t and phi t times c are both solutions. Therefore, phi of t and phi t times c are both solutions.

Solutions of the matrix of t. This equation, you see the matrix value differential equation, okay? So, phi prime equals to 80 times phi, okay? Satisfy this. So, this is the matrix differential equation.

Please remember, yes, the fundamental matrices, they satisfy the matrix differential equation, okay? Okay. So, now you see the thing is you have to also show that since the important thing is how do you check something is a fundamental matrix. First of all you check the matrix differential equation and the next thing you check is what is about the determinant if it is non-singular or not. And you see since determinant of phi of t this is non-zero right because phi of t is given to be fundamental matrix.

Therefore, determinant of phi t times c is nothing but determinant of phi t times determinant of c and since determinant of c is non-zero because c is given to be non-singular you see non-singular constant matrix okay so this is also going to be non-zero so it's trivial right so you can actually see that determinant phi t times c is also a fundamental matrix yeah very simple now we have to show the converse part so what is the converse so let's say phi of t um any two fundamental matrix is given to you yes we have to show that 1 is a constant multiple of another. So, basically, phi 1 and phi 2 are two fundamental matrices. You have to show that phi 2 is nothing but phi 1 times C. Yes.

So, conversely, let us do that. Conversely, we have to show that let phi 1 of T and phi 2 of T and phi 2 of T be two fundamental matrices, be two fundamental matrices, matrix solution, okay, of the system x prime equals to 80 times x, okay. Now, the thing is this, see if this is the case, yeah, just look at this thing, phi and phi inverse both are non-singular, right?

So, I can talk about the inverses, yeah? So, define this new matrix C of t, which is nothing but phi 2 inverse of t times phi 1 of t. Let us just call that, okay? Now, if you do that, see what is phi 1 of t in this case? Phi 1 of t is nothing but phi 2 of t times C t. Is it okay? Yes?

And if you take the, you see, phi 1 and phi 2, let us say c, phi 1 and phi 2 are both c1 maps, c1 matrix functions, right? So, c is also c1, right? So, we can take the derivative here. Now, differentiating what we have? Differentiating 1 has, 1 has, differentiating this part, okay?

I have phi 1 prime of t, right? phi 1 prime of t. Is nothing but. Phi 2 prime of t. Times c t. Plus phi 2 of t. C prime of t. Right. Now you see.

What is phi 1 prime of t. Phi 1 prime of t is nothing but. A of t. Times phi 1 of t. Yes. And. What is phi 2 prime of t. It is a of t. Times. Phi 2 of t. Times c t. Yes.

Plus phi 2 of t times c prime of t. Yes. Now you see phi 1 is, sorry, phi 1 is phi 2 times c t. Phi 1 is phi 2 times c t. This is phi 2 times c t. So basically this part and this part we can just cancel it out. So basically you can take the one matrix to be on the other side and that will actually give you 0. Now essentially what you have is this phi 2 of t c prime of t is nothing but the 0 matrix. Yes.

Phi 2 is what? Phi 2 is a fundamental matrix. So, that is a non-singular matrix. If you take phi 2 inverse on the both sides, essentially that will give you C 2 T is 0, which basically says that C T is constant. Is it okay?

So, if C T is constant, you do realize that phi 1 will be phi 2 times a constant matrix C. Okay? So, with this, I am going to end this video.