Ordinary Differential Equations (noc 24 ma 78)

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Week-03

Lecture-14: Linear System 2

So, welcome students to this video and here we are going to continue our study of fundamental matrices. See in the earlier video what we have seen is if you have let us say phi. So, first of all you consider this, consider the system. And please remember we are only talking about homogeneous system. Consider the system given by x prime equals to 80 times x. So we are not putting any initial data right now.

Just this system. And then, what we have seen is the theorem which we did is the following, that if phi of x, okay, phi of x is a fundamental solution, okay, it need not be a principal fundamental solution, okay, it is just a fundamental solution, any fundamental solution, okay. So, if phi of x is a, sorry, phi of t, phi of t is a fundamental solution, okay, of, let us call this 1, okay, of 1. Then, for any constant, any constant matrix C, right, C, which is a matrix essentially, so n cross n matrix, phi t times C here is also

is also a fundamental matrix. And any fundamental matrix, I will also show that any fundamental matrix actually looks like this. Fundamental matrix. Matrix of 1. So, I should write fundamental matrix of 1.

matrix of one clear? Okay now let's see that so basically we know that this is the fundamental now you see therefore therefore so any solution any solution of star of one sorry any solution of one okay how does it look like let's let's just write it down will will be of the form of the farm x of t so basically the general solution we talked about it in the last video right okay so x of t will actually look like the fundamental matrix what is in this case it is c of t times a constant matrix c now please remember this is a n cross n matrix and this is a n cross 1 matrix okay this is the solution not the fundamental matrix okay right any fundamental matrix will look like this this c is a n cross n matrix this is a n cross 1 vector clear Okay. So, this we proved in the last video also.

Yes, if you remember. Okay. Now, the thing is any solution will look like this. So, that is a general solution. So, basically we call such a thing as a general solution.

Now, you see, so let me write it as n cross 1. So, there is no confusion. And of course, this is the fundamental matrix. This is the fundamental matrix. And so, this is your general solution.

Let me write it like this. Now, you see, if moreover, moreover, 1 passes, so 1 satisfies, let us just put it this way, 1 satisfies n. x 0 equals to x0. I can put it at, let us just put it at t 0. So, it will be more clear.

So, t 0 can be 0. So, x at the point t 0 is let us say x 0. So, the vector, it passes through this vector. So, this is in r. Then what happens is this. You see x

At the point t 0, this is given to be x 0, right? And if this is the solution, then this will actually satisfy phi at the point t 0 as c n cross 1. So essentially, we can actually calculate that will imply this constant matrix will be given by phi inverse of t 0 times x 0. right. See, x 0 is a vector in Rn and phi inverse of t 0 is a n cross n matrix, ok.

See, phi is a fundamental matrix. So, phi is invertible. Hence, I can take the inverse of that, ok. Therefore, therefore, the solution, ok, we know that the solution, the solution of this problem of x prime equals to At times x, clear, x 0 plus sorry, t 0.

Let us just put it as t 0. You can put it as 0 also. It is not a problem, but anyways, x 0, x at the point t 0 will be of the form x of t. So, please remember, any, this problem, this is in this form, right, x prime equals to f of t x. And we are assuming here all the time in the whole entire course we will assume that a is continuous in person function. So, a is a continuous.

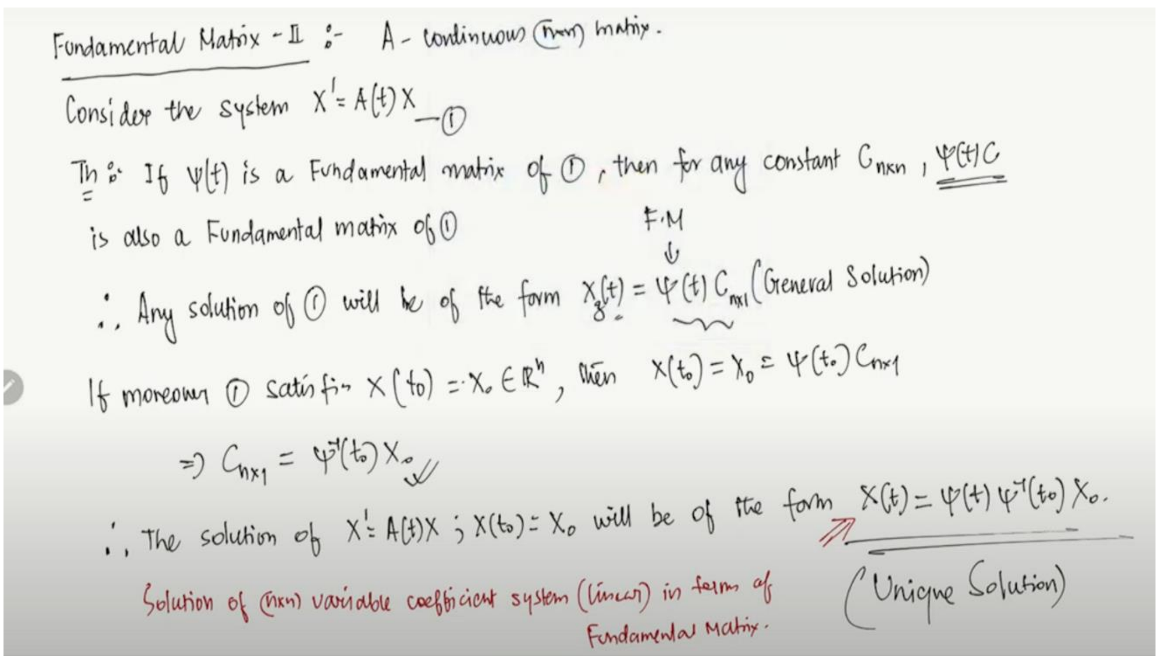
n cross n matrix, n cross n continuous, n cross n matrix, okay. So, that is as in. So, it is continuous with respect to the t variable and t1 with respect to the x variable. So, hence Picard says that in a neighborhood t 0 minus epsilon and t 0 plus epsilon, we can have a unique solution. And I have asked you to check that you can actually extend it to whole of the interval i, whatever this define, right.

So, with that in mind, so we know that there is a unit solution. Now, the thing is this, how does the solution looks in terms of a fundamental matrix? We know that, you see, any solution looks like this. General solution looks like Phi t times the constant matrix, right? And again, if it passes through x 0, then we know that what the constant matrix is, is this.

So, basically, in the solution of this problem, in terms of fundamental matrix, we look like this, P of t. phi inverse of t 0 times X0. So, this is the unique solution. This is the unique solution. How do you get uniqueness?

Unique solution by Picard. Clear? So, this is the solution in terms of, so let me put it this way. This one here, this one is the solution of n cross n, okay, variable coefficient, variable coefficient, okay, system, linear system, of course, linear, linear system.

So, you have to have linear system, n cross n variable coefficient, okay, in terms of, in terms of fundamental mathematics, okay. So, basically in terms of fundamental matrix, this will be the solution. And this is the unique solution. So, this part is here. Now, let us look at the next part.



So, now we are going to talk about something called adjoint system. Adjoint system to the equation x prime equals to At times x okay we are going to define a new system out of this old one and we are going to call it a insurance okay so let's do that see first of all we know we know So, let us start with this. Let phi of t be the fundamental matrix of the system x prime equals to At times x.

Therefore, since it is a fundamental matrix, we can talk about the inverse of that one. So, define a new matrix. Define. You do not have to define it. So, essentially what is happening is, in that case, therefore, phi t, clear?

Phi inverse of t. Phi inverse exists, right? This is going to be an identity matrix. Identity n cross n matrix that is here. Now, you see what happens is, As I have already explained to you the fundamental matrices which we are already getting it.

We can talk about differentiating matrices. So we can actually differentiate this particular thing. So if you differentiate the above relation above we get it is phi Prime of t, phi inverse of t plus phi of t, phi inverse of t, derivative of that, this is going to be 0 matrix, right? Okay.

So, therefore, what happens is you get phi inverse of t, the derivative of that is nothing but minus phi inverse of t times a of t. I hope this is clear. You see, the thing is, why this is true? Because, you see, phi is a fundamental matrix. So, since phi is a fundamental matrix of, let us call it 1, 1, right?

That will imply that phi inverse of t is equals to a t times phi of t. See, this is a matrix system. This is matrix differential equation. Let us just write it like this. Because you see this is phi prime of t is nothing but a n cross n matrix.

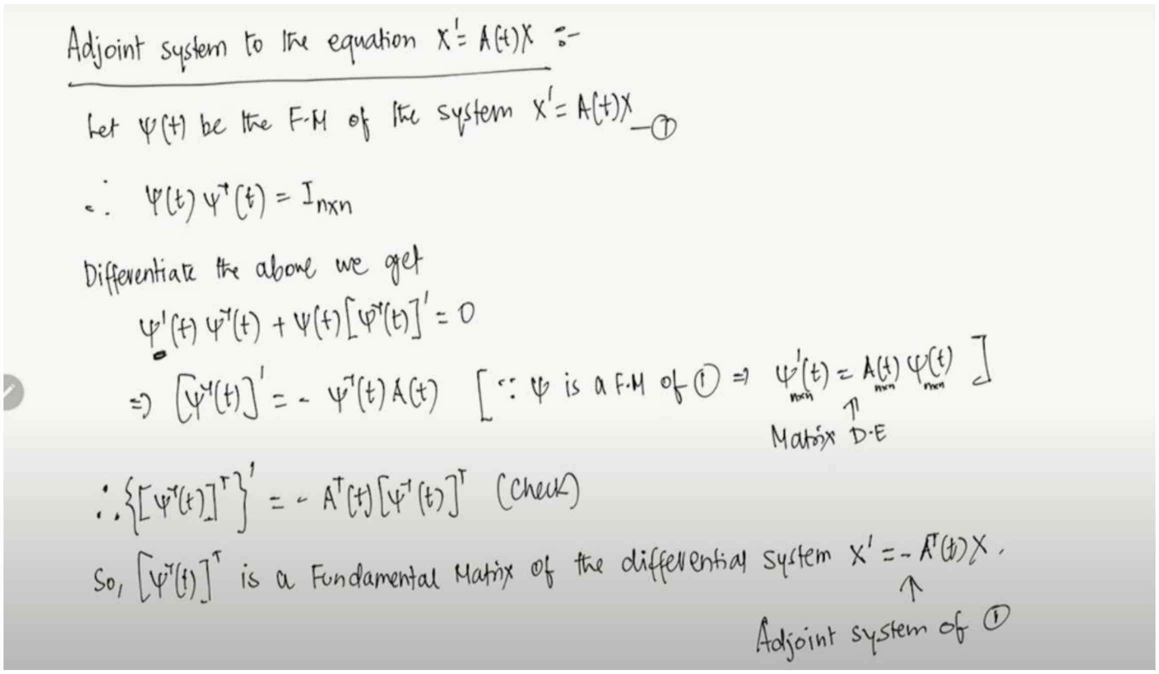
This is a n cross n matrix. This is also a n cross n matrix. So, we are not in like this thing ordinary situation. This is a matrix differential equation. So, if phi prime of t is nothing but a t times phi t, we put this phi prime of t here.

So, which is a t times phi t. So, that will give you that this whole thing, this thing, this expression is nothing but a t right. So, you can actually take everything on the other side and you can write it like this. So, therefore, we can also write as you see phi inverse of t, the transpose of it and then if we take the derivative, this is nothing but minus a transpose of t phi inverse of t. transpose of it. So basically if I take the transpose on both sides this is what I can write down because transpose the derivative goes inside.

So it is not a problem you can just write it like this. So if you are not convinced in this part you take a 2 cross 2 element simple 2 cross 2 element and please check this part. So what does this give you? You see if you define that I am not writing that part. So let us just define this is as pi of t.

So you see pi of t will satisfy pi prime equals to minus a transpose pi. So therefore one can say therefore so phi inverse of t the transpose of it is a fundamental matrix is a fundamental matrix. matrix. Of the differential equation, differential system that is a, of the differential system, what is the system? In this case, it looks like x prime of t

X prime equals to minus A transpose T X. And this system will be called the adjoint system of A. Adjoint system. of, let us say, 1. So, given a system 1, the adjoint of that system will be, so basically, x prime equals to At times x is the original given system, the adjoint of that system will be x prime equals to minus a transpose x.



We are going to talk about this thing later also. So, please remember this thing, very, very important, this one. Now, we are going to look at a very important property of adjoint system.

So, theorem. theorem. Now, you see, if psi of t, okay, is a fundamental matrix, okay, fundamental matrix of the system 1, system 1. What is the system 1? This is the system, okay, x prime equals to At times x, yeah, okay.

So, let us start with the fundamental matrix of that system. Then, Then let us try to put it this way. Phi of t is a fundamental matrix of its adjoint system if and only if phi transpose t psi of t is a constant matrix n cross n which is non-singular.

So, the natural question is this. See, the thing is you are given this system 1, right, x prime equals to At times x. And we already, we have derived another system which we are calling now. This is what we are calling it. the adjoint system of one okay right now the question is this given this system one we know that there is a fundamental matrix for the system right okay now the question is this Let us assume that, I mean, what is the relation?

So, basically, you will have another fundamental matrix, let us say, Phi of the adjoint system. Now, what is the relation between those two fundamental matrices? That is basically the question here. So, what it says is that, if Phi is a fundamental matrix, then psi is also a fundamental matrix of the original system, then phi is a fundamental matrix of its adjoint system. And when is that?

If phi P transpose phi is a constant non-singular matrix. n cross n of course. So let us look at the proof of that. Proof of this. So if phi of t is a fundamental matrix is a fundamental matrix let me write it this way.

Fundamental matrix of 1. It is given to us. Then phi inverse of T whole transpose. You see, we have already proved here.

Phi inverse of T whole transpose is a fundamental matrix of the adjoied system. Is the fundamental matrix of the adjoined system. This is what we proved just now. System. Clear?

So, you see, What we have learnt is this. See, the thing is, you remember any fundamental matrix. What is the first theorem which we talked about in the last video? So, if phi of t is a fundamental matrix of 1, then for any constant c, phi t times c is also a fundamental matrix and any fundamental matrix will look like phi t times c, right?

That is what we proved. So, you see, so if phi inverse of t phi inverse of t, whole transpose is a fundamental matrix of the adjoint system. So, what is adjoint system? Given by, let me write it down, given by x prime equals to minus a transpose x. So, you see, so now, you see, if phi inverse of t, the transpose of that, is a fundamental matrix of the adjoint system, then,

phi of t. So, this fundamental matrix which is given to us. This is psi, this is phi. So, phi of t, the fundamental matrix of the adjoining system is nothing and this is also fundamental matrix, that is also fundamental matrix of the same adjoint system. So, what did we learn? We know that it will be

The transpose of this psi inverse t, the transpose of this times a constant matrix C. Or let us just call it D. It will be better. Let us just call it D. Where D is a n cross n non-singular matrix. How do you get, how am I getting this thing? Matrix. How am I getting this thing?

I am using this theorem. If you remember the theorem which we proved in the last video, this theorem or this theorem. So, if you have a fundamental matrix, another fundamental matrix will always look like the even fundamental matrix types are constant. So, this is what I am using here. So, therefore, you see phi inverse transpose is phi transpose inverse.

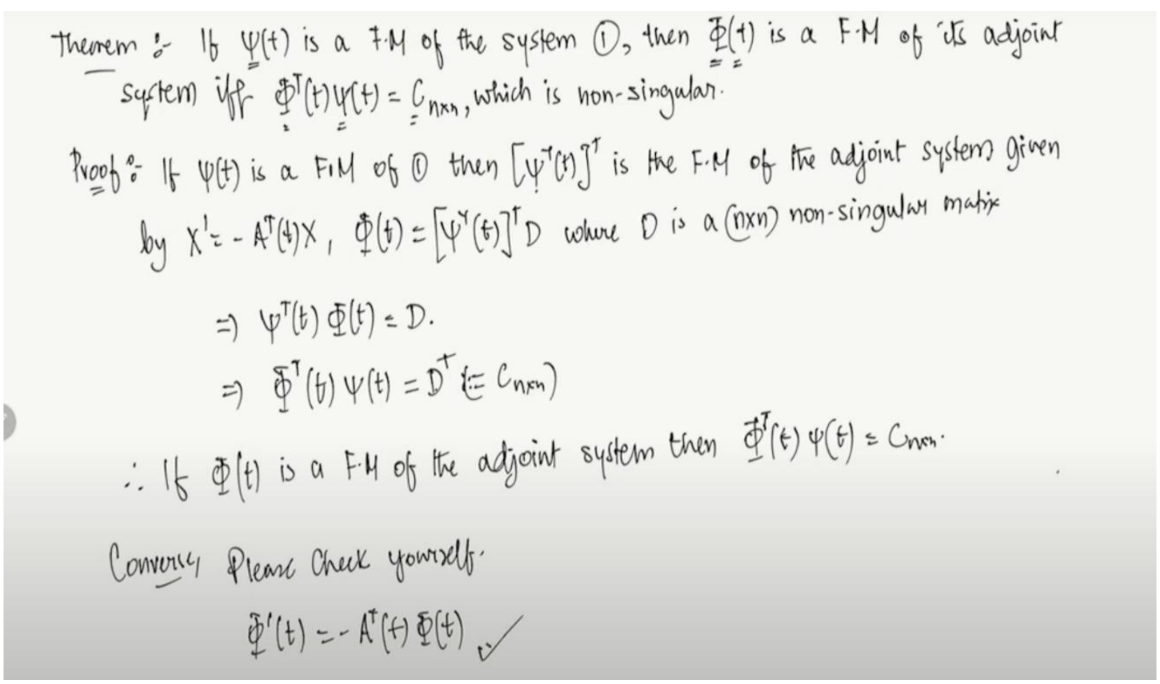
So, if I take this on the other side, what do you have is phi So, transpose times phi of t is nothing but d matrix. That will imply, so if we take the transpose on both sides, it will give me that this is phi transpose of t. psi of t is d transpose, which we will define it as our c, which is a n cross n matrix. This is what we are defining, this is our c. So, therefore, if phi of t is a fundamental matrix of the adjoint system, of the adjoint system, adjoint system,

Then you always have phi transpose of t psi of t is a constant n plus n non-singular matrix. And what about the converse? What is the converse? The converse says that if phi transpose psi is constant, you just have to show it is a fundamental matrix. So this you have to please check yourself.

Please check yours. This is not difficult, yes. Please check yours. What you need to show is, you need to check phi of t is a fundamental matrix of the adjoint system. So, basically what you have to show?

You have to show that phi prime of t is nothing but minus a transpose of t phi of t. Okay. I hope this is I mean you can easily check this part. Okay. Just put this value here and then you can actually see.

Use the fact that psi of t is a fundamental matrix of the original system and you are done. Okay. So these I am leaving it to up to you.



I hope you can do that. Okay. Right. So, with this what we are going to do is now we come to section of this video where we are going to talk about something called a variation of parameter. Variation of parameter. So, this is also known as a method of variation of parameter. Very, very, very important method and we are going to use it in a lot of time.

So, what we actually know is this. First of all, let us start with a homogeneous system. Let us call it a homogeneous system. What is a homogeneous system? x prime equals to At times x, right?

That is the homogeneous system given. And there is another inhomogeneous system. What is the inhomogeneous system? It is x prime equals to At times x. x plus b of t. Okay.

Please remember here a t and b t are assumed to be are smooth functions, smooth matrices. Okay. Smooth matrix value function that is matrix. Okay. Smooth matrices means these are basically matrices with all the entries smooth.

Smooth means infinity. You can assume it. Don't worry about it. Okay. So I have a homogeneous problem and I have an inhomogeneous problem.

We know that x of t, a general solution x of t, how does it look like? It looks like, so we know, let me put it this way, we know that any solution, any solution of the homogeneous problem, homogeneous problem in terms of fundamental solution, any solution of homogeneous problem, matrix, sorry, in terms of fundamental matrix, any solution of homogeneous equation looks like this. x of t is nothing but, let me direct this, x h means x homogeneous is nothing but a fundamental matrix, let us say psi of t times a constant c. It is a constant n cross n non-singular matrix.

Constant non-singular, right? Constant non-singular. Okay. I am sorry. This is n cross 1. n cross 1.

This is a solution, right? It is not a fundamental matrix anymore. It is a solution. So, that will be the fundamental matrix times a constant. Now, you see, I want to find out if I want to see if we can find out a solution of the inhomogeneous problem using this solution of the homogeneous problem.

So, what we do is this. So, let us assume assume that x h, sorry, x i, we will call it x i. x i is, we will call, we will say that this is a solution which will look like this. The fundamental matrix of this, this phi of t times a constant, but now constant is not, you know, it is not a constant matrix, but it is a matrix which depends on t. This c and this c has nothing to do with each other, right? So, using you know by motivated I mean getting motivated by this result we were saying that since this x h looks like phi t times a constant matrix we are assuming that the inhomogeneous problem there is a solution which will look like phi t times a c t where c is dependent on t.

Here, this is what we are assuming. We do not know whether such a solution exists or not. Let us just check if something like that can happen or not. And that is why this method is called variation of parameters. So, we are varying the parameter here, right?

So, if that is the solution, what happens then? Then, you see x i prime, what happens to this? This is phi prime of t c of t plus phi of t c prime of t. Now, you see, if phi is a fundamental matrix of this, right? So, what happens?

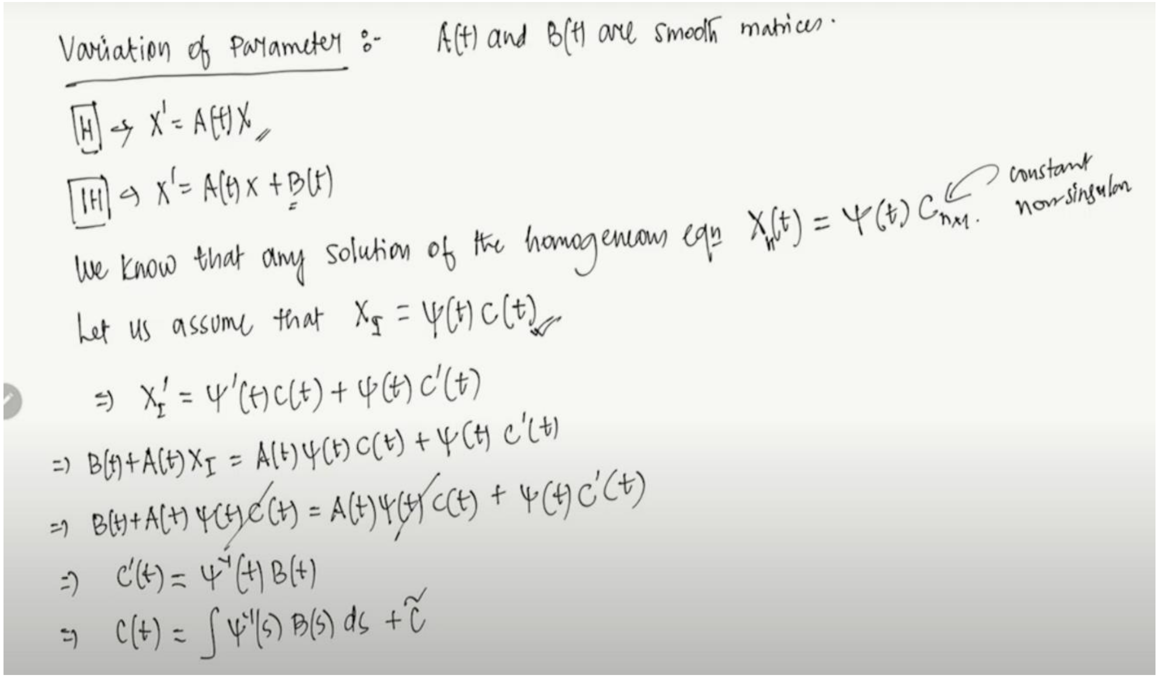
And xi prime is nothing but at times xi, right? This is phi prime of t. What is phi prime of t? It is at times phi of t. And then we have c of t plus phi t times c prime of t. I will just keep this last expression as it is. Now you see what is x of i. x of i is nothing but phi times c. So this is nothing but a t times phi t.

times c of t, which is again nothing but a t times phi of t times c of t plus phi of c phi prime of t. Now, these and these gets cancelled out. So, essentially, I did some mistake somewhere, no? 1 second, yeah. So, this xi prime, this is the solution of the homogeneous system, there is a b of t, which I forgot to write here. So, I should write it as b of t plus this, yes.

So, I should write here also, there should be b of t plus this, right. So, therefore, what we get is phi sorry, b of t. See, this is the, what am I writing here? Really sorry about it. This is c. This is c prime of t. This is c prime of t. So, I have c prime of t is nothing but phi inverse of t

I made a mess here. So, this is b, b of t. So, c time of t is nothing but phi inverse of t, b of t. So, therefore, what is c of t? This is nothing but integral, whatever the integral is from, wherever it is, this phi inverse of s, b of s, ds plus some constant c, constant matrix that is. So, constant matrix, let us just call it this.

So, now if you see that what is our original equation, the original solution which we are expecting is xi is pt times ct.



So, since therefore, what is the inhomogeneous system solution we are assuming? We are assuming it is phi t times ct. And what did we get c t to be? We got it to be integral phi inverse of s b of s ds.

So, integral phi inverse of s b of s ds. Clear? Now, you see, I can write down this c tilde or I can just integrate it between t 0 to t and then move this c in tilde. So, let us put it this way, make it to be t 0 to t. Now you see, so this is a solution of the inhomogeneous problem.

Now, so therefore, what is the solution of the, you know, general solution of the homogeneous problem? So, the general solution of the inhomogeneous problem, inhomogeneous problem, problem is given by, is given by by what is it x i general solution okay inhomogeneous problem general solution is given by phi t times c okay phi t times c plus integral t 0 to t phi of t phi inverse of s b of s Clear? Okay.

So, you see why I wrote this. This is nothing but the solution of the homogeneous problem, right? It is xh solution, the general solution of the homogeneous problem. This is the general solution of the homogeneous problem. So, you see what I wrote is this.

Essentially, the general solution of the inhomogeneous problem, okay, inhomogeneous problem is nothing but the general solution of the homogeneous problem, okay, plus a particular solution, particular solution of the inhomogeneous problem, right. You see, this solution which you got is a particular solution of the inhomogeneous problem. So, the general solution of the inhomogeneous problem, what we are doing is, we are just adding the general solution of the homogeneous problem with the particular solution of the inhomogeneous problem.

So, if you can find a one solution of the inhomogeneous problem, then you can just add it with a general solution of the homogeneous problem and, you know, put it together and that will be your answer, okay. Now, why is this true? Why is this true? Let us see. See, let us say that y, okay, so y,

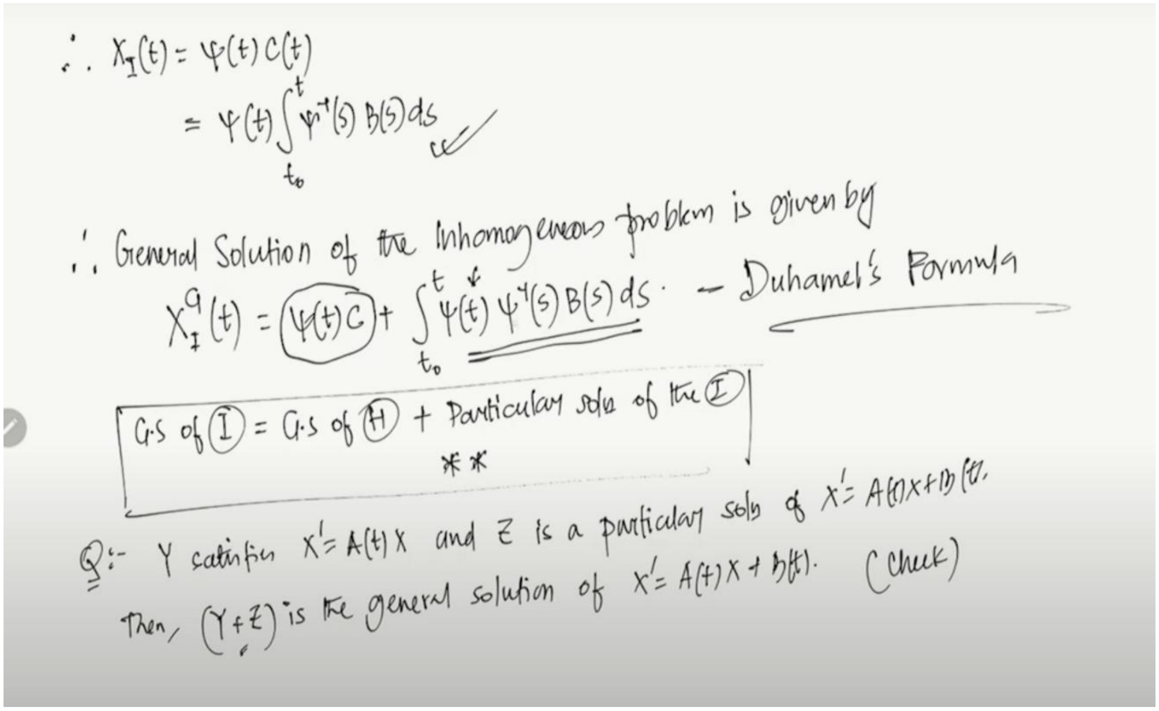
Why is this true? Question. Let us put it this way. Question. You see, if y satisfies x prime equals to At times x and z is a particular value, is a particular solution.

particular solution of x prime equals to a t times x plus b t. Then you can see that y plus z is the general solution of x prime equals to a t times x plus b t. How do you prove it? You just take the derivative of this particular thing and you can easily see that this is true. So I am not doing this thing. You have to check it yourself.

But this is trivial. This is actually trivial. There is nothing to prove here. So this is the fact which we are using here and hence we have this formula. This is also called Duhamel's formula.

Duhamel's formula. Okay. Please remember this. This is Duhamel's formula. We are going to use it all the time.

So basically any given solution, given a fundamental matrix of the homogeneous problem, any solution of the inhomogeneous problem, the general solution means any solution of the inhomogeneous problem will look like the general solution of the homogeneous problem. plus a particular solution of the inhomogeneous problem. And how is that particular solution looks like? It is t 0 to t phi of t. See, this is independent of s, phi of t. So, you can take it outside.



Phi inverse of s, b of s, ds. So, in later videos, we will compute this thing. So, we will see that how can you use this formula to a actually compute system of equation, ok. But for that we need to know something called exponential matrices and all. So, in the next video we are going to talk about that and then we use to famous principle formula to actually I mean look at the general system, yes.

So, with this I am going to finish this video. Thank you.