Ordinary Differential Equations (noc 24 ma 78)

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Week-03
Lecture-13: Linear System 1

Hello students, in this video we are going to talk about some you know relations. So basically we are going to study some relations between linear systems and linear equations. Yes and we want to also know whether you know some basic linear equations has solutions or not. So first of all I hope all of you remember what a Picard theorem is, right?

So we are basing our study from there itself. So first of all, let's just start with some examples. Let's do some examples. See. That let us say we have an equation like this.

Y double prime plus a function of t let us say a of t times y prime plus b of t right times y equals to let us say c of t. Let us say that is the equation which is given to us right. Yes and I want to know and for now for this I mean lecture let us just I mean assume that a, b and c are sufficiently smooth. are sufficiently smooth. This is what we are assuming. Sufficiently smooth.

Now you see the thing is sufficiently smooth. What I mean by this is for now you can just assume there is C infinitely differential. No problem there. But the main point is this. We want to understand that whether let us say this problem has a solution to the question.

Let us pose this question. does this does let's say let's call this one does one admit a solution a solution okay and of course for uniqueness you need to put our initial data yes i hope that you remember so for this problem so let's say uh and and does it have a does it have a unique solution so does a first of all it does add first of all whether does it have a solution or not and if it does so and whether it is unique and whether it is unique right unique clear now you see the thing is but for uniqueness you know you need some initial data given an initial data let us just put it in a bracket given an initial data initial data. So, what that initial data may be?

Let us write it down. So, y prime let us say at the point for now let us just say 0 is alpha and y at the point 0 is beta or for now let us just call y at the point 0 is alpha and y prime at the point 0 is beta. Let us just call it. Now, you see to know whether this problem has a solution or not okay or if it is unique given the initial data we can of course change it to a system of equation yeah so let us do that see for this problem what happens is ah you can actually write it down in this sense that ah we will convert it to a system so define a new function right define v of t to be y prime of t

Yes. So once you do that of course y if y prime is v then y double prime is going to be v prime. Therefore what we have is v prime is nothing but from the equation right. Nothing but minus a t times v minus b t times y. plus ct, right?

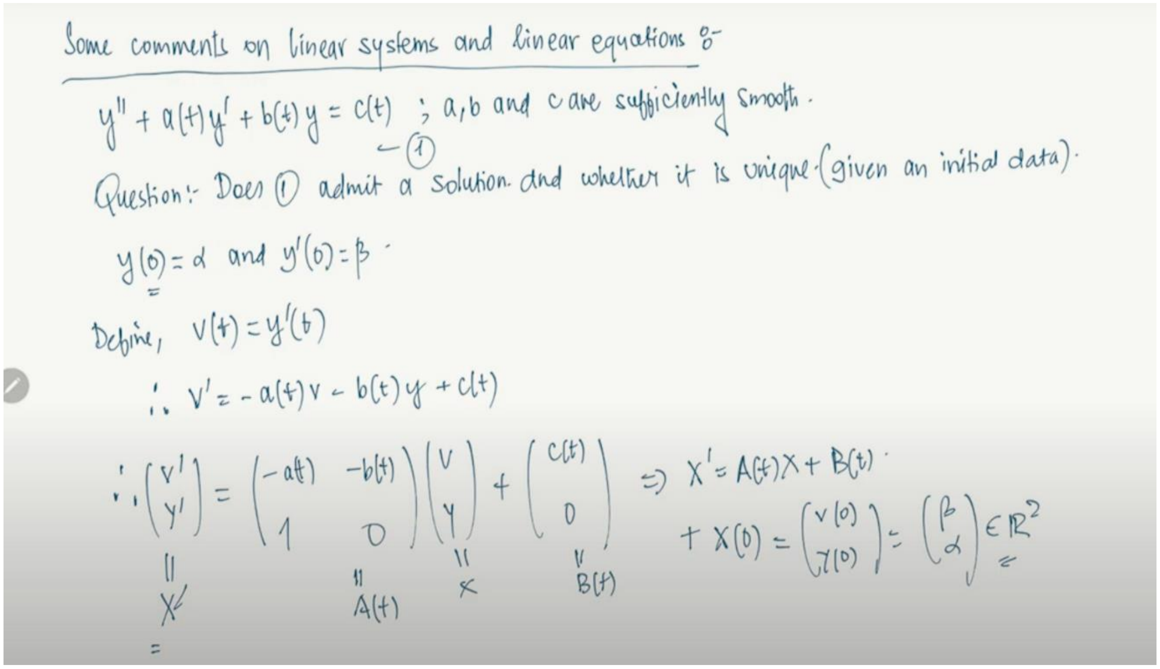
This is what we have. See, now if you write it in terms of a, sorry, if we write it in terms of a system, we have therefore v prime y prime here is equals to v times y plus ct 0, Okay. And y prime is nothing but v. So, basically it is 1, 0. And v prime is nothing but minus bt times, sorry, minus bt times y, right, and at times v. So, it is at times v.

Yes. Now, you see, if you define this thing as capital X, this thing as A, A of course depends on t, and this thing as your b t, then we can actually write it as x prime is nothing but a t times x plus b t, right, plus the initial data. What is the initial data? Initial data is y at the point 0 is alpha, and y prime which is b at the point 0 is beta. So,

What is x at the point 0? It is nothing but v prime at the point 0, sorry, y prime at the point 0, that is v, okay, and y at the point 0, okay, which is given by alpha, which is given by beta and alpha, sorry, beta and alpha. I hope this is clear. I think it is correct also, right. x0 is, sorry,

This is x, right? So, it is, sorry, this is x, right? So, this is, this is x. So, phi at the point 0, y at the point, yeah, phi. So, you see, so, and this is a vector in R2, yeah? So, it is a 2 cross 2 system, right?

As I have always explained. Now, you see, this is, so, now, whether this problem has a solution or not.



So, how does this problem look like? This problem looks like this. x prime equals to a t times x plus

b of t. And x at the point 0 is given by some vector. Let us just call it x 0. So, this is what it boils down to. So, if we want to know whether 1 has a solution or not, we need to know whether this system has a solution or not. Let us just call this system as star.

So, does star have or admits a solution or not? So, the problem boils down to this, admits a solution. If we know this, then we know that, you know, this sort of problem always has a solution, yeah. So, how do you get something like this, okay. So, let us understand this or and also the question is this, whether this problem is well posed or not, yeah, okay.

See, does it fall in this category? If x prime equals to f of x, x0 equals to x0. If it does, then we know that if f is c1, then all of this is star. But you see, star, let us call it double star. Star is not a special case of double star, right?

So basically what I am trying to tell you is this that for a particular f you cannot write it as f of x to be a t x plus b x where a and b are variable. So basically see the Picard existence and uniqueness theorem which I did that is for system which looks like double star. okay and we call this type sort of system as an autonomous system autonomous system okay no mo us autonomous system okay and the thing is autonomous systems are fairly easy to you know work around so let's say that here in star so as a special case let's let me put it this way as a special case special case let's just think or assume that a of t is independent of t. So, basically it is a constant a and b of t is also a constant, let us say, some b. So, essentially what I mean is, you see, here these coefficients are all constant coefficients. They are not variable.

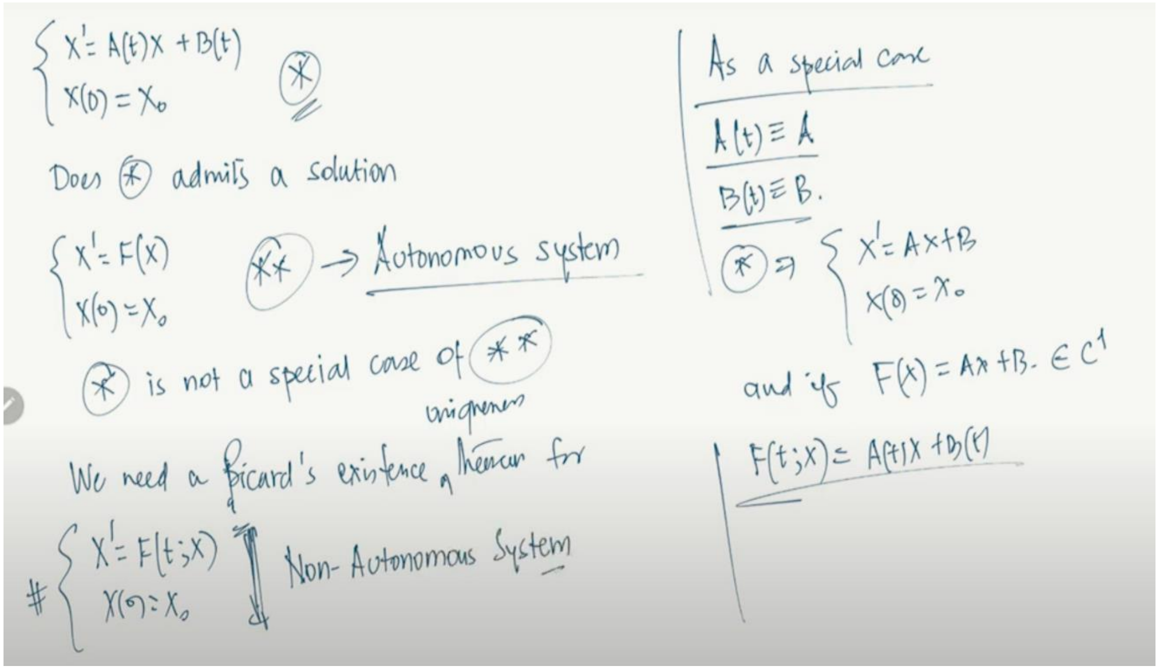
Yes. In that case, what can we say? See, in that case, a t is a and b t is b. So, therefore, star turns out to be this sort of problem, x prime equals to a x plus b and x0 equals to x0. And if you define and if f of x equals to ax plus b, you do realize that f is going to be a c1.

So, this will imply this is a c1. And then you can use Picard's theorem. Of course, we can do that in that case. But the thing is if it is dependent on b, then So, we have to, so basically we need a Picard's theorem.

So, basically the thing is we need, we need a Picard, Picard's existence theorem, existence theorem, theorem existence uniqueness theorem that is existence uniqueness, uniqueness theorem, Picard existence uniqueness theorem for this sort of system, x prime equals to f of t comma x, x 0 equals to x 0. You see, if we know that there is a Picard resistance unit, let us, you know, for this sort of system, let us call this system as hash, okay. If we know that there is a solution, existence of solution and the solution is for the system hash, then what happens is you can define f of t x to be a t times x plus b t, right?

And of course, then, you know, you can use the Picard's existence and uniqueness theorem to actually say that star has a unique solution in a neighborhood of 0 and which will actually imply that one will have a unique solution in a neighborhood of 0. Is this clear? So, essentially, the question boils down to this. Can we say that Picard's existence and uniqueness theorem also is valid for a You know, non-autonomous system.

So, this system, this sort of system is called non-autonomous system. Non-autonomous system. I hope this is clear, right?



Okay. So, for this, actually, we are in luck because Picard resistance invariant theorem does hold.

Okay. Let me put it this way. So, first of all, let me put down the fact that when, what do we mean by a solution and then we will do it for this. Okay. So, consider this system.

Consider The non-autonomous differential equation and what is differential equation? x prime t equals to f of t x, x at the point 0 is x0. okay if you have this system okay first of all i want to show i want to find out what is the meaning of a solution okay so by a solution let's call this system as hash here so by a solution of hash okay we Let me rephrase this part.

Yes. So, basically we say a solution, a solution of hash, system hash, right, is a differentiable curve, is a continuously differentiable curve. Let us just put it this way, continuously differentiable curve, differentiable curve okay x of t and where is it in rn here defined for for t in some intervals in some interval let us say i okay such that such that number one So, 0 is in I and x of 0 has to be x 0.

Of course, that should be true, right. So, the solution which you are going to get should be defined in a neighborhood of 0 here and number 2 for all t, right. So, for all t, t of t, x of t, Must be in some omega. What is omega?

I will define omega. So omega is a subset of r cross rn. Right? It is subset of open. So basically it is open subset of r cross rn and we are going to define f from omega to omega.

r n. So, that is given. So, we are starting out with the open subset of r plus r n and the vector field is given. So, basically t x t should be in omega here and x prime t has to be equals to f of t x of t for all t in i. Is this clear? So, essentially you are given this system Yeah.

What is the requirement on f? So f is a, you see this t is varying in r kind of thing, right. So basically in some interval in r. So we can think of it as an r. So f is defined in r omega which is a subset of omega which is a subset of r cross rn. Yeah. Open subset of code.

This is open. Open. Yeah. And you see we will also assume that it is continuous. Continuous.

So basically it means that it is continuous with respect to both the variables the t variable and the x variable. So first of all if there is a solution that must exist in an interval containing 0. So minus f sum plus f sum interval. must always be in omega. Why?

Because f of t x t has to be defined, right? Because f is defined in omega, right? And moreover, x of t must be equals to f of t x t and this should hold for all t in that, clear? Then we call that as solution, okay? So, with that in mind, we let us write down the local fundamental theorem, okay?

So, this is local again, local fundamental theorem, you can also call it a Picard's theorem, it is not a problem. Fundamental theorem for non-autonomous system. system okay and what is the theorem says let us just look at it so this you know i am not going to prove it first of all the proof is exactly the same sort of proof which you need for autonomous system just an extra variable t is there okay so you have to you know modify it accordingly but the thing is this is exactly the same so i am not doing it please i mean do it for yourself you have to check that part okay so what's the theorem says it says that let Omega subset of R cross Rn be open and capital F from omega to Rn is a C1 function with respect to x. with respect to x. We have seen right that you need f of x has to be c. So, f has to be c 1 because there in the earlier case it is just a function of x. So, we need that same sort of thing.

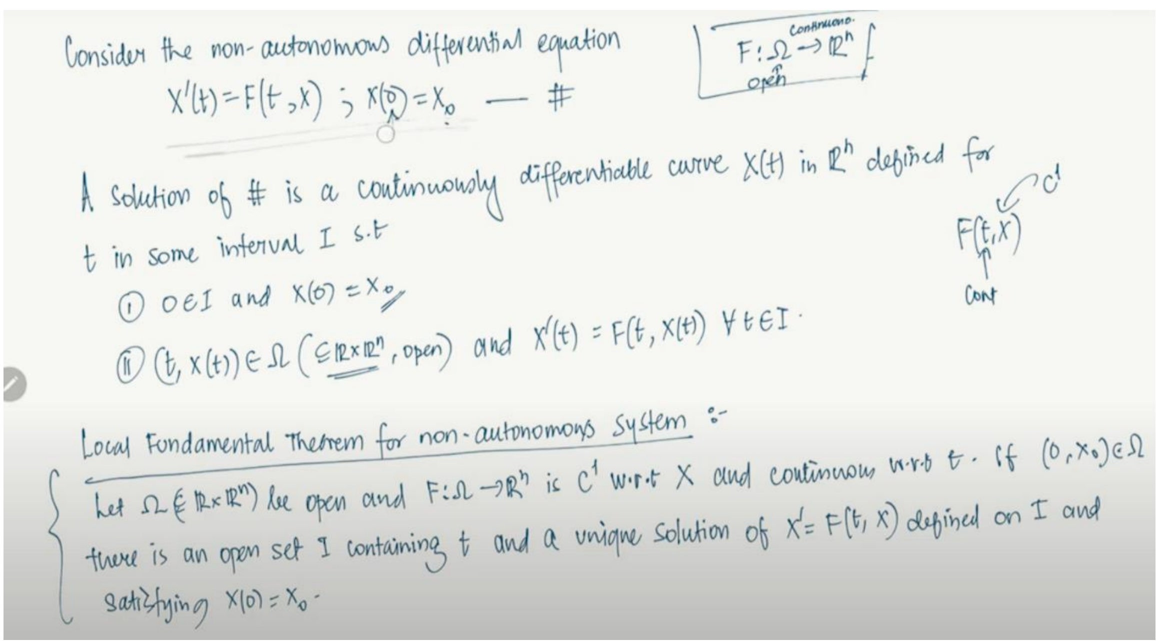
So, it has to be c 1 with respect to x. c 1 means continuous to different human right. And with respect to t what do we need? We need the very basic requirement. We just need continuity and continuity continuous with respect to t right. with respect to t right yes then if let us say t 0 x 0 is in omega okay there is an open set then there is a open set open set okay and sorry what am i writing this is a small

Let me put it this way. If 0 x 0 is in omega, right, if 0 x 0 is in omega, then there is an open set containing, one second, if this set is, there is, if you see one, right, so you will have a unique solution. Okay, this is the open set I containing, containing T. and an unique solution, and a unique solution, unique solution of x prime equals to f of dx defined on, defined on i, on i and satisfying, and satisfying x at the point 0 is x 0.

See, 0 is just a, I mean, it does not have to be, it can be any t 0, does not really matter, ok. I am just writing it to be 0 just to make, you know, look pretty, that is all. So, essentially what I am saying is this. If you have, right, so let's say this is the hash, this is what we have in question, right? This is the problem we have in question.

It's a non-autonomous system. When can we say that there is a solution, okay? So without going into all these details, so essentially what it means is, You see f is a function of t and x right. So, with respect to t it has to be continuous with respect to x1 it has to be at least c1 right.

If then happens then you will have a unique solution in a neighborhood of 0 that is what it is saying. To be very you know if I am not being very precise about it this is what it is saying.



So, basically you have to remember this particular thing here. So, if you want, I can just give you an example. Let us look at an example.

See, one-dimensional example. f of t. x and t is in interval let us say minus epsilon to epsilon minus 1 to 1 or maybe r doesn't matter. So, let us say t is in r and x also is in r. So, x of t essentially. So, now if I am taking f to be mod t plus x.

Now, you see what is your equation and then the equation will look like this. That x prime equals to mod t plus x and let us say x 0 equals to some x 0. It is just an ODE. Now, the question is this whether this ODE has a solution or not. We can actually say that there is this unique solution element root of 0 because you see here this f if you are writing this function with respect to just the t.

Yeah, this is continuous function. It is not c1, but it is continuous. And with respect to this, you do realize that this is going to be a, you know, continuously differentiable function. So, this whole thing, we can actually use in this sort of, you know, examples that it is, you know, there is a solution. Or you can also look at this problem.

Okay. Then also you can say that all of this is true. So, now as a corollary what we can say is this. So, let me put it as a corollary. So, again the proof I am not doing it exactly the same sort of proof works what we did in the because existence and you know you just have to have the single another extra you know variable and you have to take care of that variable is not very difficult same sort of thing works nothing special is happening here.

Now as a corollary let me put and this is very very very important. Please remember this thing. So let A T be a continuous family of n cross n systems. n cross n. So, what it means is all the coefficients are basically continuous. That is what I am saying.

All the entries are continuous functions, ok. So, let t 0 x 0, clear. So, let us say that is in some interval i cross r n. t naught if you are, ok. So, I will just put t 0. You can put t to be 0, does not matter.

Then the initial value problem, value problem. What is the problem? x prime equals to a t times x, right? x at the point t 0 is x 0, let us say.

t 0 can be 0 also, does not matter, okay? So, that has a unique solution, unique solution on all of i, all of i, okay? I hope this is clear. Right. So how do you prove something like this?

Let's look at the proof of this. So without, so first of all you see if you can show that if this is continuous, I think this is, I mean you can actually handle this proof. Okay. So how do you show something like this? Okay.

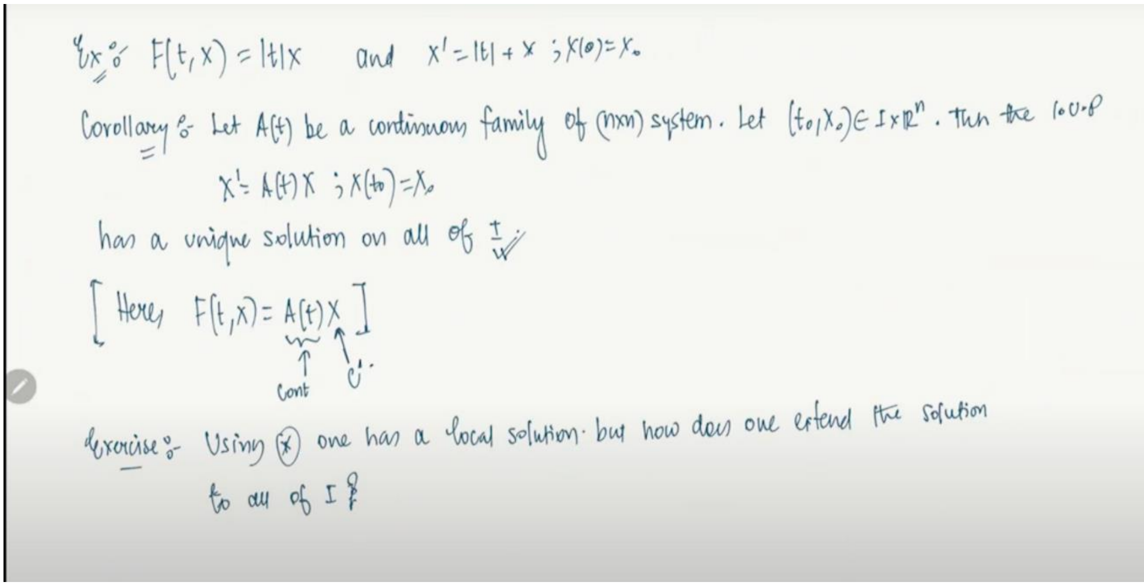
So what we do is this. See here. What is f of t x? So, see I want you guys to check the proof of this. I will just give you an hint of how to do this.

f of t x is nothing but a t times x. Here a t times x. Now, you see this function here a of t it is the continuous family of n process system. So, with respect to this variable this is continuous. right and then again with respect to this variable this is actually c 1 okay so if that is true then you have that f is actually going to be continuous right yes and so sorry f is continuous with respect to a t variable and the c 1 with respect to the x variable so we can use this local fundamental theorem for autonomous system and hence we have a unique solution in a neighborhood of t naught yeah so this problem has a solution Right. Now, this is not the end actually.

So, what you need to do is you have to show that the solution exists for all of I. You see, solution exists for all of I. How do you prove something like that, that the solution exists for all of I? Okay. So, this as a exercise, let me put it this way, exercise. Using Picard, using the, let us call it, star okay using star one has one has a local solution right local solution so basically t 0 minus epsilon t 0 plus epsilon there is a solution but how do you extend that solution to all of pi okay but how does how does one extend

the solution solution to all of i okay i hope you understand what is i what is i i you see is where you know a of t is defined so t is varying in i right okay so what i am saying is this wherever they are continuous so basically let's say a is continuous in whatever interval i In that same interval, everywhere there is a unique solution. It does not have to be on a local. It is not local. This solution actually can show that this is going to be global.

Global means everywhere. See, the thing is if you can show this. So, this is an exercise.



If you can do that, then we have already seen that any problem like 1, this problem. We have already seen that 1.

can be written as a system right which is a non-autonomous system can be written as a non-autonomous system. So, basically if we can show that this non-autonomous system you see this star non-autonomous system. If we can show that this star has a unique solution clear everywhere for all t. So, basically wherever a and b are continuous then Do you have a global unique solution? So basically if there is a solution for this problem, so let us say a, b and c are continuous in some interval i, then you can say and of course there is an initial data, you can say that you have a unique solution.

in the whole interval r. Clear? So, this you have to do it yourself. So, please check that part. Clear? So, what is our summary of this whole video?

In this video, what we have learned? And please remember very very important. Given initial value problem. Okay? This is the initial value given to us.

1 admits. Okay? So, let me put it in red. So, we should remember this. 1 let us call it 2, plus 2, okay, admits a unique solution, admits a unique solution, okay, in I, in I, and what is I?

I is the interval where a, b, c are continuous. So, basically the common interval where a, b, c are continuous, where a, b, c are continuous, ok. So, you do realize that just continuity of the, you know, the coefficients is enough to guarantee that there is a solution here, ok. So, that is what we will just close our video with that comment.

Some comments on linear systems and linear equations :-

$y'' + a(t)y' + b(t)y = c(t)$; a, b and c are sufficiently smooth. ← (I)

Question:- Does (I) admit a solution and whether it is unique (given an initial data).

$y(0) = \alpha$ and $y'(0) = \beta$ ← (II)

(I) + (II) admits an unique solution in I, (I is the interval where a, b, c are continuity)

Define, $v(t) = y'(t)$

$\therefore v' = -a(t)v - b(t)y + c(t)$

$$\therefore \underbrace{\begin{pmatrix} v' \\ y' \end{pmatrix}}_{X'} = \underbrace{\begin{pmatrix} -a(t) & -b(t) \\ 1 & 0 \end{pmatrix}}_{A(t)} \underbrace{\begin{pmatrix} v \\ y \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} c(t) \\ 0 \end{pmatrix}}_{B(t)} \Rightarrow X' = A(t)X + B(t)$$

$$+ \; X(0) = \begin{pmatrix} v(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \in \mathbb{R}^2$$