Ordinary Differential Equations (noc 24 ma 78)

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Week-03

Lecture 12- Well-Posedness of a ODE

Thank you. Welcome students and in this video we are going to talk about wellposedness of an ODE. So what does it mean? So first of all we start by considering an ODE. Let us consider a system.

So I am doing it for a system but you do realize that it can be I mean you can use the concept for any given ODE So consider the initial value problem let us say initial value problem x prime equals to f of x. Let and x 0 equals to x naught okay so this is given to us right now where and let us understand the start with x naught in rx so essentially we are talking about a n cross n system so basically we have a n cross n system okay now the point is this you see let us let us call this a star Now, the thing is given a system like this, what do you want to do with the system? So, essentially we want to study the system, but what do we mean by studying a system?

So, basically what does it mean? What does it mean? Mean to study star, to study the system star, the system, above system that is. Okay and what it means is this essentially you see we are basically looking for three main questions. So first question so that and that is called the well-posedness.

So basically you want to know whether the system is well-posed or not. Okay and what does it mean? So what are the properties which the system should satisfy for it to be well-posed? Okay so properties. So first of all A.

the system system in question star in this question star must have a have a solution okay must have a solution of course in appropriate sense okay appropriate sense you see solutions can be in many sense right appropriate sense okay must have a solution in appropriate sense okay so First of all, it must have at least one solution. That is what I mean to say. It means at least one solution. One solution.

And whatever you want to call, whatever you mean by solution, I was talking about that sort of thing. So, at least one solution should be there. So, first of all, let us just put it this way. Once you have found the solution, now the thing is if there is Exist a solution.

It should be unique. clear so basically there cannot be two solutions to start right so if you have a system or any equation for say okay and if you can find out that that system actually admits two solution okay so whatever the meaning of solution is by that meaning if you have let us say two solutions then of course it is not well posed so basically it you have to have first of all one solution and the solution has to be unique so those are the two properties and the last property is called the continuous dependence on initial data continuous dependence, dependence on the initial data, initial data, okay. So, what it means is this, see for now I am not going to make it much, we will make it much precise just in minute or so, but first for now let me just put it this way. What it means is let us say if you start with x naught, okay, you get a solution, right, let us say x t you get,

Now, what do you think what happens if you start from, let us say, x naught plus delta x naught, okay? So, let us say x at the point 0 is x naught, right? And the system does something. So, basically, you get a x t. Now, let us say, if you are starting with x 0, which is x naught plus delta x naught, you are also get a, you are probably going to get a y t, right? Because, you know, there is a solution and the solution is unique.

Let us just remember that that is true. Once those things are true, now you see for x, x naught starting from x naught you have x of t and if you start from x naught plus delta x naught you have y of t so what do you think is the relation between x t and y t okay see if x x naught and delta so x naught plus delta x naught it is a difference so let us say if delta x naught is sufficiently small okay in some sense i am not putting it like this so basically if it is sufficiently small you expect your x t minus y t should also be you know I mean you can make it sufficiently small, right? So, we are actually looking in that direction.

So, essentially what it means is if you change the data a little bit, you also expect your solution to change a little bit, right? Okay. All of these I will make it very precise just in a minute or so, but you do understand what the concept of this, okay? Right. And this is very, very important property.

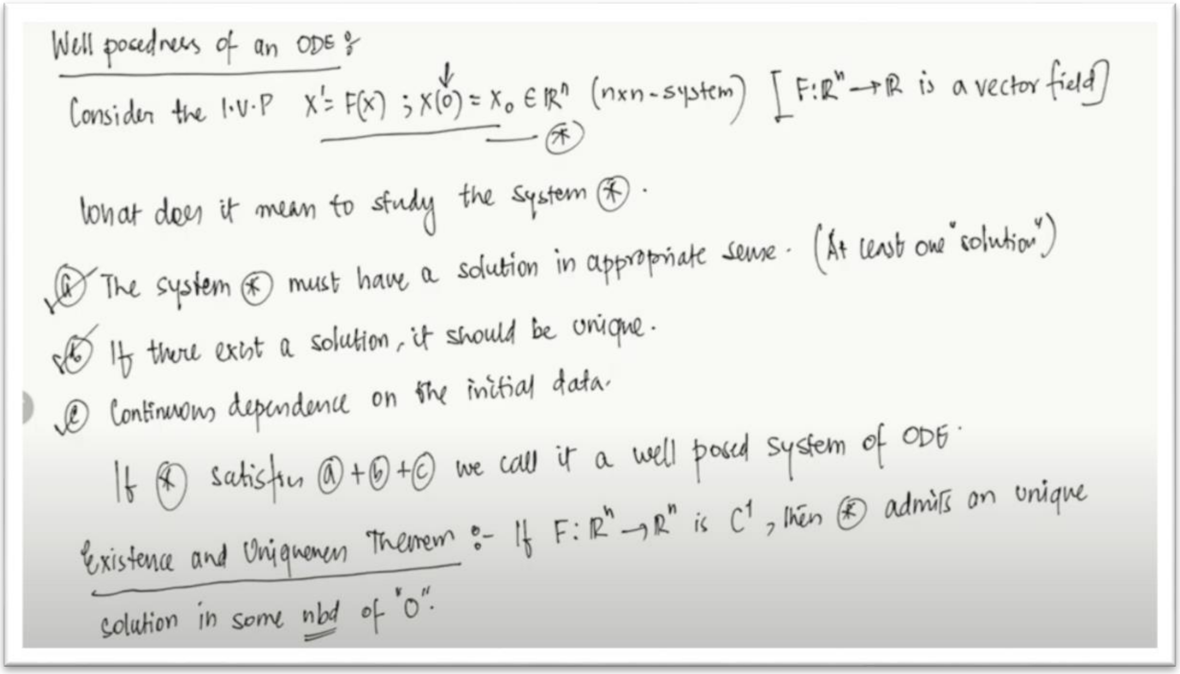
And now, if the system, so basically let me put it this way, if star Satisfies, satisfies A, B and C, we call it a well-posed, well-posed system of ODE, yeah, system of ODE, okay. So that is the main idea. Now let us say I give you a system, whatever it is, x prime equals to f of x. Some system is given, you know what that vector field f is. So here f is a vector field from let us say Rn to Rf is a vector field given.

It is given to you. clear ok now the point is this i want to study whether the system is well posed or not so what is the first thing we only to know whether there is a solution or not if there is a solution whether the solution is going to be unique or not right ok so here we already know we have seen in the last lecture the existence and uniqueness theorem right Picard's existence and uniqueness theorem which you also call the fundamental theorem basically uniqueness theorem so let me again state it here because anyways we are it is very very important and we should always know this okay so what we have is this uh let's say if capital f from rn to rn okay the vector field in question is c1 is c1 right then what do you have then then star, right, the system, you see x prime equals to f of x, x0 equals to x0, this system, given f is c1, okay, admits a unique, admits an unique solution, admits a, an unique solution, solution in some neighborhood of, in some neighborhood of 0, okay. In some neighborhood, let me put it this way, neighborhood of 0.

And what is this 0? This is 0. This is what the 0. So essentially this is t equals to 0. So 0, 0.

Yeah. Okay. So in the neighborhood of minus epsilon plus epsilon, you can actually choose an epsilon such that this system has a solution. Okay. That's what we have.

So essentially, let's say we are assuming f is c1, which is not a very, very strong assumption. It's okay. So with that, we know that they are always have some, I mean, solution, which is a unique solution in some neighborhood of the origin, right?



now what okay now what about the third property so basically a and b is taken care of if f is c1 we know that a and b is satisfied right if f is not c1 then there can be many different cases it may not have a unique solution or it may not have a solution also okay so f is c1 is okay that we know that a and b satisfied okay what about c so before we go on with the c let let us just make it more mathematical what does it mean for a continuous dependence on initial data okay So, what this means is this.

See, what I, let me, let us write down a theorem in this thing. Theorem. This theorem will actually, you know, clarify what I mean by continuous dependence on initial data. So, let omega subset of Rn. This is our assumption.

I am assuming omega subset of Rn here. And let, and F is a vector field right from omega to Rn. This is what we are assuming here. Omega to Rn is a vector field with a Lipschitz constant K. So basically we have a Lipschitz continuous function with a Lipschitz constant K. Let us just put it this way.

a. Clear? Okay. So, k is in R. You do realize that, right? Lipschitz constant. Okay.

You remember what a when it is Lipschitz. So, f is from omega to R. Basically, it means that f of x minus f of y has to be the norm has to be bounded by k times norm of x minus y. You remember that for all x, y in omega. Right. For all x, y in omega. Okay.

That is what Lipschitz means. Right. Okay. Right. So, we are assuming this thing.

So, please remember this thing. See, The thing is, for the well posedness, we did it for a C1 map. C1. C1 vector field.

Now, any C1 vector field is locally Lipschitz, right? This is locally Lipschitz, but not, may not be Lipschitz. It is locally Lipschitz. Okay? It is locally Lipschitz.

Now, here what I am saying is, you need your vector field to be Lipschitz. Lipschitz continuous, right? Now, if that happens, let, we are assuming y of t, And Z of t, and Z of t, these be solutions of, solutions of f prime equals to f of x, right? Which remains in omega.

right which should remain you see the thing is you see both the solutions y t and z you are getting two solutions right and the thing is those solutions has to be in omega because otherwise f is not defined right so we have to have that those solutions must remain in omega right because f depends c for every t x of t okay x of t should be in omega, because you also have to define f of x, right, for this equation to work, clear. So, this x of t has to be in omega. So, basically here I am starting out with y and z, those should always be omega, okay. And are defined, are defined, okay, on an interval, on the interval, let us say,

Interval, some interval, right? And let us just put the interval as t0 to t1, some interval here, okay? Then, then for all t in t0 to t1, t naught to 2 t 1, we have, okay, now this is what we mean by continuous dependence. You see, norm of y of t, see y t and z t for given t is in R n, okay.

So, y t minus z t should always be bounded by y at the point t naught minus z at the point t naught, okay, times some exponential map and which depends on k times t minus t naught. Clear? So you do understand C if y t minus z t. See t is varying between t naught and t 1. T is varying in a bounded domain. So exponential k times t minus t naught that is a bounded function essentially.

And k is fixed. That is a Lipschitz constant. Now you see. if your y t minus z t at the starting, at the point t cos t naught is sufficiently small, of course, y t minus z t, you can make it as small as you like, right? You can always find a delta.

So, given an epsilon, you can always find a delta. So, that is what I mean by continuous dependence on initial data. So, this is what it says, right? So, again, let me put it this way in case you forget. If the

Solution, it means that if the solution, what are the solutions? yt and zt, right? yt and zt, clear? If the solution, yt and zt start out, okay, start out, sorry, start out close together, right?

So, if they, you know, if yt and zt are close together, they are very, you know, the difference is very small, then, then

They remain, remain close, close for all time t, right? Sorry, for all time t close near t naught, right? Remain close for all time for all t near t naught, okay? Near t naught. So, you will have a small neighborhood of T naught where it has to be, it will be close to each other.

So, T naught can be 0 also. It does not really matter, but you do realize what I am trying to say is this. Whenever you are starting out, it does not really matter which point you are starting out with. Some T naught you are starting. And then what happens is you always have the solution actually is completely close to each other.

Okay, so to do this thing, this is the proof is extremely easy. What we are going to do is we are going to use the Gronwell inequality. And now what I am going to do is since that is done in week one, I will just quickly give the statement of Gronwell. Okay, as I told you, this is very, very important, right? Gronwell.

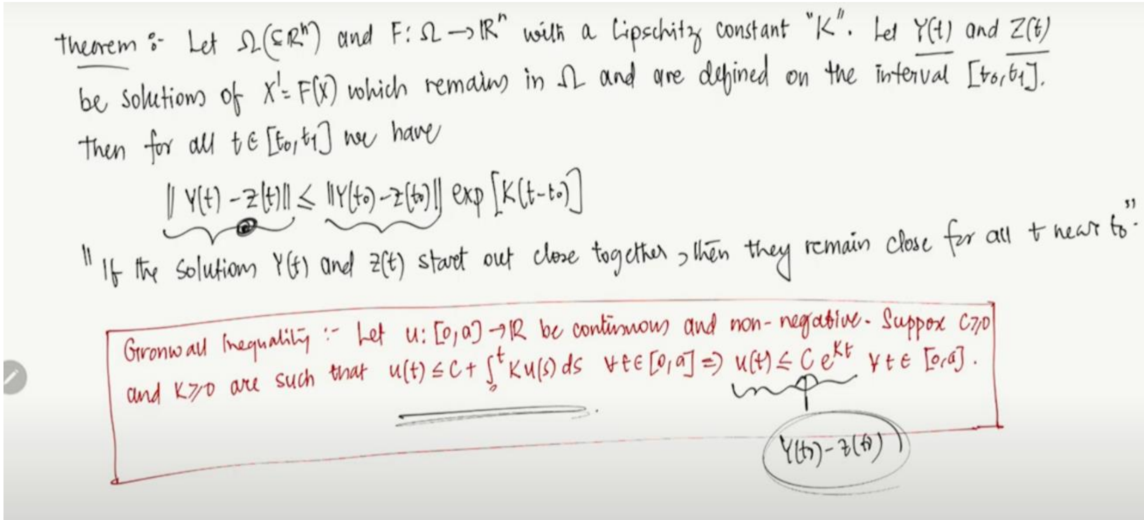
Gronvall inequality. Let me just write it down and then we are going to use this Gronwall inequality. The proof I am not doing because we already did it. So let us say that u from 0a to r be continuous and non-negative. Continuous and non-negative.

Non-negative. non-negative. Suppose c is greater than equal 0 and k is greater than equal 0. Right? Are such that are such that u t is less than equal c plus 0 to t k times u s d s.

for all t in 0a. What does that imply? Then, that will imply that u t can be made bounded by c times exponential k t for all t in 0. So, the proof, I am, of course, I am not doing the proof, you guys, we have already talked about the proof, so I am going to skip that part. Okay.

Now, let us understand that what is the proof of this theorem, the continuous dependence. Let us just look at the proof of that theorem. So, proof. See, the thing is, it is very simple actually. And please remember the trick.

This trick, we are going to use it in a lot of places. See, Here, I have to show that this quantity, particular quantity is less than equal this quantity evaluated at the point t naught, see, times some exponential, okay. So, you do realize that you see, I, here also I have something like this. So, this constant, if I can show that this is nothing but y t naught minus z naught, z t naught, okay.



And then, am more or less done right okay and ut essentially this if i if you can find this we if we can somehow write it as ut then and i can show that this constant will work then we are done right so we have to find out something like this once we have something

like this that will imply this right you understand so we are going to try and do that okay so how to do that let's do it okay so we will choose what that define sorry, define capital V t. Yes, this I am defining as the norm of y t minus z t. Here, see, the thing is t is varying between the, sorry, here t is varying between t naught and t 1, right. So, basically for that t naught and t 1, V t I will be defining to be the norm of y t minus z t, okay. If I do that,

Now, you see, since what is y t minus z t? You remember, you see, y t minus z t are the solutions of this problem, right? x prime equals to f of x. So, those also satisfy the integral equations, right? The corresponding integral equation. Now, what is it?

So, y t is given by y t naught plus t naught to t f of y s d s. And what is z naught? z t given by z t naught minus t0 to t, capital F of Zs, ds. You remember, when we did the Picard theorem, we did the exact same thing, right? Okay.

Therefore, what do we have? We have yt minus Z of t is nothing but y of t0 minus Z of t0, okay, plus t0 to t, F of ys, minus f of zs ds. This is what we have. Therefore, you see what is this?

This is vt, right? So, if I take the norm on both sides, so taking, let me put it this way, taking norms, norm on both sides, what do you have? Both sides, This is nothing but V of t and this will less, I can write it as V of t naught, triangle inequality plus t naught to t, the norm of f of ys minus f of zs ds. This is what I can do.

This is triangle inequality, plain simple triangle inequality in Rn, triangle inequality. Okay. So, now, you see, what do we have? Vt is less than equal V of t naught. You see, this is where we need the Lipschitz continuity.

You see, here, if I use Lipschitz continuity, I can say this is bound, this can be dominated by the constant Lipschitz constant k, t naught to t, and this ys, y minus z, norm of y minus z, which is nothing but V of s ds. Right? Now, you see, this is in this form. Okay, what is your constant? Constant is this, okay, and this constant is always there, u of s is, this is in this form.

So, that will imply u of t is less than equal c times kt. Here, so this will imply, therefore, therefore, we can say that u of, sorry, v of t, v of t equals to once again i have to match this part so once again let me just do this part uh we have to match it so what i am going to do is this okay so i am going to define a new function i just have to shift it a little bit so to make the you know this variable is a problem you see it is from zero to t right so here it is from t naught to t i just have to shift the variable now to make it zero to t and then i can use the Grownall right so what you have do is this you define u of t right u of t capital u of t to be v of t plus t naught okay if you do that you see u of 0 is nothing but v of t naught so this is u of 0 yeah okay so what do we have then that will imply that u of t which is equal to v of t plus t naught okay v of t plus t naught and that will be bounded by v of t naught plus t naught to t plus t naught k times v s d s u of t u of t is v of t plus t naught

right so i will just write in v of t plus t naught and that is always dominated by here you see it is i will just change the variable here so basically from t naught to t plus t naught this will change okay so i can write it like this this is what i am doing here i hope this is clear See, this holds for all t, this holds for all t. So, this holds for t equals to t plus t naught also, right. So, I will just, I am just doing that part here. And the variable is from t naught to t plus t naught, that is what, okay. So, that is there.

Now, this is nothing but v of t naught plus 0 to t, okay, k u of tau d tau t. I will just change v, this v to u and then this integral becomes 0 to t. Therefore, you see, therefore one can say from here, one can say that v of t plus t naught, t plus t naught is less than equal v of t naught times exponential k t. That is what we can say. that will imply that what is v of t then it will imply that v of t is nothing but it is dominated by v of t naught times exponential k p minus t naught. Clear? And this is what the conclusion is this is what the conclusion is so we have the you know the cartilage dependence on initial data okay.

Proof: Define $V(t) = \|Y(t) - Z(t)\|$

$$\because\ Y(t) - Z(t) = Y(t_0) + \int_{t_0}^{t} F(Y(s))\,ds - Z(t_0) - \int_{t_0}^{t} F(Z(s))\,ds.$$

$$\therefore\ Y(t) - Z(t) = Y(t_0) - Z(t_0) + \int_{t_0}^{t} \left[F(Y(s)) - F(Z(s))\right] ds$$

Taking norm on both side

$$V(t) \leq V(t_0) + \int_{t_0}^{t} \|F(Y(s)) - F(Z(s))\|\,ds. \quad \text{(Triangle Inequality)}$$

$$\Rightarrow V(t) \leq V(t_0) + K \int_{t_0}^{t} V(s)\,ds$$

Define, $U(t) = V(t+t_0) \Rightarrow U(t) = V(t+t_0) \leq V(t_0) + \int_{t_0}^{t+t_0} K V(s)\,ds = V(t_0) + \int_{0}^{t} K u(z)\,dz$

$$\therefore\ V(t+t_0) \leq V(t_0)\exp[Kt] \Rightarrow V(t) \leq V(t_0)\exp[K(t-t_0)] \quad \square$$

So, what do we have is essentially you can say that if you have a system which is let us say Lipschitz continuous in the whole whatever the domain is. So, if you have a system such that you know this vector field is Lipschitz continuous in wherever it is defined let us say from omega is defined as omega is Lipschitz constant in omega.

Do you think it is C 1? Let us say if it is also C 1. So, basically Lipschitz constant Lipschitz continuous along with that it is C 1. Then you can actually say that the problem is well posed. So, why Lipschitz continuous and C1 does Lipschitz continuity implies C1?

Of course not because fx equals to mod x as you know this Lipschitz continuous in R but it is not going to be C1 in R. So, with this I am going to end this video.