Ordinary Differential Equations (noc 24 ma 78)
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Week-02
Lecture-11
Linear Systems

Hello everyone.

Let us look at the properties of linear system.

So up till now we have looked at you know what are the system how does the linear system looks like.

So essentially we have a system which looks like this x prime of let us write down as the variable as t which is a t

x of t plus b of t okay yeah and here we are assuming that here a of t is a n cross n matrix right is a n cross n matrix

matrix with entries, entries aij.

Clear?

And these aij of course, they depend on x. So let me put it this way.

They depend on x. And bx here, okay, sorry, sorry, not x because we are writing the variable as t. So this should be t. And b of t, okay, is a n cross matrix.

and x of t x of t that will be your n cross unknown vector okay is the n cross unknown vector okay so what are the components of x of t x of t the component should be x i of t essentially with components with components x i of t

components x i of t okay so let's just write it down properly and then that will make it more precise so basically for example so for simplicity we will just do it for n equals to for simplicity we will we will throughout this whole video we will uh assuming assume n equals to .

I hope this is clear.

Exactly the same thing works for any n. So, it does not really matter.

Let us just do it for n equals to .

That makes our life much easier.

So, essentially, for n equals to , how does it look like?

It will look like this.

You see, x prime of t, it will be a of t, right?

x of t plus a of t, x of t.

So, a, a, these are the coefficients given coefficients and x prime of t is a of t, x of t plus a of t, x of t.

So, this is the system given let us call this system as for n equals to and for a general m you can write it in a compact form which is looks like this.

Here we are assuming here b of t is generally .

Speaking it should look like b of t and b of t but for our case all purposes we are just assuming this to be for now.

So this is a general form of a linear system cross system in this case you can do it for n cross n.

And what is the initial data?

So, generally speaking, the initial data which will be given to you in this kind of cases.

So, basically, it will be called an initial value problem.

Initial data will be capital X at the point t equals to .

So, X equals to is let us say x .

So, this is a compact form.

So, what does that mean?

It means that X at the point and X at the point will be given to you and that will be

x , let us say, and x , a vector in R.

Okay?

Right.

So, that is the problem.

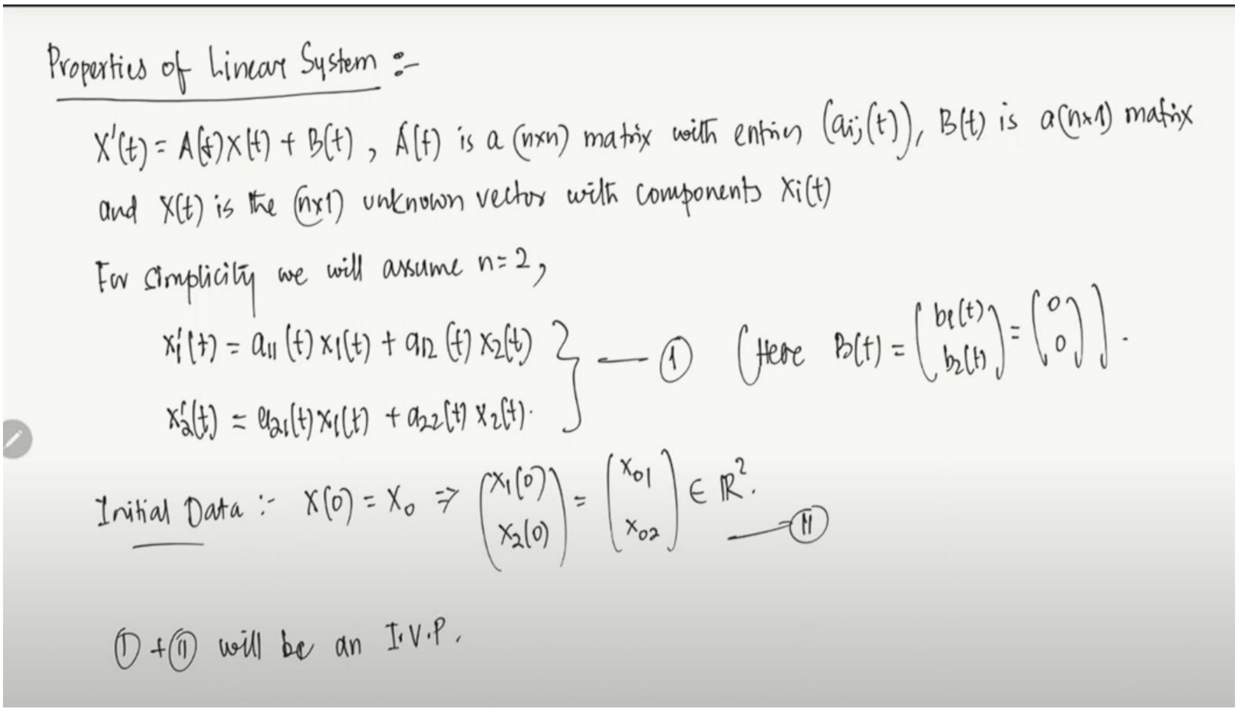
This is the, is the equation which is given to you and the, let us say, initial data is this.

So, let us call this .

So, plus , plus will be, will be an initial value

problem, initial value problem.

So, we will call it a initial value problem for a system.



Now, there are some properties which this sort of linear system satisfies.

What are those properties?

So, for now, I am not always going to write it as , but I am always going to write it as star.

The system has star.

So, you see,

The first thing is something called a principle of, principle, so this is very important, of superposition.

Superposition.

So, see, we have seen that for a linear system, when we solve linear system, we solved, we showed that if y and y is a solution, c times y plus c times y is also a solution, right?

So, that is a superposition principle.

So, basically, what this says is, let us say that let,

Let u of, okay, let x of t, let us just call it x of t and capital X of t. So these are capital X of t, X of t be two solutions, two solutions of star, okay, be two solutions of star.

Okay, so I am starting out with this equation, start the first equation.

So if there are the two solutions, now let us look at this, see what happens, x prime of t is nothing but a, t, a of t, x of t plus b of t.

right and so sorry x of t plus b of t and also x prime of t is nothing but a of t x prime of t uh sorry x of t plus b of t so those are always satisfied therefore if we define a new function x of t yeah this is a new function which is not nothing but c of x of t plus c of x of t

x of t now you see that x prime of t is nothing but c of x prime of t which is nothing but a of t x of t plus b of t right and plus c of x of t is nothing but at x of t plus bt so at x of t

plus bt, right, bt.

Now, you see, this is an observation here that it implies that x prime is equals to c times, right, c times x of t plus c times x of t times at, right, times at plus

c plus c times b, c plus c times bt.

So, you see if bt is the vector, in R,

So, basically I am assuming that B, there is no Bt.

So, it is a homogeneous equation.

So, that means that will imply, implying that star is a homogeneous equation, right?

Homogeneous, sorry, not equation, homogeneous system, homogeneous system of equation, system of equation, system of equation.

Then what does it say?

That

You see, x satisfies x of t. Therefore, x of t satisfies x prime of t equals to a t times x of t.

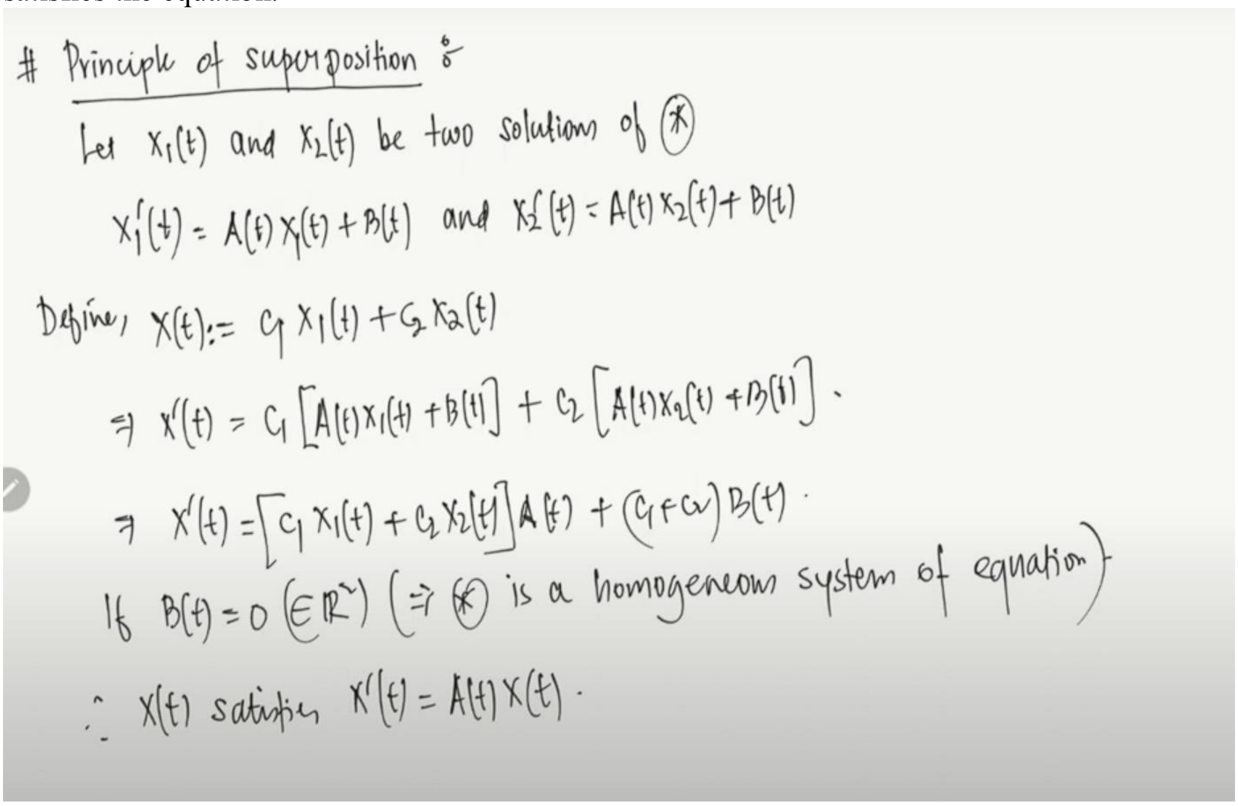
Okay, because Cx plus Cx, I defined it to be x of t. Okay, I defined it to be x of t. So, this is definition.

I should not write it like this.

This is definition defined.

Okay, so x of t satisfies this.

So, you do realize that if you have two solutions xt and x of t, okay, and if you are dealing with a homogeneous system, this constant times x, some C plus C, C times x plus C times x also satisfies the equation.



So that is always there.

So now we come to a, so with this we come to a very important property of something called a linearly independent set of functions.

So definition, definition, definition.

A vector valued function.

So, now we are talking about vector valued function.

First of all, let us quickly recall what a vector valued function is.

So, essentially we are looking for a function, we are basically saying phi is a vector valued function.

If phi, the, you know, the image, sorry, the domain of phi is R or a subset of R.

And it goes and takes it to Rn.

Okay.

So that sort of function we will call it a vector valued.

So basically phi of t is a vector in Rn.

You understand.

So the value is taken in as a vector.

So that is a vector valued function.

This is called a vector valued function.

valued function.

So, you do realize here whenever we are saying something is a solution, this is a vector valued function, this is the vector valued function, okay.

So, you are dealing with vector valued functions here, okay.

So, let us say the vector valued functions, vector valued functions, functions and what are the vector valued functions?

Let us just call it x of t, x of t,

xn of t. So I am working with like n sort of functions.

And they, let us say they are defined, defined on, they have to be defined on a common interval, defined on i, let us say, some interval i, the common interval, are said to be linearly independent, said to be linearly independent.

linearly independent.

So, this is basically the same sort of idea what we do in vector spaces for general vectors also, but here for just for some sort of function.

If c times x of t plus c times x of t plus cn times xn of t equals to implies as you know

for linear independence, we have c equals to c equals to cn has to be .

And please remember, the c, c, cn, these are all real numbers.

And x of t, for fixed t, x of t has to be a

vector okay in Rn for n equals to so if you are dealing with Rn it is n Rn it is Rn so basically what I am trying to say is this if n equals to it is basically a vector in R okay.

So now when is it a linearly dependent?

Of course, if one of the ci is a non-zero, then we have a linear dependent.

So they are linearly dependent.

If there exists constants

constant ci's, okay?

So, basically, let us say c, c, cn, okay?

Not all , not all .

So, at least one non-zero, not all , such that, such that this c times x of t plus c times x of t plus cn times xn of t, this should be .

Clear?

So, very simple.

Now, let us look at some examples and then we will proceed.

So, the first example which I am going to give is this.

So, the most simple problems which you can think of is the functions which you can think of it as polynomials, right?

Okay.

So, let us look at this thing.

Let us look at this set , x, x square and x power n.

OK, so we are saying that this is linearly independent, linearly independent.

This set is linearly independent.

OK, see, I am saying set is linearly independent.

I mean, in most books, they say the functions are linearly independent.

OK, that's what I mean here.

OK, so it means that the functions are linearly independent.

I'm just doing it like this.

But of course, we should say it like this.

The functions are linearly independent.

OK, linearly independent.

Okay, I am writing it like this just for, I mean, avoiding any confusion here.

Okay, so these functions are linearly independent in any, every interval, let us say, every interval, interval I, whatever the interval is, does not really matter, but if they are linearly limited, if they are defined and is defined in whole of R. Okay, so how do you prove something like this?

see c plus c times x plus c times x square plus cn times x power n this is right okay and if at least if any ck is not so let's say if let's say ck there is a k which is not for some for some

k, okay, less than equals to n. For some k less than equals to , let us assume that it is non-zero.

So basically you see, if I have to show it is linearly dependent, this should imply, this should imply c equals to c equals to cn equals to , okay.

So if this is not the case, then at least one of the ck's has to be non-zero.

And then the equation, you see, then this equation must, could hold for at least m minus values of x, right.

You understand?

See, if one of the ck's are non-zero, so essentially what is happening is this.

Then this equation c plus cx plus cmx power n equals to zero.

This will hold for at most.

for at most k values of x, right, of x. So basically what I'm trying to say is this.

So there will be like k values of x for which they are like, I mean, this relation should hold, right?

And then what happens is, but this relation in this case has to hold for all x, right?

But it should hold for all x.

Should hold for all x, right?

So that's a contradiction and hence, okay.

See, this is one way of looking at it.

If you are not convinced, you can do it like this.

Let's say, okay, let me put it this way.

Let's say and x, are they linearly independent?

How do you prove something like this?

So please check this part.

Check.

Let's say this equation, and x, these two functions are linearly independent.

How do you prove something like this?

Linearly independent.

Let me write it like this.

How do you prove something like this?

Okay.

So you start with this c plus cx equals to .

Clear?

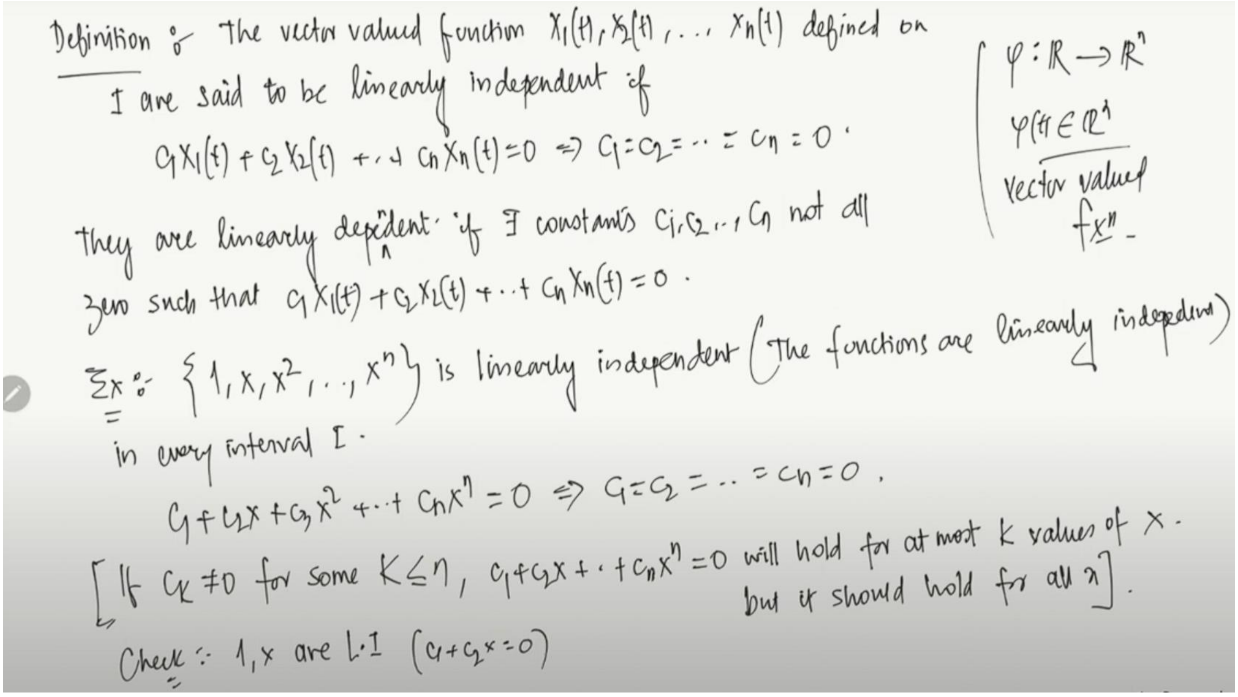
And now if you differentiate both sides, you do get that c has to be .

Right?

Yes?

And if c is , then of course c has to be .

So c and c has to be .



So they are linearly limited.

You can argue like that also.

It's not a problem.

Okay.

And so, I mean, you can do it for any n. So, basically, x x square x power n linearly independent is going to be the case.

Okay.

Now, let us look at another example of a vector valued function, right?

So, x of t given by e to the power t, e to the power t, clear?

And

x, sorry, of t and x of t given by e to the power t and e to the power t.

Let us just assume that.

Okay.

So, they, do you think, the question is this, do you think they are linearly dependent or independent?

Okay.

You see the thing is if they are linearly dependent one vector has to be multiple of another but here clearly you can see that this does not work.

You see even it may be multiple for one particular value.

It may be multiple for one particular t but for this thing to work.

Here the difference with you know the general vectors and this one is since they are they should hold for all t it makes life complicated.

So basically c times e to the power t e to the power t plus c times e to the power t and e to the power t equals to less than .

So basically you do realize this is in R.

okay this is zero part not like zero zero right okay so this will actually imply two things right c e to the power t plus c e to the power t has to be zero this is in r this is zero so this is in r so let me put it this way this is zero and of course again by equating you have c e to the power t plus c e to the power t this has to be zero

Now you do realize that if both the cases has to hold then c has to be equal to c has to be equal to .

So therefore x and x are linearly dependent.

Dependent, right?

So please make sure that what I'm trying to say is, I mean, you understand that this relation must hold for all t in wherever the interval is, right?

So that has to be true.

Okay.

So from now on, I will always write it as LD or LI, meaning linearly dependent or linearly independent, right?

Okay.

Now we define something called, so definition.

We define a new sort of function, which we are going to call a Wronskian function, right?

We are going to define, so given, let us say, given n vector valued function, vector valued function, valued function.

Okay.

And how is it given?

Let us say it is given by x of t, xn of t. Okay.

And then you define.

So given all this function, okay, the determinant, the determinant, determinant w, I will write it like this, determinant of x, x, xn.

Yeah.

See, this, of course, depends on t, right?

This, so we can also write it like w of t, right?

So, given those functions, w is a new function which depends on all these functions, okay?

And that will be nothing but the determinant of this.

So, essentially, you see, x is a vector, x or xi, right?

This vector.

at any point t is nothing but an element of Rn, right?

So, xi of t, if you think about it, it should look like the component.

So, basically, let us say xi, the first component, xi, the nf component, right?

It should look like this.

So, we will write all these things together and then we will just look at the determinant.

So, basically, it will look like this, x of t

x of t x n of t okay and this goes on this goes on and then this is again x n of t and x n n of t

Okay, so you just put every you see x you put it in the first column x you put it in the second column like this.

So put in the column, but it doesn't really matter you have to do it the other way also.

But anyways, so because since we are dealing with determinant, but so essentially this is called the Wronskian.

So this is called the Wronskian.

Okay.

So, what is a Wronskian?

Basically, it is a function.

Please understand this.

This is a function.

What sort of function is it?

Wronskian is a function.

So, Wronskian is a function from t. See, this depends on t. So, basically, it is taking values.

Given the n functions, it is taking values in whatever interval t is defined.

So, let us just assume that t is defined in i. And what is the end product?

W of t is going to a determinant.

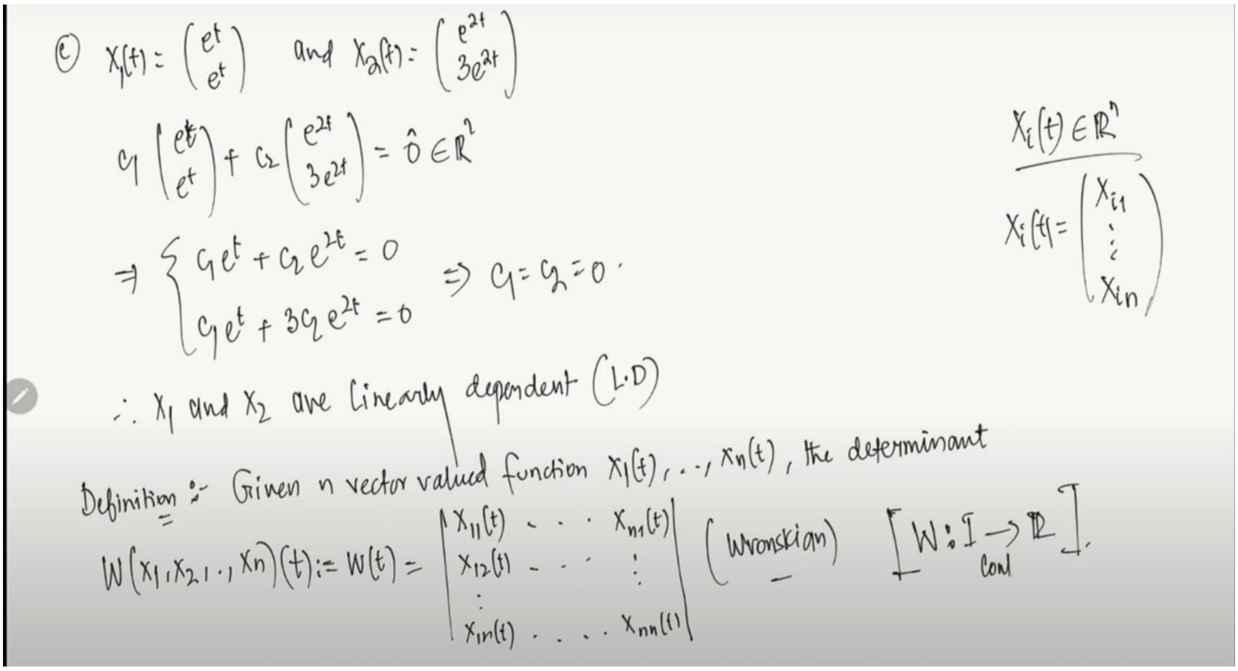
So, basically, i is a real valued function.

Clear?

And now, let me ask you this question.

See, you do realize that this is basically, I mean, since this is a determinant, this Wronskian is going to be a continuous function.

continuous function, okay?



That is very easy to see, right?

It's just a determinant function.

So that's a continuous function, right?

Now, the important thing which we need to understand is this theorem.

This is a very, very important theorem.

So what does this theorem say?

So please remember this theorem.

We are going to use this theorem a lot.

So if, if the Wronskian, the Wronskian, okay, and for now I will just always write Wronskian as W of t, without any exception.

This is Wronskian, Wronskian of n vector valued functions, n vector valued functions, functions, okay, what are the functions?

x of t.

xn of t, okay, is different from , is different from , from , clear?

So, basically what I am trying to say is the determinant is non-zero for at least one point in the interval, okay, for at least

At least one point on the interval.

One point.

One point of the interval.

One point.

Of the.

Interval.

Okay.

I let's say.

Then.

These functions.

These functions.

Are linearly independent in I. Okay.

Linearly.

independent in I. So basically what I am trying to say is this see, the Wronskian doesn't have to be, it's not like it has to be everywhere or non-zero for all T.

Just for , t, if you can show that is different from .

So basically it is non-zero for at least t in i, then you have that the functions are linearly independent.

So how do you prove something like this?

Let us look at the proof of this.

So, you see, first of all, let us assume that since I am saying that we have to prove they are linearly independent.

So, if they are not linearly independent, it can be linearly dependent, right?

So, let x of t, x of t, xn of t, let us say, be linearly dependent in I.

So, if that is the case, then there exists, so that implies that there exists constants, constant or maybe s constants, we do not know, c, c, cn, not all , at least one of them is non-zero, such that

Such that, you see, summation, right, i equals to to n ci, okay, xi of t, this is , right, in i, whatever the interval is, right.

That is the definition, that is the definition of linearly independent, okay.

Okay.

Now, what does this say?

See, if you break it up, so if you break it up, let us do it here.

Just a little remark kind of thing or maybe a little something to note.

You see, for , n equals to , let us just see n equals to .

What does it mean?

It means c times

x of t, x of t plus c times x of t, x of t is nothing but , .

So, what does that give you?

That gives me that the homogeneous system of equation.

So, this will imply that the homogeneous system of equation,

system of equations.

What are the system of equations?

c x plus c x is and c x plus c times x equals to .

This system of equation has a non-trivial solution, right?

Okay, homogeneous system of equation.

Summation x

i k okay k is the component the kth component okay of t c i this is i equals to to n yes you understand i am just writing it in terms of a you know shorthand notation so that will imply you see ah

c x plus c x is and c x plus c x is .

This is what I am just writing, okay.

So, this homogeneous system of equation has a non-trivial solution, right, non-trivial solution.

See, at least one of the ci's are non-zero.

So, basically it says that this system of equation has at least one non-trivial solution, okay.

Right.

Now, you see, we know that this homogeneous system, okay, this homogeneous system, this homogeneous system has a non-trivial solution.

This we talked about in the first week, right.

Okay.

So, this homogeneous system has a non-trivial solution when if and only if the Wronskian is zero.

Okay.

So now that will imply that the, this should imply, both sides, that the Wronskian at the point t has to be , right?

What is the Wronskian?

Wronskian is nothing but x, x, x, x, right?

This should be .

That is the only way it can happen.

See, this is a general, this is just a ODE, sorry, this is just a system of equation, right, for a fixed T, let us say.

It is a system of equation.

So, essentially, what is happening is, you are basically showing that if the Wronskian is , then C, at least one of the C and C has to be non-zero, okay.

So, this is what I am just saying in n cross, for our general n. So, that will imply that Wronskian of T has to be , okay.

But,

You see, it is given that Wronskian of t is different from for at least one point.

But this says that Wronskian of , t is for all t in i, right?

But Wronskian of t is not for at least one t. One t in i. So, that is given, which will actually give you a contradiction.

I hope this is clear.

So again let me put it like this.

Here in this sort of thing since we are dealing with n cross n systems things look a little complicated.

What you can do is you break it up into.

 cross systems, and you can be rest assured that if it works for cross , this sort of theory, at least in these basic levels, whatever theorems are there, if they work for cross , generally speaking, of course, I mean, most of the time, they work for n cross n. Okay?

Right.

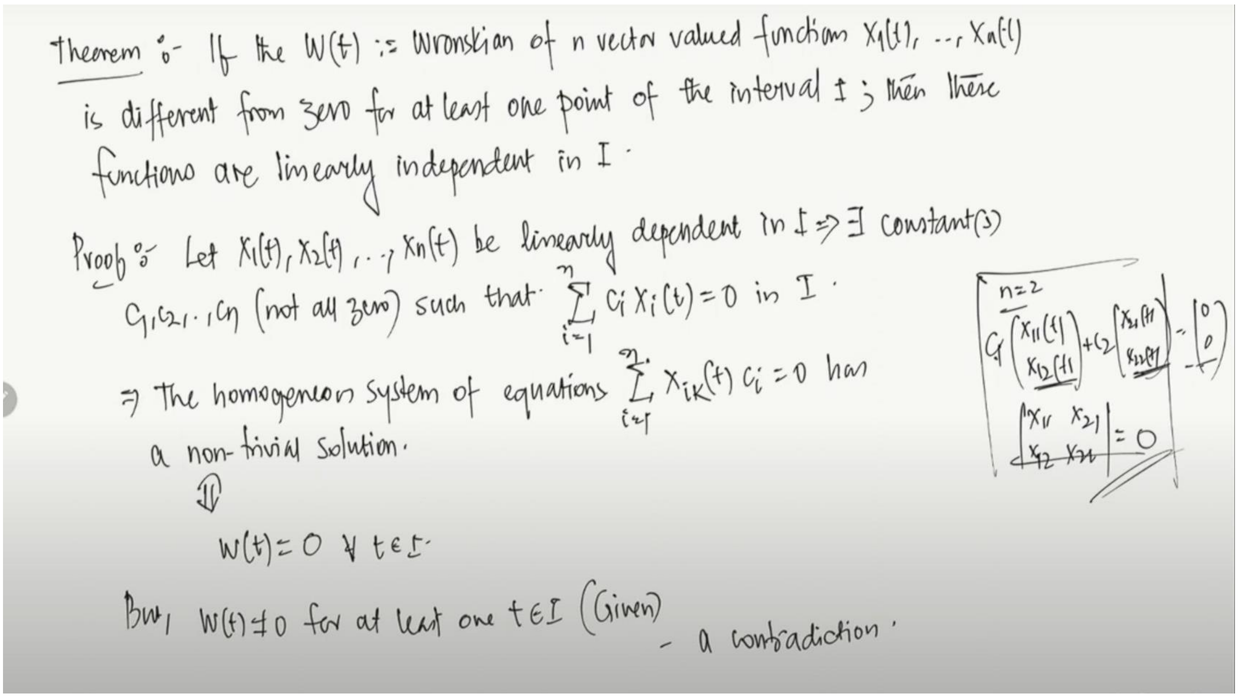
Right.

So, this is true.

So, what did we prove?

We proved that if you have a Wronskian, so given n vector valued functions, you define a Wronskian.

If that is non-zero at at least one point, then you are basically dealing with a linearly independent set of functions.



Clear?

Okay.

Now, what about the converse?

Converse.

What is the converse?

So, basically the converse says that if

If you have a set of vectors x of t, vector valued functions that is xn of t are linearly independent.

Does that imply that the Wronskian of x xn of t is in I?

Do you think that is true?

Although it may look like it is true, it is actually not true.

So, let me give you an example.

So, basically let x of t is nothing but t and x of t is nothing but t square and t. Clear?

It is definitely not difficult to see that x of t is equals to t times x of t, right?

And definitely that is not linearly dependent.

So, hence it is linearly independent.

That is trivial.

So, it is trivial to check that x and x are linearly independent.

independent.

So, that is not a difficult thing.

What about the Wronskian?

As you can see, the Wronskian of x and x, if you calculate it, it is t t square t. So, which will be for all t, right?

For all t in whatever interval i. So, you see,

That this example shows that Wronskian non-zero in I is not necessary for linear independence.

So, now we are going to look at this converse.

When is it valid?

So, essentially we have this new theorem.

So, this theorem says that let x of t, x of t, xn of t be linearly independent, be linearly independent.

independent solutions, independent, not any more, any functions, linearly independent solutions, okay, of the differential equation, of the differential system, that is, of the system x prime t equals to a t times x of t, the homogeneous system, that is.

So, if they are linearly independent solutions of a homogeneous system,

in I then you can guarantee that the Wronskian will be

non-zero for all x-in i. Yes, I hope this is clear.

See, what I am trying to say is this, if the Wronskian is , it is not a necessary condition for linearly independent.

So, the example which we just saw, we saw that they are linearly independent, but not necessarily the Wronskian is non-zero.

So, Wronskian can be .

So, essentially what they are saying, what this theorem says, that

If there is not any function but linearly independent solution of some system, then there is a guarantee that the Wronskian is non-zero.

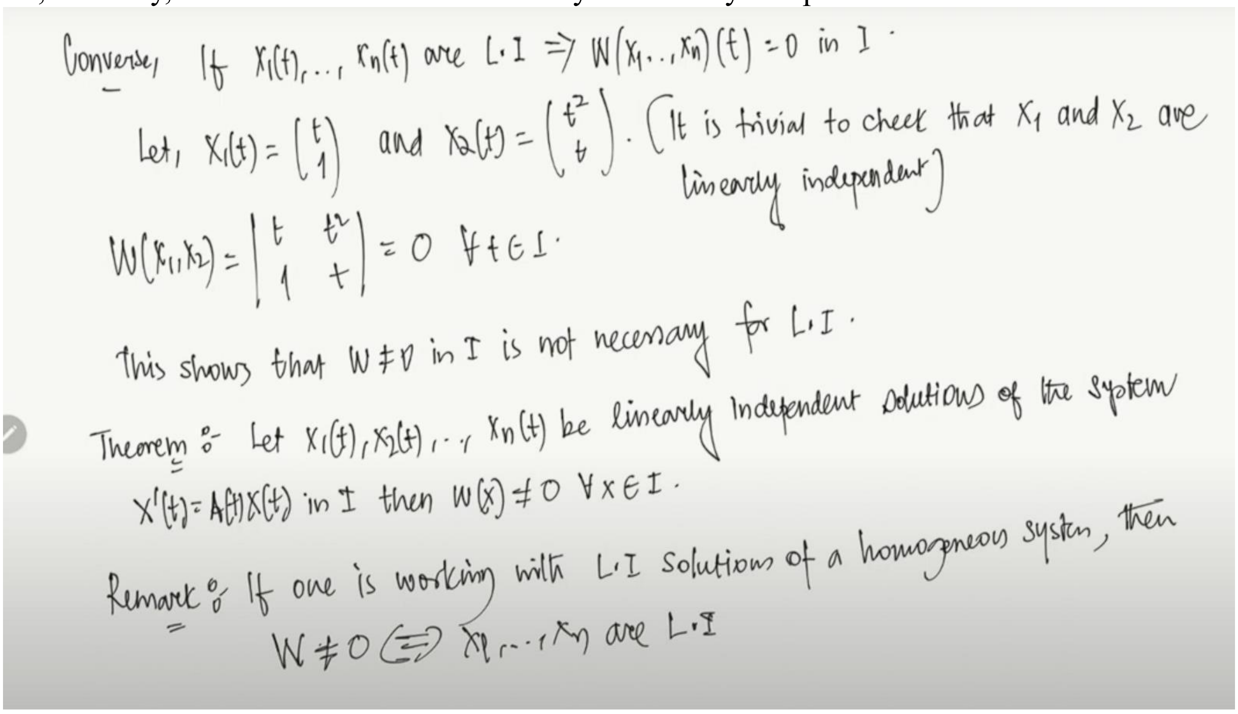
So, basically if you have linearly independent, so as a remark, if one is working with,

working with linearly independent solutions of a homogeneous system, then what can you

Wronskian non-zero will imply that x, xn, they are linearly independent.

And vice versa.

So, basically, wronskian non-zero means they are linearly independent.



Converse, If $x_1(t),\dots,x_n(t)$ are L.I $\Rightarrow W(x_1,\dots,x_n)(t) = 0$ in I.

Let, $x_1(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $x_2(t) = \begin{pmatrix} t^2 \\ t \end{pmatrix}$. (It is trivial to check that x_1 and x_2 are linearly independent)

$W(x_1,x_2) = \begin{vmatrix} t & t^2 \\ 1 & t \end{vmatrix} = 0 \ \forall t \in I$.

This shows that $W \neq 0$ in I is not necessary for L.I.

Theorem: Let $x_1(t), x_2(t), \dots, x_n(t)$ be linearly independent solutions of the system $X'(t) = A(t)X(t)$ in I then $W(x) \neq 0 \ \forall x \in I$.

Remark: If one is working with L.I solutions of a homogeneous system, then $W \neq 0 \Leftrightarrow x_1,\dots,x_n$ are L.I

If x, x, xn are linearly independent, it means wronskian non-zero.

So, both sides are true.

So, let us look at how to prove something like this.

Okay, so another theorem, okay, so this theorem, okay, so the proof.

So, let x is in I, yeah, be such that, so I am looking at a point such that the Wronskian at x , okay, or maybe let me put it as t , it will be better.

t .

This is non-zero.

So, sorry, this is zero.

So, see what I have to prove?

I have to prove the Wronskian at

Sorry, I should write it as t. Because I am writing t everywhere, I should not write it as x. It will be confusing for you guys.

Let us do it this way.

So, let t is in I. So, what I need to prove is this.

See, if they are linearly independent solutions of some system, not any function, but solutions.

This is important, solutions.

Then the Wronskian is not .

So basically what I am doing is let us start with the point t where the Wronskian is .

Let us see what happens.

Then there exists, if the Wronskian is , then there exists c, c, cn, not all , such that

Okay, the system summation i equals to to n ci xi of t, yeah, not is .

clear, this is okay.

See, the thing is, if the Wronskian is at the point t, so if you look at this system at the point t, so, you know, this particular thing, this particular system at the point t, okay?

So, we looked at this thing, right?

This system at the point t, you see, that will actually guarantee that you will have at least one ci, which is non-zero, such that the system is satisfied, the system works, okay?

So, since now you see what is happening with this, I will define a new function x t. Now, x t v is nothing but summation.

So, basically what I am doing is this.

I started out with this system and since the Wronskian is , then we can say that at least one of them, at least one of the CIs are non-zero.

I hope this is clear.

Let me do it here so that you are not confused.

C, x, let us say, is the first component of x of t plus C, the first component of x, , , sorry, once again, I am doing some mistake, C, x, sorry, x, the first component of x, t,

equals to and again c of x, sorry , second component of x plus c x, the second component of x t is .

See, the thing is, this equation can be written like this, right, x at the point t, x at the point t, x at the point t and x at the point t.

times c, c is , .

Now, this is a system, right?

And this system is basically this one.

This is what I wrote in a compact form, this one.

Now, see the Wronskian.

This is nothing but, this quantity is nothing but the Wronskian of x, x at the point t.

So if this Wronskian is given to be , you see, at the point t, it is given to be .

What does that say?

It says that this system has a non-trivial solution, so at least one of the ci's is non-zero.

So that is what I wrote here.

So, you have this set of c, c, cn such that at least one is non-zero, right?

So, I will use this set of ci's, okay?

And I will create a new function x of t, which is based on this c, c, cn.

So, basically, it is cy of ci, xi of t. I will define it like this.

Is it clear?

This is not any CI.

Please understand.

For that P, I get a set C, C, Cn such that at least one of them is non-zero.

I will take that set of C, C, Cn.

I will define a new Xt using that.

Okay.

Now see, this by the superposition principle solves the homogeneous problem, homogeneous system.

Homogeneous system.

Right.

Yes, because that we solve for any c, c, cn.

And xi is our solution, right?

xi is our solution for all i. So, ci times xi sum given by which we are defining to be x of t, that is also solving the homogeneous system, principle of superposition.

So, this is principle of superposition.

Principle of superposition, right?

Okay.

Now, you see, what happens x at the point t ?

What happens to that?

That is nothing but summation i equals to to n, c i, x i at the point t , right?

See, and for these c i's, summation i equals to to n, c i, x i at the point t is , okay?

So,

you have x of t satisfying the homogeneous system.

So, which is given by x prime of t equals to a t times x of t. And x at the point t is .

So, by the existence and uniqueness theorem, existence and uniqueness theorem, uniqueness theorem, Picard basically, okay, we can guarantee we have x of t is identically equals to , right, for all t in I. I hope this is clear, okay, and therefore,

Therefore, since x, x, xn are linearly independent, since x, xn are linearly independent, what happens?

What can you say?

That each ci has to be equal to .

That will imply

c equals to c equals to cn has to be equals to .

There is no other option, right?

Okay.

So, this actually is a contradiction, right?

Because we just assume that at least one of the ci's are non-zero, but we are showing that all has to be zero.

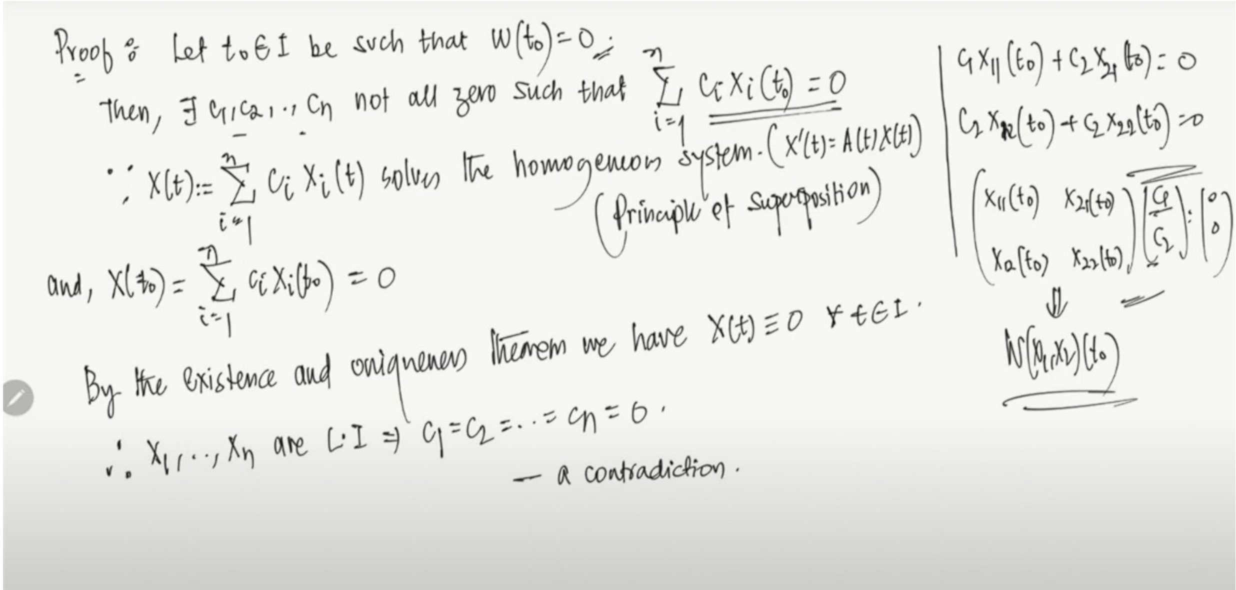
So, a contradiction.

Contradiction.

So, what happens is if you have this linearly independent solutions, then Wronskian non-zero implies that they are linearly independent.

And if they are linearly independent, it means that Wronskian is non-zero.

Now, we will finish this lecture with a very, very important property.



Property is called as the Abel's formula.

It is called Abel's formula.

Abel's theorem or formula, whatever you want to call it.

So, essentially, I have whatever the statement I gave just before this, that statement I gave using Abel's formula without mentioning it.

So, essentially, let me do it this way.

Let x of t, xn of t be the solution of the homogeneous problem, solution of

x prime equals to a t times x. Right?

And, let us say, I am assuming there is a point x, sorry, t .

There is a point t in I. Here, then, for any x in, sorry, t in I, t in I, one has

Wronskian at any point t will depend on Wronskian at the point t times exponential of integral t to t trace of a t dt.

What is the trace?

It is basically the sum of the diagonal elements.

Now how do you show something like this?

So essentially there is this small remark.

Before we show this, let us do this small remark.

See if Wronskian at the point t is .

Okay, so basically for some point if the Wronskian is , since the exponential is never going to be , that will imply that the Wronskian is for all points, for all t and i. So that is why I was saying that if they are linearly independent, it will be everywhere and if it is everywhere, then they are linearly independent.

Okay, it does not have to be at one point.

If it is non-zero at one point, then it will be non-zero everywhere.

Okay, right.

So, how do we prove the abel's theorem?

So, let us do that.

You see, if you remember, in the first week, we talked about the derivative of a determinant function, right?

You remember?

So, you see, Wronskian is nothing but a determinant function.

So, w prime of t is nothing but

sum, you remember that we did it, right, to n, rho y is this thing, the derivative thing.

So, let us do it, x, of t, x, sorry, n of t,

So, the first component x first component the nth component of the first thing x n n of t and x the first component the first function x the ith component t derivative of that x n i of t.

Derivative of that.

Right.

Yes.

So this is from week .

If you forget it, please look at week .

We did this.

Okay.

So once you have this, you see xi.

Okay.

Prime of t.

See, x, x, xn of t is nothing but the solutions of that equation, right?

So, x i prime of t is nothing but sum, if you write it properly, it is a i k of t, okay, x j k

of, right?

And this is k equals to to n, clear?

See, x prime is nothing but that, so basically if you break it up, okay, x prime, x prime will be nothing but that matrix times x.

So, you can compute that.

So, essentially, this is what we are going to get, clear?

And now if you put it in the, this above expression, clear?

And multiply the first row with a i.

So multiply the first row, the first row with a i, let us say, okay, of t.

The second row with a i.

Second row with a i.

Of t of course.

And you go on doing this.

Except the ith row and subtract their sum from the ith row.

So except the ith row and subtract their sum.

sum from the ith row.

Then what happens is you get that w prime of t is nothing but summation i equals to to n a ii of t wronskian of t.

Clear?

This is a equation, you see, this is a basic ODE, very simple first order ODE, right?

So, that will imply that this is nothing but W prime of t is trace of A t times W of

Now, you just integrate it and you get your result.

What is the, one second, where is the, you get this result.

Once you integrate it, you get this result.

I hope this is clear.

Again, I am telling you, if you find this complicated, because there are lot of indices which are involved, what you do is, you break it up into, break it into,

a two cross two system okay two cross two system and then of course you can use induction or that sort of thing but the thing is by doing a two cross two system you can just you know figure it out why why these sort of indices are working okay and it will be much clearer for you so please check this part check this what is what are you what do you check you do this exact same theorem but do it for two cross two system if you want you can just explicitly write everything break it down write everything and do it and then it will be very clear to you

So, with this we are going to end this video.

ABEL'S THEOREM :- Let $X_1(t), \dots, X_n(t)$ be the solution of $X' = A(t)X$ and $t_0 \in I$, then for any $t \in I$ one has

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \operatorname{Tr} A(t)\, dt\right) \checkmark$$

Break it into 2x2 system. (Check)

Remark :- If $W(t_0) = 0 \Rightarrow W(t) = 0 \ \forall t \in I$.

$$W'(t) = \sum_{i=1}^{n} \begin{vmatrix} x_{11}(t) & \dots & x_{n1}(t) \\ x'_{1i}(t) & \dots & x'_{ni}(t) \\ x_{1n}(t) & \dots & x_{nn}(t) \end{vmatrix} \quad \text{(Week 1)}$$

$$\therefore x'_{1i}(t) = \sum_{k=1}^{n} a_{ik}(t)\, x_{jk}(t)$$

Multiply the first row with $a_{i1}(t)$, second row with $a_{i2}(t) \dots$, except the i^{th} row and subtract their sum from the i^{th} row

$$W'(t) = \sum_{i=1}^{n} a_{ii}(t)\, W(t) \Rightarrow W'(t) = \operatorname{Tr}(A(t))\, W(t).$$