Ordinary Differential Equations (noc 24 ma 78)

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Week-02

Lecture-10

Picard Existence and Uniqueness Theorem

Welcome students, and in this video we are going to talk about the existence and uniqueness theorem.

In other words, this is also may be called as a fundamental theorem, fundamental theorem

of ODE we do not call it like that but till you know why i want to put emphasis on fundamental theorem because this is probably the most important theorem which you can learn in ODEs right okay so first of all what am i trying to do here first of all we start with the system right so let us consider this system let us consider a n cross n system clear n cross n system system

given by given by x prime equals to a right which depends on right sorry we are not looking for a linear system but we are looking for x prime equals to f of x that's a system right okay and see here f is a given a given function, clear?

f is a given function and we are looking at a system x prime equals to f of x and now we want to see whether you know can we say something about the existence of solution or not.

So before I say the existence of solution first of all we need to understand what a solution what do I mean by a solution.

So, let us call this equation of x prime equals to f of x. What do we mean by a solution?

So, you see a solution of this system essentially of x prime equals to f of x.

And let us say at the initial condition, x at the point t is x .

This is the whole system.

So essentially, x prime equals to f of x is the rule according to which the particle moves.

And initially, at the time t equals to t , let us say, you can take t to be also, x equals to t, t equals to t , the position is essentially at x , which is in Rn, right?

So which is in Rn.

x is in R. So basically, we are talking about an n-crossing system, right?

Okay, so a solution of this system is a differentiable function, is a differentiable function, continuously differentiable, sorry, let me put it this way, continuously differentiable function.

You do not need continuously differentiable, but for all purposes, we are just, we will be using it, continuously differentiable function.

function.

And what is the function?

This is a function x from i to Rn.

What is i?

i is some interval in, i is some interval in, let us say R, right, i subset of Rn.

clear okay so essentially if you have a function i from rn x from i to rn okay what does this mean this means that this is nothing but actually you are looking at a curve such that such that for all t in i t in i clear x prime of t

So, basically, it has to satisfy for all t, right?

So, it is capital F of x of t, clear?

And moreover, x at the point t , x should be such that x at the point t has to be x , clear?

So, this has to satisfy for all t in I, and at the initial point, they should pass the vector x .

So, essentially, what, you know, very quickly speaking, what it means is this.

See, x prime equals to, so basically, let us say that there is a particle, right, which is moving with time, which is moving with time in some, you know, direction.

r or r okay in some space r let's say r it is moving in r okay with time it is moving so at t equals to t naught let's say okay it is somewhere here so this is t equals to t okay and what happens is it is actually moving along let's let's just assume for now that it is moving like this here so as t increases it goes on doing this now you see what is happening is this what is x prime equals to f of x

is actually the rule, right?

See, F is a vector field.

F here is a vector field from Rn to Rn.

It is a vector rule.

Now what f does is, so n equals to in this case.

What this thing says is the rule.

So it will actually dictate how the particle moves.

So at every point it will give a specific direction.

And it will say you have to move in that direction.

So this is the rule according to which the particle moves.

And the initial condition x at the point t equals to x means you are basically saying that the particle has to start from this point.

So, this is the rule according to which the particle moves and this is the x at the point t equals to x will actually tell you that where to start.

So, basically the question of finding a solution is this.

you have a particle, yeah, you are basically looking for a particle which starts from, I mean, whose position at the point t is, let us say, x , and such that at any point, okay, at any point t, whatever the domain, so basically t is in a, b, right, so any point t in a, b, x prime of t, so the, you know, the tangent vector at that point, right, t,

has to be matching with the value of the vector field at that point.

So, I hope this is clear.

This is just an intuition, not very mathematical, but the thing is, I think the idea is clear.

Okay, so that is there, right?

Now, and you see here, I have to write down f is a given vector filter, given not function, but vector, I should write this vector filter, given vector field.

Okay, given vector field.

see ah what happens is this let us say that ah so let us take some examples and see examples of existence you will get so let us come back to existence so now that the concept of solutions is clear okay examples so what are the examples which you are looking for so is there a solution or not first of all existence and uniqueness says the first thing is existence of course as the name suggests whether you have a solution or not and secondly

you also need to know that if there is a solution is it unique clear okay so let's look at some examples see let's say that look at this example y prime equals to just a first order equation okay so first order order ODE ODE right now you see you do realize that if i am writing y prime equals to that will imply y of t equals to constant right for any constant any constant will work

So, essentially the solution, so therefore the solution is a family of curve, the solution of such an equation, an equation is a family of curve, is a family of curves.

This is family of .

Now, so, does there exist a solution?

Of course.

How many solutions are there?

They are infinite because, you know, C is varying in R. Now, the thing is this.

So, uniqueness is not here.

So, generally speaking, if you are not specifying the initial data, right, the uniqueness part won't work.

So, basically, most of the times, maybe there will be a solution, but the thing is you cannot guarantee the uniqueness.

So, basically, now, you see, if I am putting, so, additionally,

Let us say that y at the point is .

So basically I am specifying that at the point , y has to pass through .

Then you see y identically equals to is the only solution.

It is the only solution.

It is the only solution.

So, you do realize that once I put a specific, I mean, you know, point that I am saying that my curve has to pass through this point, right.

So, at t equals to , it has to pass through .

So, then what happens is the uniqueness is there.

Now, the question is, see, I am saying it is the only solution.

How do you know this is the only solution?

That is another question.

Let us just quickly see what we can say.

See, let us say that if, so this is how generally we try to prove this.

If y and y are two distinct solutions, let us say, two distinct solutions, distinct solutions,

of this equation.

Let us say y prime equals to . y at the point equals to I wrote.

So maybe let us say y .

y can be .

It does not matter.

y is in R.

So, let us see what happens.

If there are two distinct solutions, then define, if one defines, it is very trivial, right, y to be y minus y, then that will imply what?

That y prime is y prime minus y prime, which is again .

And that will actually imply that y of t is constant.

okay and since y is so that constant has to be y right that constant has to be y but you see the thing is constant is not y here because y is y minus y and y at the point so y at the point is y and y at the point is y so y minus y y minus y is essentially so this is nothing but .

. So, y is , sorry yt is .

Now, that will imply that yt equals to y of t. The question is this, again is it clear that if yt equals to is the only solution, there are no other solutions.

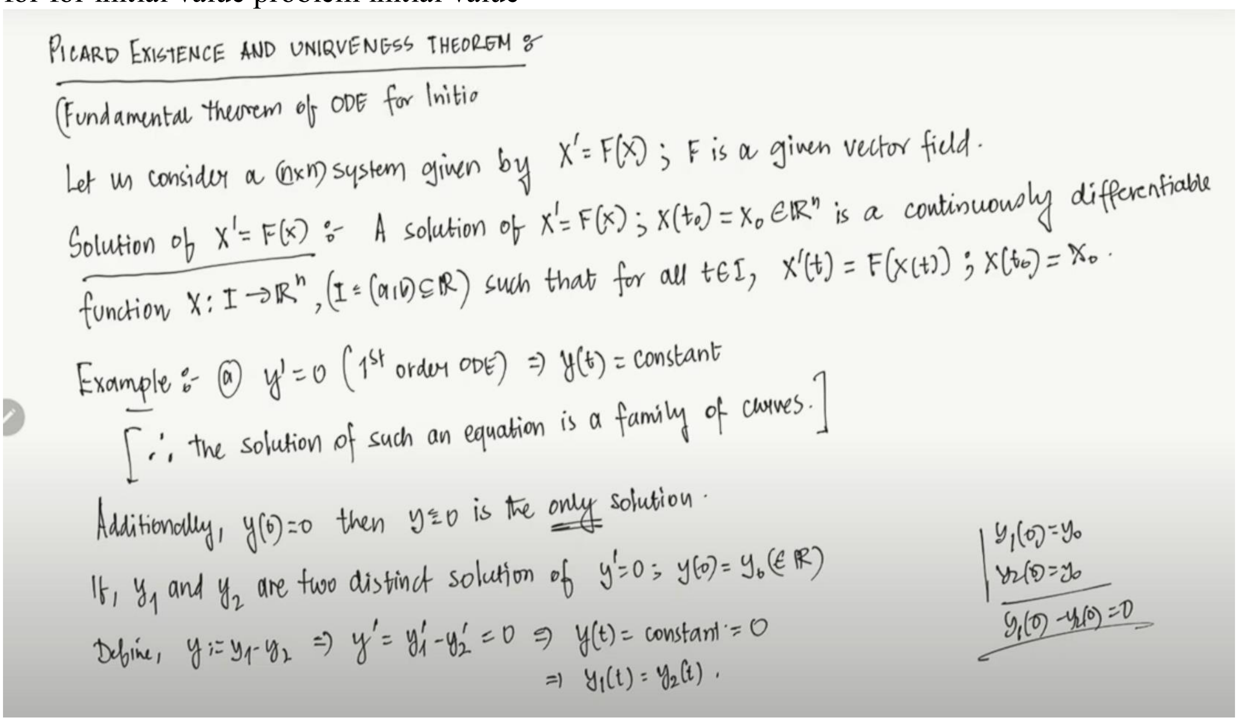
So, from here it is quite clear that that has to be the case right there is no other option.

So, in this case the thing is this problem has a unique solution.

So, that is more or less done, but the thing is this for a general problem finding the uniqueness even though you have found the existence finding a uniqueness is a very very difficult question.

So, we need to know how to

do that okay so for that we are going to look at some uh a very very important thing but before i do that uh let us look at another example okay so the question is this let's say that if i am putting a initial value okay so please remember this thing the existence in theorem this holds for so uh for for initial value problem initial value



Very, very important, initial value problem.

I hope you understand what initial value problem is.

Initial value problem means that you are essentially looking at one point.

So, you are specifying that the particle is starting from some point.

That is the only data which you have, initial value.

So, let us look at another.

So, basically, let me put it this way.

That is all initial value problem.

has a unique solution.

Let us just put it this way.

So, they may not be.

So, for example, let us say that y prime of t equals to y to the power by .

Let us look at this equation.

I hope you guys can solve this equation.

Of course, you do realize that y equals to .

Let us just put this condition.

This is the kind of problem.

So, clearly, clearly,

y identically equals to is always a solution, right?

Always a solution.

See, if y is identically equals to , y prime is .

And again, this part is also .

So, left and right hand side is for all t. And of course, y is identically .

So, y at the point has to be , right?

So, that is always the right.

Now, you see, also, also, y

of t let's call another thing y of t equals to tq also solves the equation solves the equation okay so you see

We have actually and you can easily check.

So, please check this part.

Check this part.

That y t cos t cube also satisfies the equation.

Okay.

Right.

So, and actually moreover you can actually show that moreover for any tau greater than .

Okay.

You can show that this family u tau of t. Okay.

y tau of t sorry given by if t is less than tau and t minus tau cube

If t is greater than tau, okay, this problem, so this family also satisfies the, you know, equation.

So, you do realize that, I mean, it does not have to be that the initial value problem has to have a unique solution, right, unique solution, okay.

So does it always have to have solution?

What do you guys think?

Even that is true.

So let us look at this is b. Now see another example.

Let us look at this problem.

Let us say consider this problem.

y prime equals to and minus .

 if t is negative.

If t is greater than or equal to , that's the equation.

Now, this you have to do it, you guys have to do it yourself.

See, first of all, here the source term, source term is the right hand side.

So, see, remark, we put it as a remark.

The right hand side is not continuous, not continuous.

It is a valid function of course, but it is not continuous in R. See, if it is not continuous, if it is a solution, then y prime t has to be equals to and minus .

 for t negative and minus for t greater than or equal to .

So, this is not continuous at .

So, y prime t, if there is a solution, then that solution, the derivative at the point ,

Doesn't have to exist, right?

So, this, sorry, it has to exist, but what I am trying to say is this, it doesn't have to be continuous, okay?

So, that will imply that Y, as a solution Y is only differentiable, differentiable.

Doesn't have to be continuously differentiable.

Of course the derivative exists.

But the thing is the derivative at the point is not continuous.

So basically y is only differentiable not c.

Not C.

So it is not continuously differentiable.

So that is the issue here.

So not issue but this is the idea of solution.

So in this case the solution is not continuously differentiable function but just a differentiable function.

Now you can check that there does not exist, there does not exist, this is the sign does not exist any solution.

Yes.

So you understand what do I mean by solution of this problem.

There does not exist any solution.

And see here whenever I am saying that there does not exist any solution.

So solution in this sense.

Solution in this sense.

Satisfying.

Satisfying y at the point is .

Okay.

See the thing is what I am trying to say is this y at the point is .

That sort of solution if you are looking for there is no such solution which will be satisfied in this case. Clear?

Okay.

That is what I am trying to say.

Okay.

So now that that part is clear.

So essentially what we learnt up till now is this.

Let us do a quick summary.

Summary.

So, essentially we are looking for a initial value problem.

Initial value problem.

Let me put it this way.

Initial value problem.

And what sort of problem are we looking at?

We are looking at a first order equation, f prime equals to f of x. So, we are looking for a system and for a special case, it becomes like a usual ODE.

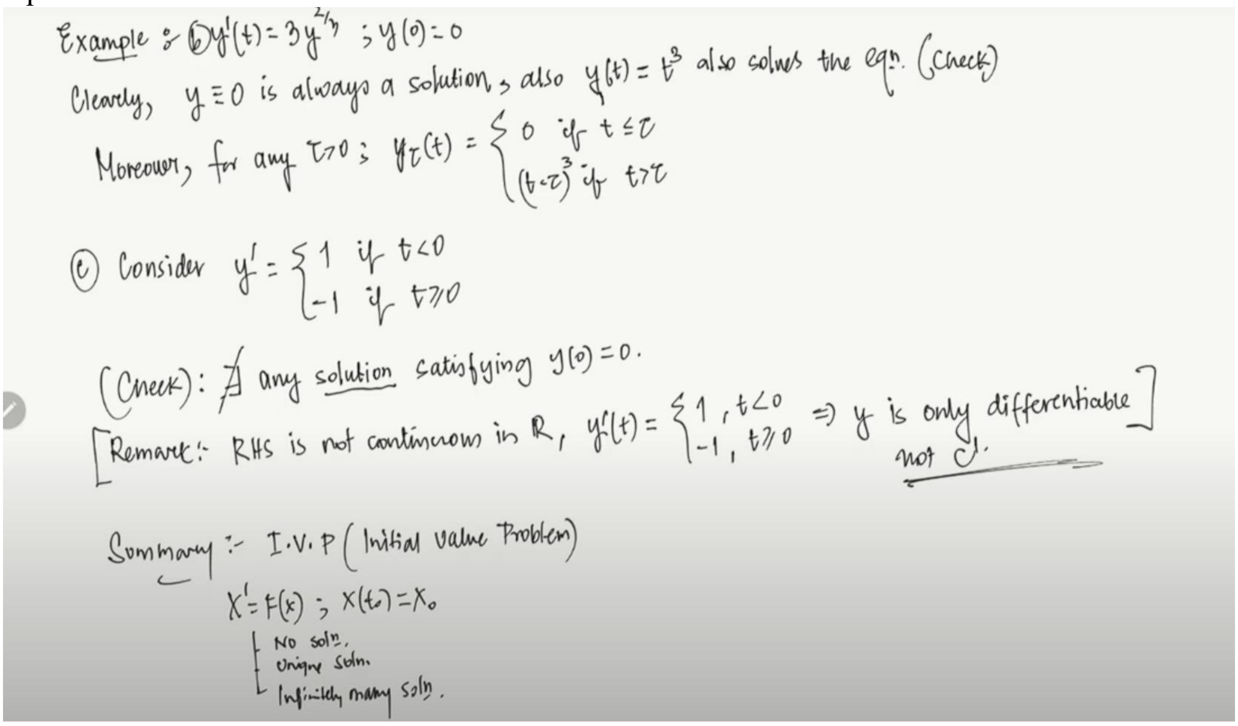
So, for this equation, what we have seen for this system essentially, what we have seen is this, x at the point t is x .

There may be no solution

There may be a unique solution, unique solution or there may be like infinitely many solution,

solution.

So, what is the guarantee that if I give you any problem, you can actually say that let us say that equation has a solution and if the solution is uniform.



So, that is the theorem which we are going to do.

So, this is called a fundamental local theorem.

So, this is the theorem is local.

This is called a fundamental local theorem.

Fundamental local theorem.

Why local?

Because this is not the theorem which I am going to give you is not a unique, like a, you know, global theorem.

Local theorem of ODE.

Okay, let us look at the theorem here.

So this is called Picard's theorem, Picard's existence and uniqueness.

So consider

the initial value problem initial value problem and what is the initial value from x f prime equals to x prime equals to f of x x at the point is x is x clear which is in rn which is in r so basically we are looking for n crossing system suppose and this is the condition which we need f is

F from Rn to Rn, let me put it this way.

Suppose F from Rn to Rn is continuously differentiable.

Continuously differentiable.

So F is a vector field which is continuously differentiable.

So basically what it means is,

You know, all possible, so basically the first order derivative will be this, and that is going to be continuous.

Right.

So, this is the only condition.

So, you are given this equation, and the condition which I am just asking, this is very, very minimum condition.

What I am saying is, f has to be just continuously interchangeable.

If that is the case, then you can guarantee that there exists a unique solution, a very, very important, unique solution.

Let us put it in a box.

Unique solution.

Let me call this as a .

Unique solution to .

Unique solution to the initial value problem .

to the initial value problem .

Clear?

So, in a more mathematical context, what does it mean?

Let me later, let us put it this way.

So, that is, there exists a positive

Some a, this a and this a is not same.

Let us call it b, let us say, or whatever.

Let us say epsilon.

There is this epsilon positive.

And a unique solution, solution x from minus epsilon to epsilon to x.

R n. Clear?

So, you see, I am looking for a solution.

So, I am saying that I am looking for a curve which passes through the origin.

Right?

And at the origin, at t equals to , the curve should pass through the vector x .

That is what I am saying.

equals to x .

That is what it means.

It means that at t equals to .

So, when the time is , the curve is at the vector x .

Yes?

And then the curve is moving.

yes according to the rule x prime equals to f of x i want to know whether there is a such a curve or not what you can say is in a small neighborhood of p so basically minus epsilon and plus epsilon okay time in a small time minus epsilon and plus epsilon you can actually say that there is such a curve given f is continuous differential so basically a unique solution like this satisfying

satisfying x() equals to x here okay so uh the proof of this thing we have to do and before we go on uh i think this is okay right yeah okay so

Yes, before we go on, in the first lecture, yes, in the first, we talked about this, right.

So, but again, let us just quickly recall.

I am not going to do the proof.

Now, we are going to recall some concepts and then we are going to do the proof, okay.

See that let omega subset of Rn be an open set.

Open set.

So I am looking for an open set.

A function, a function capital F from omega subset of Rn to Rn here is said to be Lipchitz continuous.

Lipschitz continuous.

We are going to need this concept here.

That is why we are just saying this is Lipchitz continuous.

On omega, if there exists k, this k of course is positive such that, such that

f of y minus f of x. So, please understand this, f of x and f of y, f is a Rn to Rn function, right?

It is a vector field.

So, whenever I am saying f of x minus f of y, this element is in Rn, this element is in Rn.

So, I am taking the difference of two vectors and I am taking the modulus, the norm of that.

So, this norm, think of it as a norm in Rn, okay?

Okay.

So, that will always be dominated by constant times norm of y minus x. So, again y minus x, x and y are in Rn.

So, this is also the Rn norm, okay.

This should hold for all x, y in x, y omega, okay, right.

the the k the constant k this is very important this k depends only on depends only on omega yeah and not on any particular point okay so k only depends on omega and is called is called the lipchitz constant this we already did so this is constant

constant for F, for capital F. And you can also recall that we call, so we say also F is locally Lipchitz, Lipchitz.

If each point

in omega has a neighborhood, what is the neighborhood?

Let us say omega prime, which is subset of omega, such that when you restrict f to omega prime, so when you restrict f to omega prime, it is going to be lipchitz.

clear okay and you know you do realize that in this case there is a lipchitz constant but that actually is going to vary according to omega prime yes ah so we have already covered all of this in the first week so i am not going to elaborate this but basically ah this is just to recall yes okay

Now, you see why the question is this.

Why are we suddenly talking about Lipschitz continuity here?

Fundamental Local theorem of ODE :-

Consider the I.V.P $X' = F(X)$; $X(0) = X_0 (\in \mathbb{R}^n)$ — (1). Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable.

Then $\exists$ a UNIQUE solution to the I.V.P (1).

i.e., $\exists\ \epsilon > 0$ and a unique solution $X : (-\epsilon, \epsilon) \to \mathbb{R}^n$ satisfying $X(0) = X_0$.

[Recall :- Let $\Omega (\subseteq \mathbb{R}^n)$ be an open set. A function $F : \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ is said to be Lipchitz/continuous on Ω if $\exists\ K > 0$ such that

$$\|F(Y) - F(X)\|_n \leq K \|Y - X\|_n \quad \forall X, Y \in \Omega.$$

The K (depends only on Ω) and is the Lipchitz constant for F.

Also, F is locally Lipchitz if each point in Ω has a neighbourhood $\Omega' (\subseteq \Omega)$ s.t $F|_{\Omega'}$ is Lipschitz.]

Because the statement of the theorem does not say anything about Lipschitz continuity.

But we are going to use this.

This is a very, very important property in that sense.

So, before we do that, again we have to go through a small lemma.

So, what is the lemma?

So, it says that let

capital F from omega subset of Rn to Rn.

So, we are looking for a continuously differentiable vector field.

So, let us say this is continuously differentiable.

Then, capital F is locally Lipchitz.

Lipchitz, I should write it properly.

It is capital L

So what it means is, see, it is saying that if you have a continuous differentiable function, see the statement of the theorem which we are working with, it has that if the vector field is continuously differentiable, c, that is, this is c.

So, what we are saying is if the vector field in question is continuously differentiable, there we can say that the vector field is locally lipchitz.

So, what is the proof?

The proof is this.

So, see suppose

that capital F from omega to Rn is C.

Let us just say it is C.

And we start with x which is in omega.

See, f is c given to us, right?

And I have to show locally lipchitz.

So, basically, I will start with a point x, which is where I will show that there is a neighborhood around x where f is restricted to that neighborhood is.

That is what you have to show, okay?

So, now let epsilon positive be small enough such that enough such that omega epsilon bar.

So, this is a closure of omega epsilon.

So, this I am defining as the closed ball.

So, this is x minus x .

So, this is the set of all those x in omega such that

x minus x .

So, this is normal.

This is Rn, Rn.

So, norm of x minus x is less than equals to epsilon.

Clear?

So, this is, I am defining this thing here.

And for now, I do not want to use that bar all the time.

So, let us just say this is omega o epsilon.

Let us just call it o epsilon.

That will do it.

okay now you see o epsilon what is happening is this f is c right f is c and this is defining whole omega right so you can actually have that the norm of d f of x okay d f of x okay this is always dominated by some upper bound okay so essentially why what i am trying to say is the c f is c right that will imply that the derivative of f at some point x is uh

A linear map from Rm to Rn, right, which is again continuous because this is C, right.

So, dfx is continuous, is continuous.

See, if f is C, df, df, that map is continuous map.

Again, df at a particular point x is a linear map.

That is different.

But anyways, here in our case, dfx is continuous, right?

And you see, this O epsilon, this set is compact.

Compact.

Okay?

So, if you are choosing your omega, the point x from O epsilon, okay?

So, this holds for all x in

O epsilon.

So, if you are choosing your x from O epsilon, what is happening is this, that df

You are basically looking at, you are restricting it to a compact set.

And we know that a continuous function on a compact set is going to be bounded.

So, basically it attains a maximum somewhere on the compact set.

So, that is why, that is what I wrote.

So, the norm of E fx is bounded.

Clear?

Okay.

Also, note that since this is closed set also, O epsilon is convex, right?

Very, very important.

It is convex.

So, what is happening is this.

For, if we start with two points, y and u in O epsilon, if we do that, that will imply y plus epsilon u, y plus epsilon u will be in O epsilon, where

u will be defined as is nothing but z minus y. See what I am trying to do is this o epsilon.

So let me draw this part.

This is x .

The ball is at center at x .

This is o epsilon.

O epsilon.

Clear?

Now, you see, this is compact.

So, you start with any two points, y and z. You start with any two points, y and z. Okay.

Then, the straight line joining y and z. Okay.

That should be in O epsilon because O epsilon is convex.

Okay.

So, the thing is, I am defining as new straight line.

So, sorry, that's like segment containing y and z must be this.

Sorry, like segment, not like this.

Now, you see the straight line which is given by y plus epsilon u. This straight line.

This should be also in O epsilon.

And what is u here?

u is nothing but z minor.

I did a small mistake here.

u is not in O epsilon.

For y in O epsilon.

So, basically what I am trying to say is this.

Since y O epsilon is convex, that will imply that...

y plus epsilon let us call it u okay is in o epsilon where where u is in z minus y okay y is z minus y and less than equal s less than equal okay why is this true see let us let us check

You see y plus epsilon u is nothing but z minus y, right?

So, it is nothing but, this turns out to be minus epsilon times y plus epsilon z, okay?

So, you do realize if epsilon lies between and , this will lie in y epsilon.

So, if y and z is in O epsilon, then epsilon times z plus minus epsilon times y has to be in O epsilon.

So, this is what I am writing here in a short hand.

Yeah, that is fine.

Now, you see, we define, let phi of x, yes, this is defined as capital F of x.

y plus su.

See, I am defining a new function.

The function is defined from r and is taking values in rn.

Clear?

So, this is our new definition of psi.

Then, you see, this

inside this function as a function of f is differentiable of course this is a linear function is differentiable as a function of s yes and capital f is a c function so that is also continuously differentiable so phi is nothing but the so since phi is the composition composition of two c function of two

C function.

Since it is a composition what happens is you can talk about the chain rule.

C prime of s is nothing but derivative of f which is df acting at y plus su.

df at y plus su acting at u.

Because you know that by chain rule the derivative of this is going to be u because derivative is with respect to s. So this is what we are going to get.

So therefore what is happening is this.

See therefore what is f of z minus f of y.

You remember, for any y and z in that ball, I need to show that f of y minus f of z is dominated by y minus z, the norm here.

So I am looking for f of z minus f of y. See, if we do that, this is nothing but si of .

What is si of ?

Psi of is nothing but f of y, okay, minus si of .

What is si of ?

Si of is nothing but, one second.

sorry, si of is nothing but f of z and si of is nothing but f of y, right.

So, this is can be written as to , the fundamental theory of calculus, si prime of s ds.

And again si prime of s, we already know what it is.

It is to df of y plus su

Acting at u.

Okay.

So please understand df y plus se is nothing but a linear map from Rm to Rn.

Okay.

Rm to Rn.

And this is acting at an element of, so this is a map.

Okay.

And this is acting at an element of Rn.

Okay.

So this is what it means.

Right.

Now, you see what is happening is this.

This u, this u is nothing but an element in O epsilon.

u is the element of O epsilon.

And in O epsilon, so if we are just restricting our f to O epsilon, what is happening is this.

So, let us put it this way.

See, if we take the norm of fz minus f of y,

Okay, that can always be dominated by to , the constant times norm of u ds.

Okay, why can we do this?

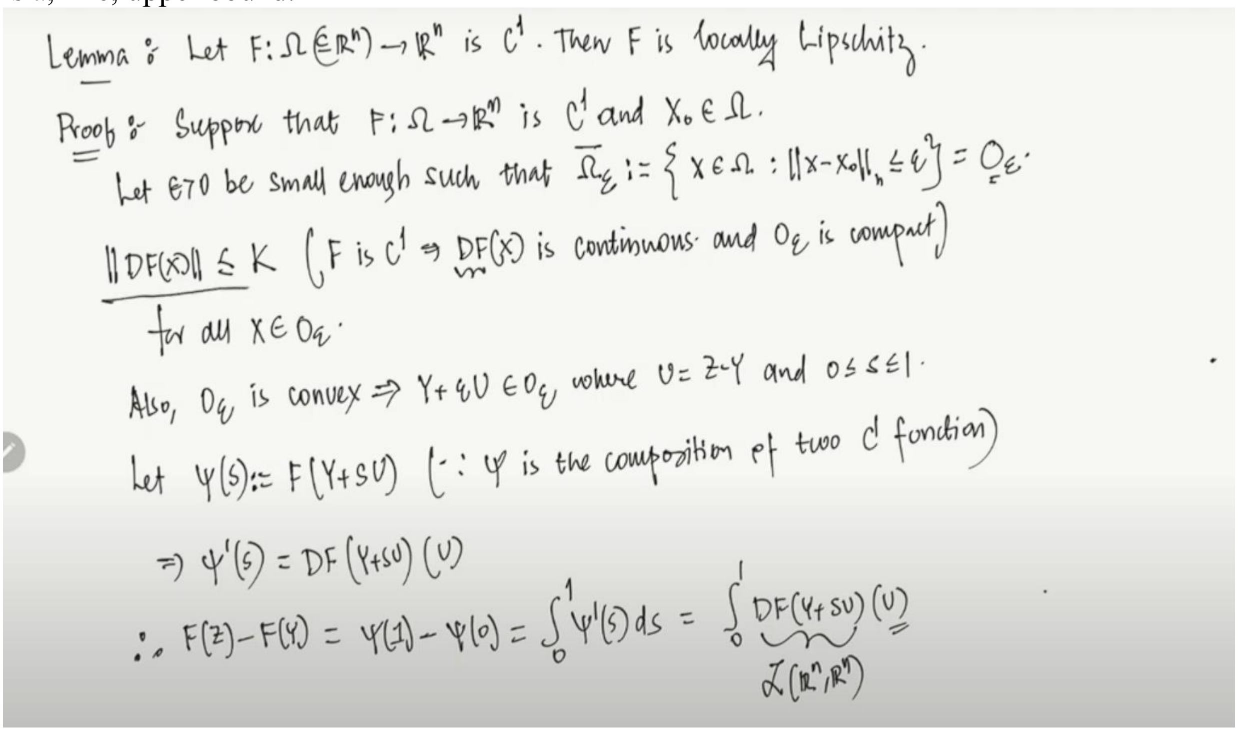
See, let us say that I want to show that it is locally leachous, right?

So I do not have to look at the whole omega, but in a neighborhood, right?

And what is the neighborhood?

The neighborhood in our case is O epsilon.

And in O epsilon, since O epsilon is compact, right, we showed that this is true, right, that there is a, like, upper bound.



Lemma: Let $F:\Omega(\subseteq\mathbb{R}^n)\to\mathbb{R}^n$ is C^1. Then F is locally Lipschitz.

Proof: Suppose that $F:\Omega\to\mathbb{R}^n$ is C^1 and $X_0\in\Omega$.

Let $\epsilon>0$ be small enough such that $\overline{\Omega}_\epsilon := \{x\in\Omega : \|x-x_0\|_n \le \epsilon\} = O_\epsilon$.

$\|DF(x)\|\le K$ (F is $C^1 \Rightarrow DF(x)$ is continuous and O_ϵ is compact)

for all $x\in O_\epsilon$.

Also, O_ϵ is convex $\Rightarrow Y+\epsilon U\in O_\epsilon$, where $U=Z-Y$ and $0\le s\le 1$.

Let $\varphi(s) := F(Y+sU)$ ($\because \varphi$ is the composition of two C^1 function)

$\Rightarrow \varphi'(s) = DF(Y+sU)(U)$

$\therefore F(Z)-F(Y) = \varphi(1)-\varphi(0) = \int_0^1 \varphi'(s)\,ds = \int_0^1 \underbrace{DF(Y+sU)}_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)}(U)$

The maximum is at it somewhere.

So that maximum, I can, if I am taking the norm here, see, if I am taking the norm here, so what do I have?

Let me put it this way.

Fz minus f of y equals to to df y plus su df.

acting at u ds.

This is what we had, right?

This is a linear map again and this is an element of Rn.

Element of Rn and this is a linear map from Rm to Rn, okay?

See, if I put a norm in both sides, so there will be a norm on both sides, right?

And then this will be dominated by to .

So, this is just a integral, this inequality is coming from the integral inequality.

So, this is norm of

df of y plus ac, okay, times norm of u. So this norm is in, is from ln to ln norm and this is ln norm, right, ds.

Now this is always dominated by k, this we saw, okay.

So that is what we are going to use here, okay.

And you remember how we are getting this thing.

So this we already talked about in week .

So if you do not remember, please go through that.

Okay, so this is the case and now you see if this is the case, this is nothing but k times k comes out, it becomes to norm of u ds and what is u?

u is nothing but k times norm of z minus y.

Okay.

And to ds is essentially.

So you see these holds for all yz in O epsilon.

Okay.

Now the question is.

So this is proved.

So this is locally Lipchitz.

So basically what I am trying to say is this.

If it is continuous difference between locally Lipchitz.

Now think about it this way.

Let me ask you this question.

Question.

Why suddenly locally Lipchitz?

Why not Lipchitz?

So if f is c, does that imply f is Lipchitz?

We saw it is local ellipsis but it is ellipsis.

So what about the same proof, the proof does it work?

Do you think it will work?

Of course it won't work.

Why it won't work?

Because you will not get this bound only.

The bound which you are going to get here, you see since it is local, what is happening is this, there is a neighborhood which is going to be compact and in that compact neighborhood you can have a bound on DA.

That is very important here you see.

Yeah.

So without this bound you can't, the whole thing does not really work.

Okay.

So at least this proof won't work.

Okay.

But the question is this, is it true that whether it is lipchitz or not?

Okay.

The answer is no.

I won't tell you the answer.

So you have to do videos.

Okay.

Now let's look at the proof of this.

What is the proof of what?

Proof of our fundamental theorem.

Again, let me recall what the fundamental theorem is.

We are basically looking at x prime equals to f of x. And x restricted at the point t equals to is x .

Clear?

What I am saying is this.

Given f is continuous equivalence, just that, only continuous equivalence ability will guarantee a unique solution in some neighborhood of .

Clear?

Okay.

So, you see, so proof, proof of fundamental theorem, let us put it this way, of fundamental theorem.

This is also called Picard's existence and uniqueness theorem.

Okay, so you see, if

Let us say, let me put it this way, a remark.

Before we look at the proof, let us do this remark and then we go to the proof.

So, if x from some i to r satisfies x prime t,

equals to f of x of t. If it satisfies this with x equals to x, what does that imply?

That will imply that x of t is equals to minus x.

So, if you are integrating both sides, minus x at the point is nothing but to t f of x of s ds, right.

So, this is integrating

both sides.

Clear?

Okay.

So, this form, this form is called the integral form.

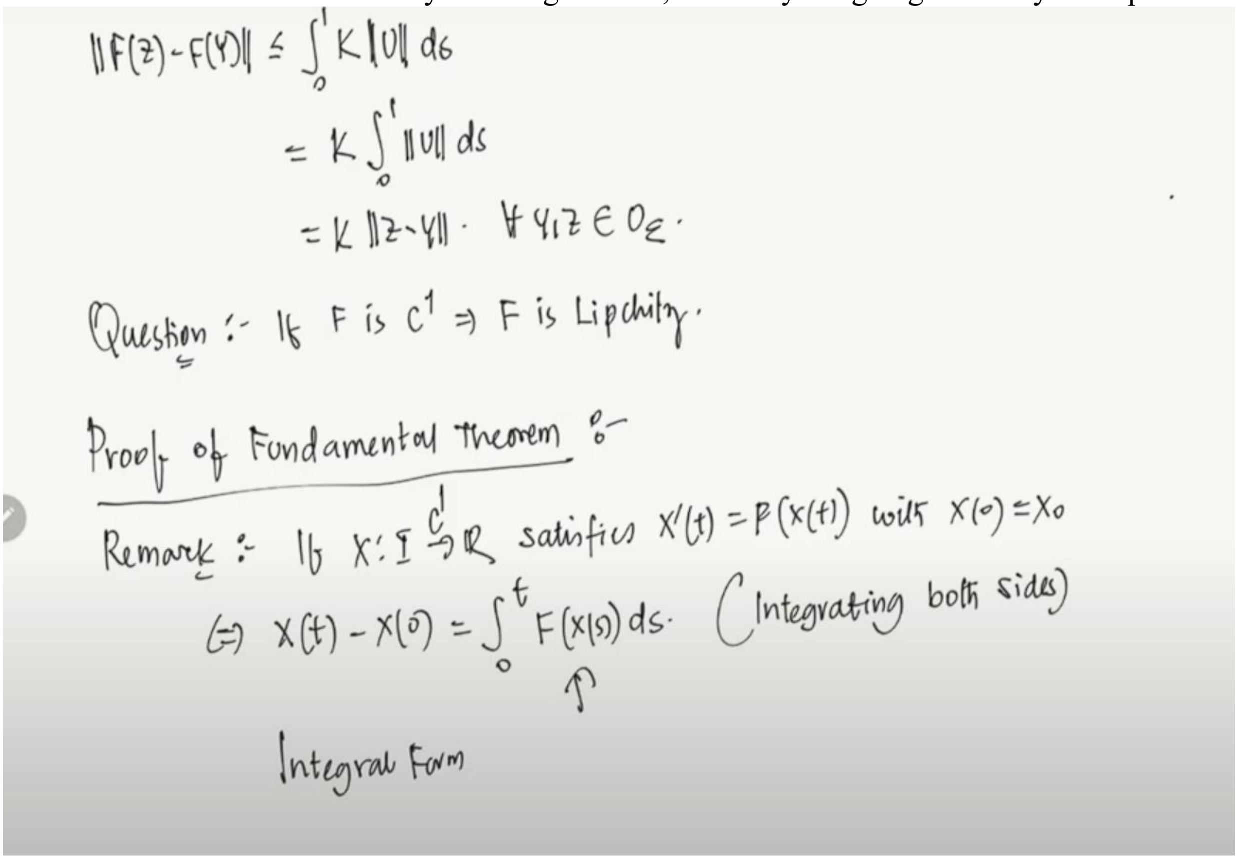
Integral form.

And you can see that it is actually both ways.

Okay.

So, if you have a c function x, then I am satisfying the equation, then it will satisfy the integral form.

If there is a function which satisfy the integral form, then they are going to satisfy the equation.



So, integral form of the differential equation x prime equals to f of x. Clear?

Let us look at the, now we come to the proof of this.

So, you see what I am going to do is this.

I will show that there is a, you know, function which will satisfy the integral form.

Once I do that, then you see that if there is a function which satisfy the integral form, then that function has to be the solution of the equation, okay?

So, proof.

So, we will do some assumptions here.

First of all, I will write that O rho is a

closed ball, okay, closed ball of radius rho, rho positive, and it is centered at x , and centered at, centered at x , okay.

Now, what is x ?

x is a point through which our curve passes at the time t equals to , clear?

Okay, A. These are some assumptions, huh?

b is there exists a lipchitz constant lipchitz constant constant okay k for f on oro here see oro is closed part of course it's bounded so what i am trying to do is this is compact set and

f, you are given to be a c function, right, in omega.

So, when f is restricted to o epsilon, o rho, sorry, then we know that this is locally Lipchitz and hence there is a Lipchitz constant, right, okay.

Now, norm of fx, okay, this is always greater than, bounded by m on

Again, O rho is compact, right?

F is a continuous function.

Compact set, there is a maxima, which is called as M. And the thing is, we choose, okay, choose A here, which is less than equal minimum of, so this is just a technical thing, nothing special.

I will show you how, why we need this, but this is just a technical thing which we need.

Rho by M.

by k. And we will write the interval i to be minus a comma a. So, these are our assumptions.

These are our assumptions.

So with these assumptions we are going to start by writing down something called a Picard effect.

So what did I tell you?

You see we saw that if there is a c map which satisfies this equation then it has to satisfy this integral form and vice versa.

So, essentially I will form, we will construct the sequence of function which will actually converge to the integral form.

And then that solution will actually be a solution of our equation.

That is the idea.

So, how do we do something like this?

See, we start with some iteration scheme.

So, we start with the iteration scheme and this iteration scheme is called scheme.

Called Picard iteration.

And what is it?

You see the thing is this Picard iteration.

So the first iterate we are going to define is u of t to be sorry x of t to be identically equals to x.

See, I am trying to somehow approximate the solution.

Clear?

Now, what do we know?

The only thing which we know about the solution is at the point t equals to .

So, at time t equals to , it has to pass through x .

So, even though it is a very crude approximation, but you can always guarantee that x

identical equals to x so basically the first function which is identical equals to x should always i mean at least you know is a good approximation right because it passes to the initial point that's all okay now so define let okay now define the next approximation x of t how do you define the next approximation you see i want my approximate

functions to do this x at the point t minus x to equals to to t f of xs ds right this we want okay we actually want this you see you can write this thing as x t equals to what is x it is x plus to t f of xs ds

So these are all equivalent.

So I want my iterate to converge here.

So I will define it accordingly.

I will define it using x which is x plus to t f of you see xs right.

So I will use the previous x here which is x in our case.

x naught of t sorry s.

Okay.

So, this if you write it, it is nothing but x plus what is x of s?

It is x , right?

So, it is f of x .

So, t times f of x .

f of x is essentially a constant vector.

So, I can take it out.

So, to t ds is t. Now, since t is less than a, right?

And

You know how we are getting this, right?

And the norm of fx is less than equal m. Clear?

Okay.

Therefore, x of t minus x.

okay see what we need to do is this essentially i am starting out at p equals to we are at x right and the second iterates or the nth iterate which you are going to get okay or the solution which you are going to get must be in the ball o epsilon right so basically the x of t which you define for all t should be no epsilon okay how do you guarantee that so x of t minus x i want to make it less than rho can i do that this is nothing but

less equals to mod t norm of f x .

This is just basic calculus.

So, t is dominated by a because mod t is always less than equal a. That is given.

See, i is between minus a to a. I am taking t from there.

So, it is always dominated by a. And what is f of x ?

That is given to be m dominated by a.

Why this is given?

You see this is given here.

So it is less than a m. Now you see a m is less than rho.

See I chose rho.

such a way that a is less than minimum of rho and m. So, a is always less than rho by m. So, a m is always less than rho.

So, that is what we are using here.

So, as I told you, this condition is a technical condition.

So, this is less than rho.

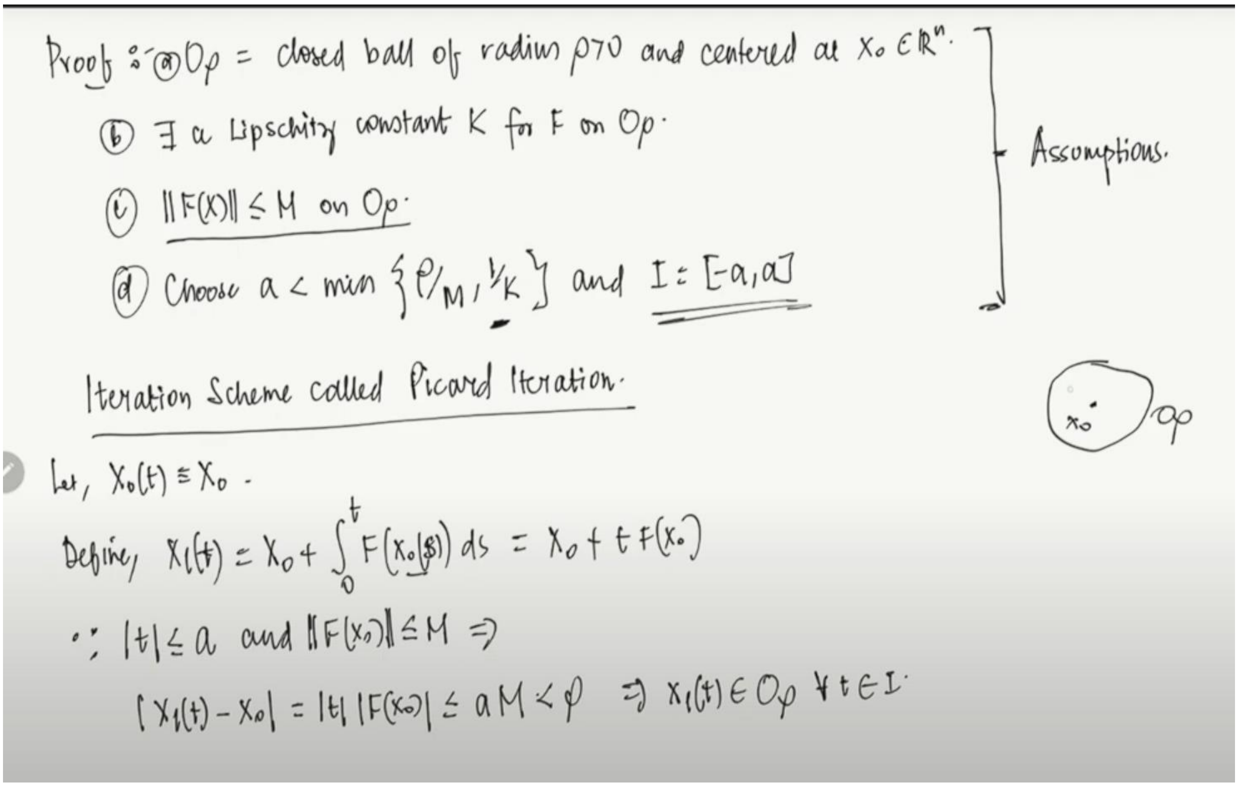
What does that imply?

It implies that x of t is always in o rho, this set for all t i.

C, x is a point, is a vector x .

And O rho is this.

O rho is a ball, closed ball, centered at x .



The iterate which we are finding is such that at all time t, x is always .

In that interval.

That is the point.

And not only x, we want all our x's to be in that interval.

Okay?

See, now, by induction, let us assume, by induction, induction, assume that, assume that,

xk Of t. Is defined.

Defined.

And.

The norm.

Of x k t. Minus x .

Is less than.

Rho.

For all t. In I. Clear.

So essentially I am saying.

It is again in O epsilon.

So then.

Let define x k plus of t is x .

So, this is our definition, x plus to t, the Picardy rate, the k plus Picardy rate, f of xs.

I want this to converge to f of xs, right?

So, I will use the earlier function which is available to us, which is xk in this case, xk of s.

So now you see this will make sense.

Why?

Because f, see xk of s, so let me put it this way.

This makes sense.

Since xk of t is in O epsilon, right?

yes, for all s, for all s. So, you see, all of this f acting at xk of s, all of this is defined properly.

So, it is not a problem.

Now, you see, if you take the norm xk plus of t minus x , let us look at this norm.

This is always dominated by to root t norm of f of

xk of s ds.

Clear?

Now, you see xk of s is in O row and f restricted to O row is always bounded by m. So, I will just use it like m times to t ds.

Clear?

Which is ma, which is less than .

Why ma?

Because it will be mt, right?

And t is again dominated by a. t is varying between, you see, t is varying between minus a to a, okay?

So that's bound, that is the bound which I am using here.

Okay, so that's fine.

Now, so what we have is this.

See, by induction we have, therefore, you have this sequence, xn of t, let's say, Okay, which are well defined, well defined in O rho.

Here, now what we will show is, so next we will show, we prove, so we have to prove the convergence, right?

So now we will show that this xn, that actually converges to some x of t, which actually happens to be the solution.

So, next we prove there is a constant L greater than such that for all k greater than equal norm of u k plus t minus u k of t

is less than equal a k whole power small k times l. This is what we need to show.

So, if we show this thing, we basically show it is Cauchy equation, it is a Cauchy sequence, and then all of we can get what we want.

So, how do we show this?

See,

Let, so we will take the maximum of u minus u.

By induction, assume that $X_k(t)$ is defined and $\|X_k(t) - X_0\| < \rho$ $\forall t \in I$.

then let, $X_{k+1}(t) := X_0 + \int_0^t F(X_k(s))\,ds$

[This makes sense since $X_k(s) \in O_\rho$ $\forall s$]

$$\|X_{k+1}(t) - X_0\| \le \int_0^t \|F(X_k(s))\|\,ds < M \int_0^t ds = Ma < \rho.$$

$\therefore \{X_n(t)\}$ which are well defined in O_ρ.

Next, we prove there is a constant $L \geq 0$ such that for all $k \geq 0$

$$\|U_{k+1}(t) - U_k(t)\| \le (aK)^k L$$

So, let L is the maximum of the norm of x t minus x of t, t lies between minus a to a. That is possible.

Why it is possible?

t is varying between minus a to a, x and x are both continuous function, right?

So, the maximum exists on a compact set, right?

So, the maximum exists and that maximum we will call it as a. Now, you see this l, this l, okay, you can of course see that and l is less than equal a, right?

You see, we proved it, right?

Where is it?

Yeah.

This one.

This one.

This is norm, right?

Okay.

So, this is your maxima I am choosing.

And I am saying that is L. So, L is always bounded by a m, which is less than rho, right?

So, that is always there.

That is what I wrote.

L is less than equal a m. Now, let us look at this.

Norm of x of t minus x of t.

This is nothing but to t f of x of s minus f of x of s ds.

The norm.

Now you see, I know that this is again dominated by to t k times norm of x of s

minus x of s ds okay this we already know why this is true because you see x and x since x of s and x of s is in oro for all s right so that is why we need it in oro for all s so since these are in oro for all s f is drifted contiguous

In O rho, f is locally Lipschitz and when f is restricted to ORO, then f is Lipchitz continuous.

So, I can take the Lipchitz constant out.

Let us just say that constant is k. So, that is out and then we can write it as k times norm of x minus x.

Now, if this is the case, this is nothing but less than equals to a times k capital L.

But it is capital L, the maximum of x and x.

So, this can be dominated by L and then to T ds will be dominated by A. Now, I want you guys to do something.

This part I am not doing this thing.

So, you use induction.

Use induction.

This you already can do I think to show that

norm of u k plus of t minus u k t is bounded by a k power k times l. Okay.

This, I think, you guys can show it yourself.

Please do this part.

Okay.

Right.

Now, so, this is fine.

Now, let alpha equals to a k.

Alpha equals to ak.

So, we know that alpha is less than .

This is our assumption.

This is our assumption.

Yes.

See, k we chose it in, see, a is less than equal minimum of this and by k. So, ak is always less than .

Clear?

So, this ak I am choosing it to be alpha, which is again less than .

So, now, you see, given any epsilon positive, we may choose n large enough, large enough, okay, so that for any

r greater than s greater than n. What do we have?

We have xr of t minus xs of t norm is bounded by summation k equals to n to infinity norm of xk plus of t minus xk of t.

This is again dominated by summation a equals to n to infinity alpha k times l. This we can do.

So this is like the usual calculus stuff.

I can just write it like this.

And once I can do that, this is always less than equal epsilon.

See, the thing is why can we do this?

Because it is basically a geometric series.

It is basically a geometric series.

And what do we know?

We know that since the series converges, the geometric series can be made as small as possible.

Now, what we do is this.

So, we show that this is true.

Let $L = \max_{t \in [-a,a]} \|X_1(t) - X_0(t)\|$ and $L \le aM$.

Now, $\|X_2(t) - X_1(t)\| = \left\| \int_0^t F(X_1(s)) - F(X_0(s))\, ds \right\|$

$\le \int_0^t K \|X_1(s) - X_0(s)\|\, ds$. ($\because X_1(s)$ and $X_0(s) \in O_\rho$ $\forall s$)

$\le aKL$.

Use induction to show that $\|U_{k+1}(t) - U_k(t)\| \le (aK)^k L$.

Let $\alpha = aK$, so $\alpha < 1$ (assumption)

Given any $\epsilon > 0$, we may choose N large enough so that for any $r > s > N$,

$$\|X_r(t) - X_s(t)\| \le \sum_{k=N}^{\infty} \|X_{k+1}(t) - X_k(t)\| \le \sum_{k=N}^{\infty} \alpha^k L \le \epsilon.$$

↑ Geometric series.

So, basically what we have is this.

We have this sequence of function.

So, let me write it this way.

You see, we have a sequence of function.

xn is a sequence of function.

Of function.

We have shown that this is Cauchy.

So, which converges, right?

Converges uniformly.

uniformly to a continuous function to a continuous function x from i to rn okay this is a causal sequence and the convergence is uniform so basically we know that uniformly converging sequence this will actually converge to a continuous function x from i to rn clear okay

Also, see that x k plus of t is nothing but x plus to t capital F of x k of s, sorry, yes, x k of s ds.

Now, if you take, taking limits on both sides, take limits on both sides, taking limit on both sides.

both sides, what do we have?

We have x of t is nothing but x plus limit t tends to infinity to t capital F of x k of s ds.

Clear?

Yes.

I can take the limit inside because of the uniform continuity.

And then I have, this is x naught plus to t

limit t tends to infinity f of xks ds.

Right?

Now, see, f is continuous function.

It is continuously differentiable.

So, I can take the limit inside.

So, I can say this is to t f of xs ds.

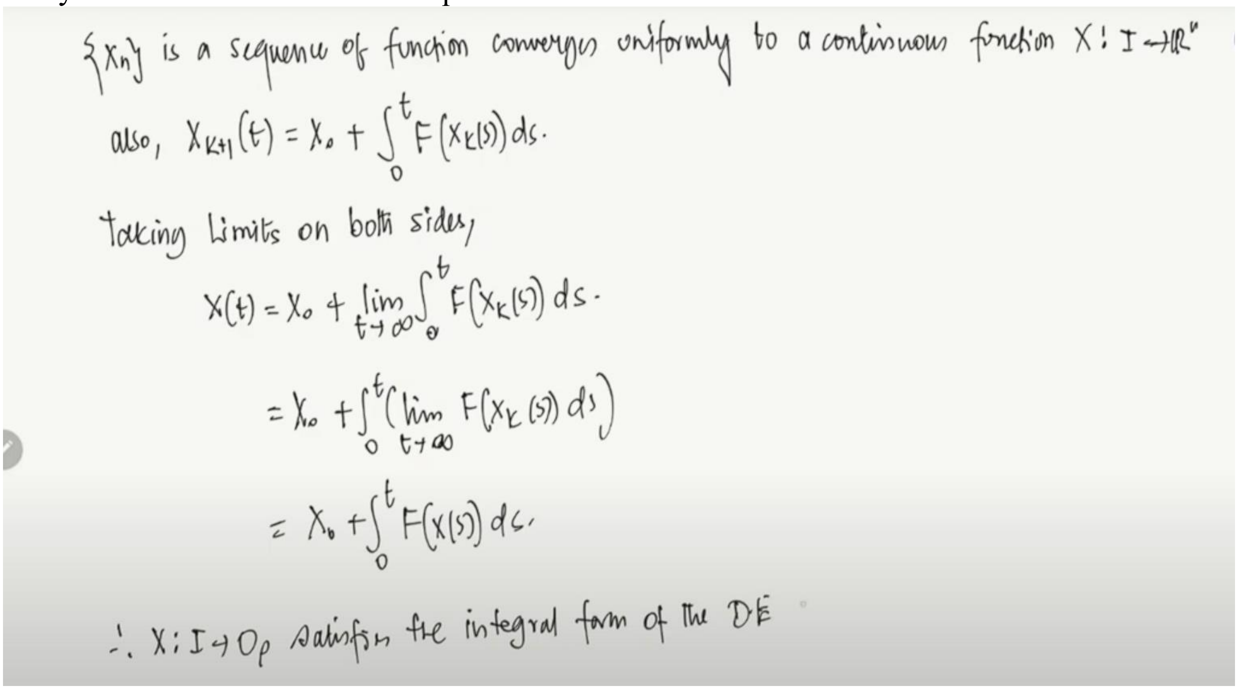
Clear?

So, therefore, what you have, therefore, x

we have found the x from i to O rho, okay, which satisfies, satisfies the integral form, the integral form, form of the differential equation, right, differential equation, okay.

So, what does it say?

It says that it is a solution of the equation and hence x is in



c here so um we actually got that you have a you know existence of solution at least once yes okay that is there now please remember this thing that the solution which here we are getting is in i i is minus a you see it is in where is it i i minus minus a here and this minus a a a is

Once again, I did some mistake here.

No, no, it is okay.

A I shows A. So, now you see.

So, essentially the solution which you are getting is a small neighborhood of .

So, that is why we are saying this is a local .

So, basically we are saying that we have a curve.

which is defined in a small neighborhood of zero, which satisfies the equation and passes to the initial point.

Now, the important part is the existence part.

So, existence part is done.

Now, the important part is the uniqueness.

Uniqueness of solution.

Now, this is not very difficult to prove.

Let us do this part, uniqueness of solution.

So, let us say, let

xy from i to r are two solutions.

Sorry, i to x of t is in omega are two solutions of the differential equation x prime equals to f of x.

satisfying x equals to y equals to x, okay?

See, why, x and y is from i to r, x and y, this basically x of t and y of t, that should also satisfy f of xt, right?

And f, the domain is in omega.

So, basically xt has to be in omega, yeah?

in that sense essentially okay now you see i have to show that x is equal to y right so we have to show we have to show to show x of t equals to y of t for all t in i so let q let

q is nothing but the maximum okay of norm of x t minus y t and where is t varying between t is varying between minus a to a okay so this is minus a to a right here t is varying between minus a to a okay so that is there now this maximum

See, let us say, and let me put it this way.

Let us say that this maximum is attained somewhere at the point t.

So, t is in minus a to a. So, therefore, what is happening is this.

First of all, if the maximum can be attained, of course, it can be attained.

See, norm, what did I tell you?

Norm is a continuous map, right?

How do we show?

See, norm.

From sum x to r is a continuous function.

Show this first of all.

Check this part.

Function.

Check.

Okay.

Right.

I already did it.

If you remember properly, think of one inequality.

Yes, think of one inequality.

I already proved it.

Yes.

So it is a continuous function while doing Lipchitz continuity.

It is a continuous function.

And it is more or less Lipchitz continuous actually.

And this x and y, these are continuously differentiable.

So x minus y is continuously differentiable.

norm x minus y is basically a composition.

So, it is a continuous map and on a compact set minus a to a continuous map there is a maximum.

So, now you see q is I can write it as to t x prime of s minus y prime of s

Why?

Because you see, this is nothing but mod of x t minus y t.

I can write it just like this, right?

Clear?

Fine.

Now, you see, this particular thing, I can always do it to be less than equal to t f of

x of s minus f of y of s ds.

I can do that.

Now, this is always dominated by, you see f is dominated by k. f has a bound k. So, this is always dominated by k times to t mod of, sorry, norm of x of, so this is norm.

These are all norm because these are all vectors.

these are all norms so x of s minus y of s yes okay so that is always dominated by a k q here now since c a k is less than

then this has to be a k is less than equal a k times q. So, since a k is less than , the only option is q should be .

There is no other option.

Therefore, if q is , then the maximum of this x t minus y t, the norm of that is going to be .

That will imply that x t has to be equals to y of t for all t in minus a k.

So, the unique test follows.

So, with this I am going to end this video.

Uniqueness of Solution :-

Let $X, Y : I \to \Omega$ are two solution of the D.E $X' = F(X)$ satisfying $X(0) = Y(0) = X_0$.

We have to show $X(t) = Y(t)$ $\forall t \in I = [-a, a]$.

Let, $Q = \max_{t \in [-a,a]} \|X(t) - Y(t)\| = \|X(t_1) - Y(t_1)\|$. [$\|\cdot\| : X \to \mathbb{R}$ is a continuous function (Check)].

$$\therefore Q = \left\| \int_0^{t_1} (X'(s) - Y'(s))\, ds \right\| = \|X(t_1) - Y(t_1)\|$$

$$\leq \int_0^{t_1} \|F(X(s)) - F(Y(s))\| ds \leq K \int_0^{t_1} \|X(s) - Y(s)\| ds \leq aKQ$$

$\because aK < 1 \Rightarrow Q = 0$

$\therefore X(t) = Y(t)$ $\forall t \in [-a, a]$.