Ordinary Differential Equations (noc 24 ma 78)
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Week-01
Lecture-01
Vector Spaces

welcome students to this first lecture on sub preliminaries which we need to know so essentially this we I am supposing that we know and i am just giving you the basic facts okay so first of all we need to know what a vector space is right and of course I am assuming that we know but still it's just a revision

that you are looking at a set, of course, a non-empty set.

So, V, you have a non-empty set, non-empty set, right?

And this set, we will put some operations on it.

So, first of all, this non-empty set with a binary operation, okay, binary operation, which we will call addition,

addition and then we will also have a binary function on it okay and a binary function on it binary function which we will call as scalar multiplication so the set along with these two

operation and a binary function, and it has to satisfy certain assumptions, and it satisfies certain assumptions, satisfies the below assumptions.

So, what are the assumptions?

Assumptions.

So, first of all, associativity is there.

So, what is this?

It is basically

U+(V+W) you can actually write it as(U+V)+W okay that is associativity okay right now this is associativity of vector addition okay now we have something called a commutativity is this (U+V)=(V+U) this is commutative

Vt, okay, of vector addition, okay.

And then we will have an identity element, which we are basically going to call it as a .

So, identity element is , and what is the property?

So, ∃ ∈ $V(Zero)$, we will call it as , okay, in V, and we will call it as , such that V+=+V=V, yes, and

this is equals to V, this should hold for ∀ V∈ V(Identity) in the vectors, in the sets V, right.

So, if you take any element of the set, if you add with it, does not really matter in which way you are adding, this always will be .

This is the identity element, okay, identity element.

And then, of course, you have the inverse, right, inverse with respect to addition, vector addition.

So,

for every element V∈ V okay you will ∃-V∈ V such that v+(-v)= so this is always given to you right so that is going to give you the identity element . Okay so that is the inverse so if you have a element v you have always have a inverse minus v right now we so these are the four properties which addition so with respect to addition it should satisfy and then

The other properties which we need is the scalar multiplication.

So, a(bv) should look like (ab)v. And what is A,B here?

a,b ∈ ℝ .

or whatever field you are choosing.

For this course, we are always going to choose the set over R, right?

So, that is your scalar.

So, here I am saying scalar multiplication.

It means the elements which we are multiplying with the vectors.

The vectors will be from element V. So, the scalars will come from ℝ, okay?

It can come from c also, but for this course, at least now, we will always consider it to be ℝ, okay?

So, the scalar field,

is ℝ and then you have this property that a(bv) so bv is an element of V right so when you multiply it by a it means that (ab)v okay and then you have this property that identity element with respect to the scalar multiplication right so *V=V okay so this is the multiplicative identity actually multiplicative identity

identity okay and then you also have distributive property distributive property is a*(u+v)=a*u+a*v so this is distributive okay and then you also have distributivity of scalar multiplication with respect to the addition

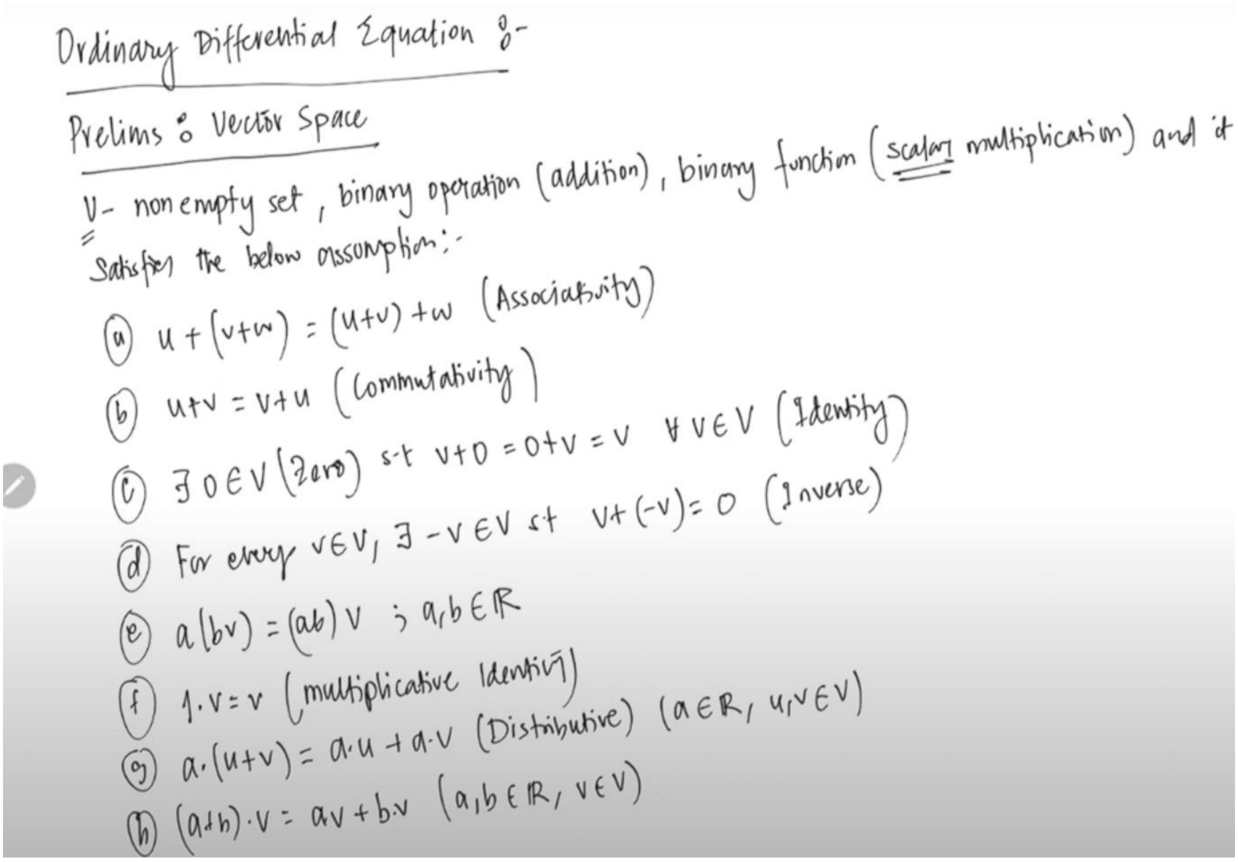
So, that will be given by (a+b)v=a*v+b*v. So, essentially, here, you see, I am assuming a to be in ℝ and u,v is in V. ℝ can be in any field f. You can replace it with there.

And again, here, a,b ∈ ℝ and v ∈ V, for any V in V. It does not really matter which one.

This is...

This is not distributivity, but this is distributivity of scalar multiplication with respect to the field addition.

So, these are the properties and this as I am assuming you already know this stuff.



So, let us move on to the next part.

So, essentially what we have is let us look at some examples which we can build out of this.

So, we always know that the first example which we can consider say example.

So (a) is of course ℝ, right?

So ℝ over ℝ. So I am choosing V=ℝ(real) is a vector space.

And then the field which we are choosing is of course ℝ. Then ℝ over ℝ, this is a vector space.

Vector space, right?

And so another one, of course, you have V. Let us say you have V=$\mathbb{R}^n$.

So this is $\mathbb{R}^n$ over ℝ . That is also a vector space.

This is also a vector space.

And we call it a real vector space.

because it is over $\mathbb{R}$, okay, so $\mathbb{R}^n$ over $\mathbb{R}$,that you can do, and of course, so these are the, these are the preliminary vector spaces, which you already know, I am assuming that, and of course, there is another vector space, which is very important in our context, which is $P_n(\mathbb{R})$, what is $P_n(\mathbb{R})$, so this is the set of all those polynomials, what are the polynomials, so basically, you are looking at this thing,
$P_n(\mathbb{R})$=$\{a \quad + a \quad x + \cdots + a_n x^n\}$, right, and here, [$a$,..., $a_n \in$ $\mathbb{R}$]this I am choosing it from $\mathbb{R}$, right.

So, this is a polynomial over $\mathbb{R}$, that is the coefficients are basically chosen over $\mathbb{R}$ and this is all possible polynomials of order less than equal n, okay.

So, of, so this is basically polynomials, polynomials of order

less than equal n. And with this, you can of course see that this is also a vector space.

And once you have this, we can also talk about what is the dimension of all these spaces.

So, what is the definition of dimension?

Let us just go over this quickly, dimension of a vector space.

So, we will define the number of linearly independent elements.

So, this is the minimum number.

The minimum, let us just say, put it this way, the minimum number of linearly independent elements required to span V. So, what does this mean?

Let us understand this.

First of all, what is linearly independent elements?

So, linearly independent, first of all, we have to understand this, right?

Linearly independent elements.

What is it?

So, essentially, you are basically looking at a subset of V, right?

So, given a vector space V, you are looking at a subset.

So, what is the subset?

Let us just call it S and the subset is given by some elements of V. Let us just write it like this,S=$\{v\ \ ,v\ \ ,\ldots.\,v_n\}$ }, okay, be a subset of

V yet now we will call we call $\{v_i\}$ the set is linearly independent when is it? If you look at this relation so, that is $C\ \ V\ \ +\ldots+C_nV_n=$

this will always imply that every C_i, every C_i, that is, $C\ \ =C\ \ =C\ \ =\ldots=C_n=$.

So, essentially, as an example, you can, of course, think of it that they do not depend on each other.

So, essentially, if you just think of two elements, V and V .

So, as an example, let us just look at it.

You see, if V and V , V and V are linearly independent.

What does it mean?

Linearly independent.

If I say something like this, that will imply, $C \ V \ + C \ V \ =$.

cannot be written as $V \ = -\frac{C}{C} V$ right you can't write it like this you can't write it like this because for a linear independence you have C and C both has to be so this division is not possible okay so what does that mean it means that one vector should not be

scalar multiple of another.

And what does that mean?

It means that let us say V and V are vectors.

So if you think, if one thinks, one thinks that they are

vectors in $\mathbb{R}$, let us say.

So, V and V are vectors in $\mathbb{R}$, let us say.

So, V is not a scalar multiple of V.

It means that they are not parallel to each other.

So,

So, if they are parallel to each other, it means that they are scalar multiples and in that case, V and V are linearly dependent.

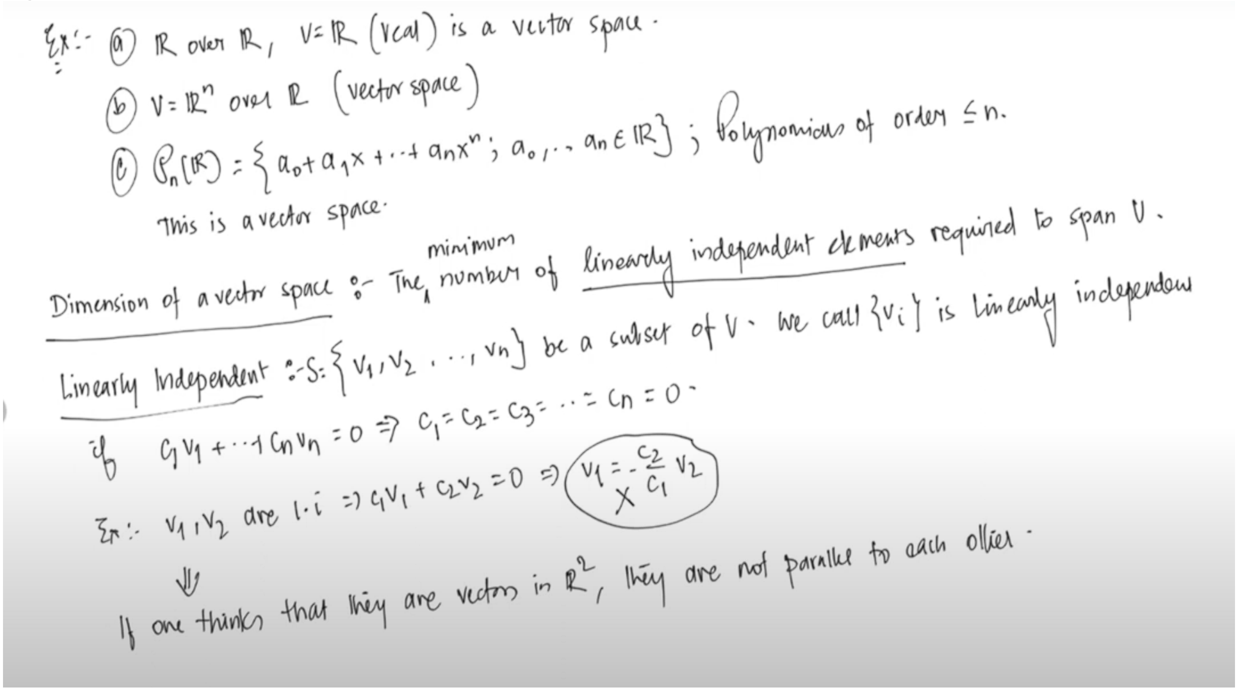
So, if they are parallel, they are linearly dependent and in this case, since you can see that one is cannot be linearly dependent of another, then they are not parallel to each other, right.

So, that is your linear independent.

And now the thing is the minimum number of, so what is the dimension of vector space?

This is the definition which we are going to use.

We are going to use the minimum number of linearly independent elements required to span V, right?



Okay, so what does that mean?

It means that what is span of V?

So let us just define that also, span of V.

Okay, so one thing I must say, you see, I am skipping most of these concepts because I know that all of you know this thing.

Okay, I am just telling you the basics in case you have some forgot or something.

Okay, so what is span of V?

Span of V is this.

So basically, given any element, let us say v∈V.

Okay.

So, what happens is you can actually have a set.

Okay.

So, let us say S is a set.

This S, let us just say this S which I defined here.

This S spans V if any element V can be written as,v= $\sum_{i=}^{m} C_i V_i$.Is it okay?

So, basically i equals to to

m does not matter what m is.

m∈N. So basically you can take some elements of this set and after that you can combine them to form any element of this initial set V and then we call that S pans V.

And it does not have to be linearly independent.

You see it can be more than that.

It does not really matter.

But the thing is what it means is basically if you are given some elements with the help of those elements you can linearly combine those elements to produce any element of your original space field.

That is your span.

If you have the minimum number of element which is required to span V, then we call that as the dimension of a vector space.

So, for example, let us say (a), example, the dimension of $\mathbb{R}$ =

Why?

Because any element, let us say r∈ $\mathbb{R}$ can be written as r=r*.

So,

What is your basis element here?

So what is the set which actually spans R?

The set will only contain .

So the basis set, let us just call it S, that will only contain the element . S={}.

And then again, let us say, what is the dimension of Rn in that case?

Dimension of $\mathbb{R}^n$=n, right?

And what is the basis element?

Basis, let us say, let us call that basis element.

See, what happens is, let us just try to find out what is the basis.

So, any element, let us say, V∈ $\mathbb{R}^n$ can be written as

V=V (,,,....)+V (,,,....)+V_n(,,....).

So, the nth element is .

So, here V, if the coordinate of V is given by V , V ,..., V_n, we can write it like this, right? V=$(V\ ,V\ ,\dots,V_n)$

V can be written like this.

So, basically, you see, if I denote these things, (, , ,...) as E , so let us just denote this as E , (,,,....)this as E , t(,,,,....)this as E_n, then this is nothing but summation V_i, E_i, right? V=V (,,,....)+V (,,,....)+V_n(,,....)
=$\sum_{i=}^{n} V_i\, E_i$

And then, you can see that, and please prove this, so please check this part, check that there cannot, there cannot be

not be (n-) linearly independent number of elements, number of elements which can span, which can span, span V, right.

So, essentially what is the basis element here?

Basis contains E ,

E to E_n.

Once you prove this, and then the dimension will be n. So, similarly, you can check, so please check this part, that the dimension of $P_n(\mathbb{R})$=(n+).

And why is that?

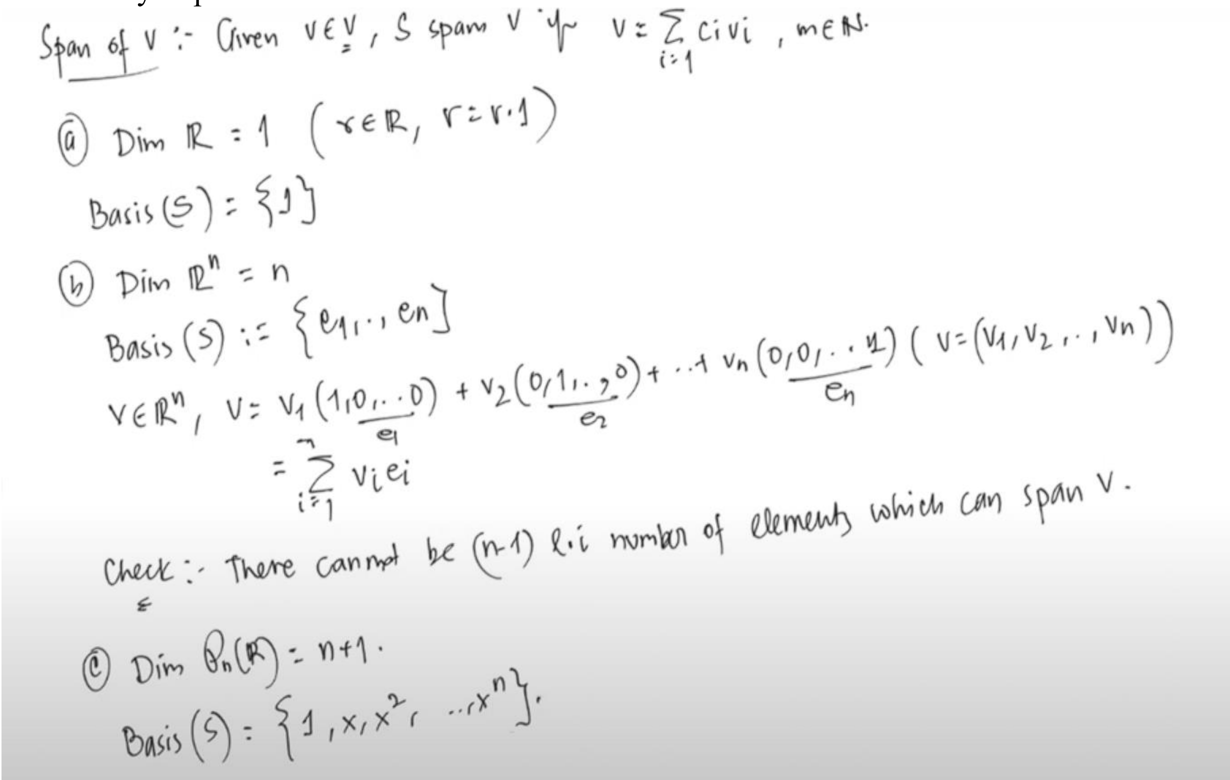
Because you can see that the basis element, let us just call the basis element, the set as, so basically, it will contain this element, Basis(S)={, x, x ,..., x^n}. You see, this element, all of this, if you combine it together, this will actually produce, first of all, you have to show this is linearly independent.

And then you can, once you show that, you can show that this can actually span any element of this set.

So, there, basically, you can show that this is going to be the basis of $P_n(\mathbb{R})$,right.

And this is very important for us.

This is very important for us.



So, why it is important?

So, here, this is the first example, which I think is going to be a little different than what you have known till now about vector spaces.

So, D, an example of vector space.

So, first of all, I will start with the interval I=[a,b](Closed bounded interval in ℝ.

So, in most of this course, whenever I am saying I is an interval, just think of it as closed bounded set [a,b].

So, this is a closed bounded set, closed bounded set.

And bounded interval, right?

Interval in $\mathbb{R}$. Interval in $\mathbb{R}$. And of course, you guys know that this is nothing but a compact set.

Compact set in $\mathbb{R}$, right?

Compact set in $\mathbb{R}$. Okay, right.

Now, we define, okay?

You define C[a,b].

What is C[a,b]?

C[a,b]={set of all continuous functions in[a,b]}

So, the function is defined on [a,b], sorry, on [a,b], okay, and then we call those as C[a,b].

So, basically, you take all those functions together and you form a set, this is called C[a,b].

Now, these, I can actually show that this is going to be a vector space, okay, and what is the addition and the multiplication?

So, basically, what I mean is this, see, if you start with two elements, $f,g \in C[a,b]$, right, if you start with two elements, then you

$(f+g)(x):=f(x)+g(x)$

See, this x is from I. You can check that this operation actually satisfies all the properties of vector space.

So, that is your first one.

And the second one is $(cf)x=cf(x)[c\in \mathbb{R}, f\in C[a.b]]$

So, please check this part.

Check that these two operations on C[a,b] forms a vector space over ℝ. Over ℝ, why?

Because here C is in ℝ and f is in

C[a,b]. You do realize that you see I am saying this is a vector, right?

C[a,b] is a set of all continuous functions on [a,b]. You see I am not, I am saying that any element of this set

So, this is a vector.

So, these are actually functions, continuous functions, but we are calling them as vectors since this actually satisfies all the properties of a vector space.

So, we can call them as vectors.

Now, one important thing is this.

see, okay, what are the examples?

Let's just look at some examples of C[a,b], some elements of C[a,b].

So, of course, if you define, let's say $f\ \ (x) = \exp(x)$, or any functions which is like a, you know, proper trigonometric function, so sines, cosines, let's say $f\ \ (x) = \sin(x)$, okay, so that can also be element of this set.

And then,

Of course, you also have the polynomials, right?

So, let us just write like $f\ \ (x) = (C\ \ \ + C\ \ x + \cdots + C_n x^n)$, right?

x∈C[a,b],and $C_i \in \mathbb{R}$.

So, this is a polynomial, any polynomial that will also be a continuous function.

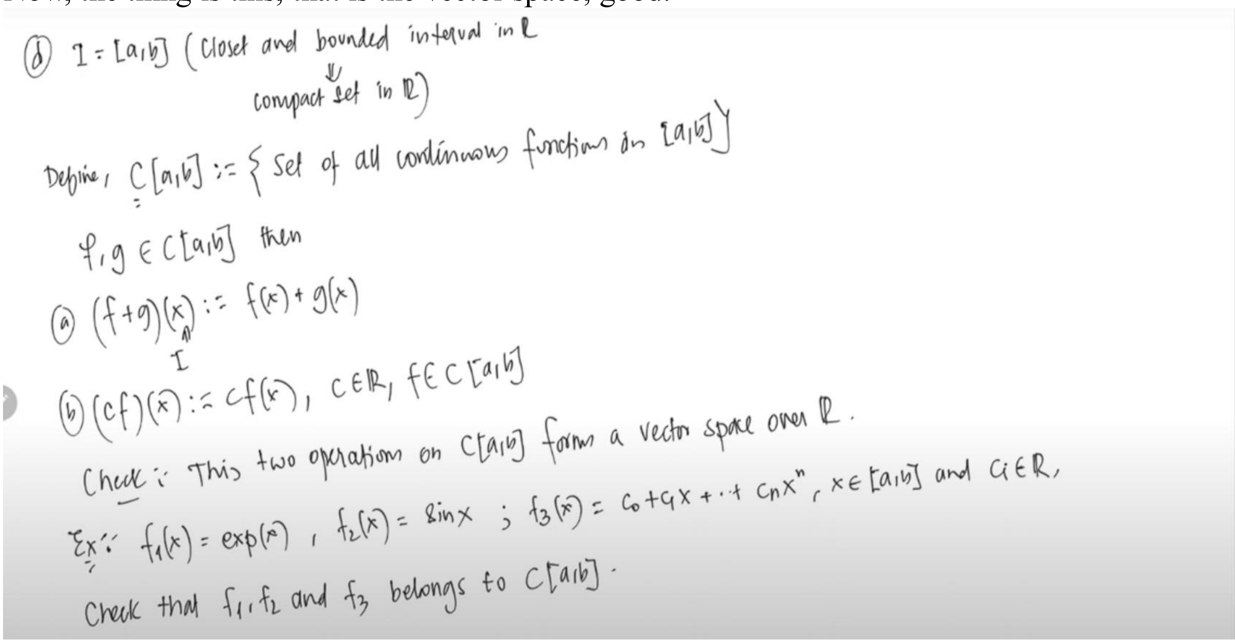
So, if you do not know this, please, I mean, what you can do is please check this part.

Check that f , f and f belongs to C[a,b].

C[a,b].

So, these forms have vector space, very nice.

Now, the thing is this, that is the vector space, good.



Now, can we talk about the basis of this, basis of C[a,b]?

So, what is the basis of C[a,b]?

See, C[a,b] is

is a very different set than all of this, whatever I have defined, $\mathbb{R}$, $\mathbb{R}^n$, $P_n(\mathbb{R})$. Why?

Because the thing is, this is an example of an infinite dimensional vector space.

So, basis of C[a,b], what is the basis?

This is the question.

So, basically, what the dimension we can say, dimension of C[a,b] is infinite.

So, this question is out of the context for this course, but basically what I am trying to say is what does it mean?

It means that there does not exist, there does not exist, there does not exist any finite

any finite subset of C [a,b]

which is linearly independent okay linearly independent let's just call it L.I., linearly independent this is how I'm going to write in this course okay because it saves time and you know I don't want to write so much so L.I. okay so basically what it means is you cannot get a finite subset of C[a,b] okay so you cannot get a finite collection of continuous functions which are linearly independent

and which spans C[a,b].

So, what I mean by this, vaguely speaking or not very mathematically speaking, intuitionally it means that if you cannot have a finite collection of continuous functions defined on closed interval [a,b],

Such that you can span any element of C[a,b] with those right finite combinations of what I mean.

So and how do you prove such a thing?

So this please you have to do this part yourself.

So you check this see,

So, can you give me a subset of C[a,b]?

So, first of all, any subset of C[a,b], you can think of it as $P_n([a, b])$, is a subset of C[a,b], right?

Is a subset of C[a,b] for any n∈N, right?

So it does not really matter, all polynomials, it does not really matter if it is first order polynomial, second order polynomial, nth order polynomial, whatever it is, they are all continuous functions.

Okay, so basically any polynomials defined on closed interval [a,b], okay, that is always contained in C[a,b], right, that you can of course see because any polynomials are continuous functions, okay.

So and since this works for any n∈N, it does not really matter what n is.

So you can see that of course the dimension of C[a,b] is going to be infinite.

It cannot be possibly finite.

But the thing is this is not a precise way of putting it.

I mean this is just an intuition.

Okay.

And please I want you to do that you just show that the dimension of C[a,b] is infinite.

Okay.

So what sort of infinite?

Countable, uncountable?

We do not know that.

We do not need to know that right now.

But the thing is you just need to know that the basis is the number of elements on the basis is infinite.

Okay.

Now, can we produce more vector spaces out of it?

Of course, we can.

So, we define this.

So, define, we define C^k[a,b].

What is C^k[a,b]?

C^k[a,b]={set of all, set of all kth continuously differentiable function on [a,b]}

Okay? Right; so, uh you can define this so basically what I am doing is what do I mean by continuous differentiable so first of all as an example let's say what is C ? C [a,b]={f:[a,b]→ℝ;such that the function is continuous, f' *exists and* its continuous.}

See, C^k is kth continuous differentiable function, right?

So, up till kth order.

So, basically, it is like, so C is up till first order, continuous differentiable.

So, it means that f is continuous, f' exists and is continuous.

So that is what it means that the first derivative also exists and it is going to be continuous.

We call it a C .

So similarly C also you can think of it as the second derivative.

So f is continuous, f' exists, f' will be continuous and then f'' will exist and f'' has to be continuous and then you have C [a,b].

And as you can see, I mean, you can guess that this Ck of [a,b], these states are always going to be infinite dimensional.

So, this also you need to check that C^k[a,b] is infinite dimensional.

So, $C^k[a,b]$ is infinite dimensional.

So, this is basically the ideas of vector spaces which you are going to need.

Now, another question I want to pose here and I want you to understand this that is there any inclusion between all these spaces $C^k s$.

So, essentially what I mean is this.

You see, you can actually see that C[a,b] is the biggest space, right?

So, basically what I mean is, let us say C [a,b] and C[a,b], what is the relation between these?

See, if a function is continuously differentiable, once continuously differentiable, it means it is going to be continuous, right?

So, the continuous function space is actually bigger, right?

So, C [a,b] is contained in, so let me put it this way, C^{k+}[a,b], [a,b] is contained in $C^k[a,b]$ for any k=,,,…

Right? Okay; and similarly uh you can see that right because if it is if the kth derivative exists and this is going to be continuous then (k-)th derivative also exists and it is continuous so basically one is included in another right and uh we can also define this you can also define C^{∞}[a,b]

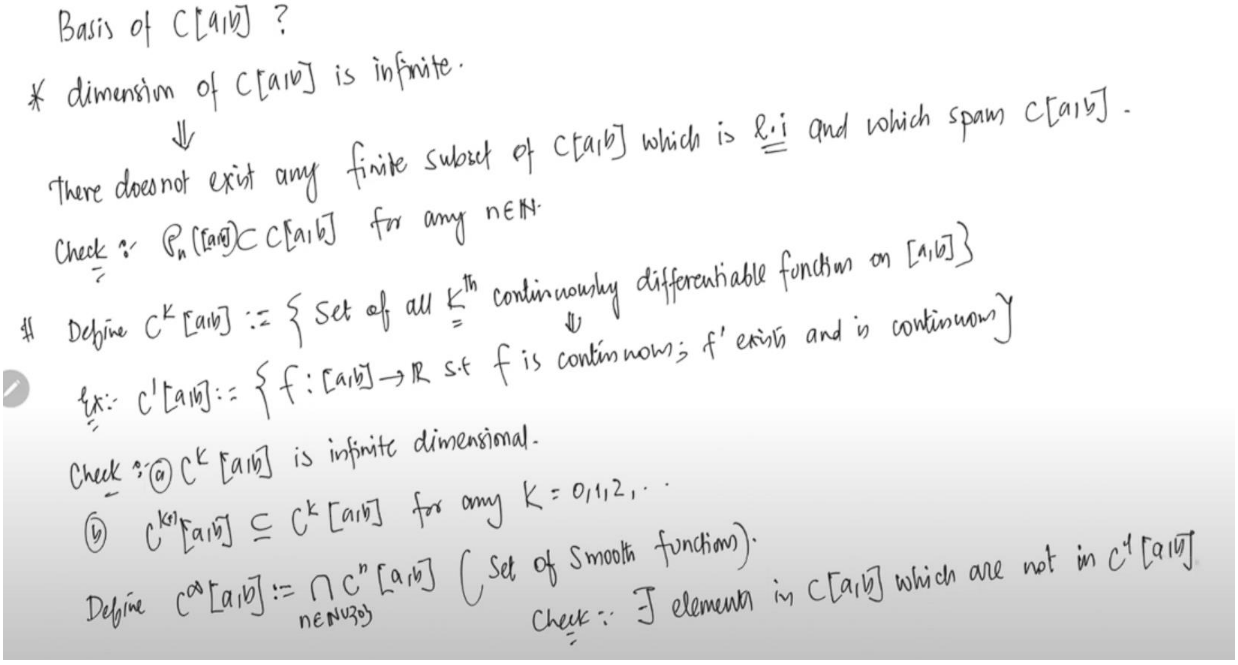
C^{∞}[a,b]=$\bigcap_{n\in\mathbb{N}\cup\{\}} C^{n}$ [a, b] (Set of smooth functions)

So, basically what I am trying to say is this, you take all C^{k}s and you take the intersection of that, that will give you C^{∞}.

So, this we will call it as a set of smooth functions.

So, with this we are going to conclude the lecture, but before I do that, just a quick question.

There exist elements in C[a,b] which are not in C [a,b].



okay yeah and one small details also which I missed and I should add that part also that see here this I am using all this inclusion as a part of subset okay but you can also define something called a subspace right this you already know but still let me just do it subspace so basically you have a set V subset of V okay so given a vector space V this is given okay V is called

called a subspace if, and this is the easy way of doing it,

Let me put it this way.

$V\ \ W \in V$ =V+W∈V.

and $v \in V$,C∈ ℝ= CV∈ V

Okay?

So, if these two properties are there, so of course, if it is V∈ V ; V,W∈ V , it has to be in also in V. But basically, what I am trying to say is this, V inherits the properties of, you know, the scalar multiplication and the vector addition, those two operations, those two binary operations, those the properties are inherited by V also, and then we call it a vector.

So, V will form is also a vector space.

Also, a vector space, right?

It is also a vector space over whatever the field is.

Let us say here the original field is in ℝ , right?

Vector subspace over ℝ and we call it a vector subspace.

subspace of V. And as an example, of course, you can see that let us say C [a,b] is a vector subspace of C[a,b].

It is a vector subspace.

So, let us just define C^{∞}[a,b] is also a subspace of C[a,b].

Now, one question and this is where we are going to end the lecture.

One question is this.

Can you find, can you find

A finite dimensional subspace, finite dimensional subspace of C[a,b].

So, that is the question, and with this question.

Subspace :- $V_1 (\subseteq V)$ is called a subspace if

↑ Given

$v, w \in V_1 \Rightarrow v + w \in V_1$

and $v \in V_1$ and $c \in \mathbb{R} \Rightarrow cv \in V_1$

V_1 is also a vector space over $\mathbb{R}$ and we call it a vector subspace of V.

Ex :- (a) $C^1[a,b] \subset C[a,b]$

↑ Subspace.

(b) $C^\infty[a,b]$ is also a subspace of $C[a,b]$.

Question :- Can you find a finite dimensional subspace of C