An Introduction to Hyperbolic Geometry

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Lecture – 36

The Role of Quasi-Geodesics in Hyperbolic Metric Spaces: Stability and Connectivity

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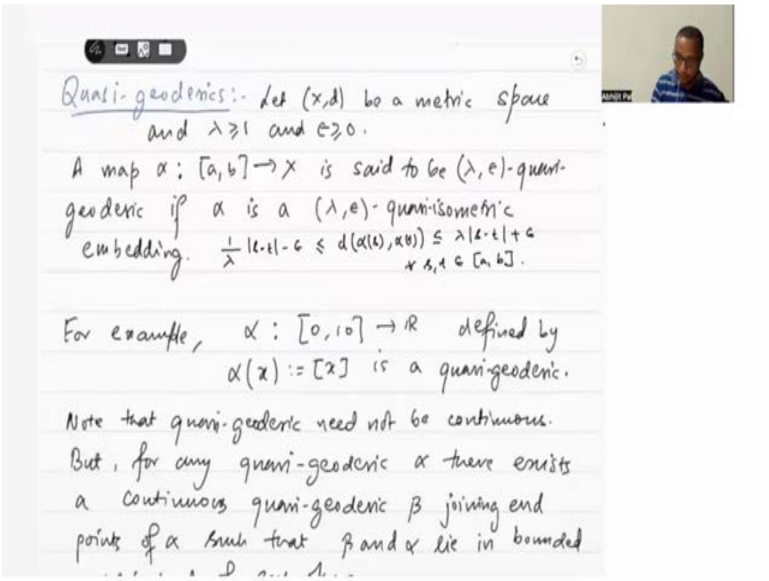

Hello, and welcome to today's lecture, where we will explore the definition of quasi-geodesics. Quasi-geodesics can be understood as quasi-isometry embeddings of a geodesic. We will demonstrate that in a hyperbolic metric space, geodesics and quasi-geodesics that connect the same pair of points lie within a uniformly bounded neighborhood of one another. This remarkable property is known as the stability of quasi-geodesics. Utilizing the stability of quasi-geodesics, we will prove an important result: if there exists a quasi-isometry between two metric spaces, X_1 and X_2, and if X_1 is a hyperbolic metric space, then X_2 must also be a hyperbolic metric space.

Let's begin with the definition of a quasi-geodesic. Consider a metric space X and let λ be greater than or equal to 1 and ϵ be greater than or equal to 0. A map from the closed interval [a, b] to X is said to be a (λ, ϵ)-quasi-geodesic if it is a (λ, ϵ)-quasi-isometry embedding.

What does this mean? It means that for all s, t belonging to the closed interval [a, b], the following inequalities hold:

$$\frac{1}{\lambda}|s-t| - \epsilon \leq \text{distance}\big(\alpha(s), \alpha(t)\big) \leq \lambda|s-t| + \epsilon.$$

As a practical example, let's consider a map α from the closed interval [0, 10] to R (which could also be any R^n. We can define α(x) as the greatest integer function, commonly denoted as ⌊ x ⌋. In this case, the image of α consists solely of integers, making α(x) a quasi-geodesic. This is quite straightforward to see.

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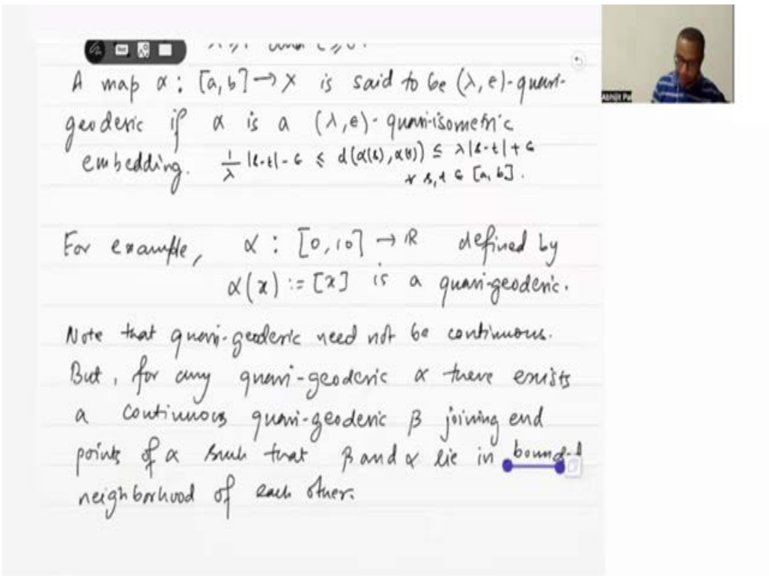

It's important to note that the map α is not a continuous map; however, it is indeed a quasi-geodesic. This illustrates that a quasi-geodesic does not necessarily have to be a continuous map. Additionally, if we examine the definition we discussed earlier, by setting λ = 1 and ϵ = 0, we obtain the definition of a geodesic, which is continuous by nature.

However, we can further demonstrate that for any quasi-geodesic α, there exists a continuous quasi-geodesic β that connects the endpoints of α. Notably, both β and α will lie within a uniformly bounded neighborhood of each other. This relationship highlights the inherent stability of quasi-geodesics within metric spaces.

This brings us to the proposition we have formulated. Let's consider α to be a λ, ϵ-quasi-geodesic defined on the closed interval [a, b]. We can then find $\lambda' \geq 0$, $\epsilon' \geq 0$, and $K \geq 0$, along with a continuous path β defined on the same interval [a, b].

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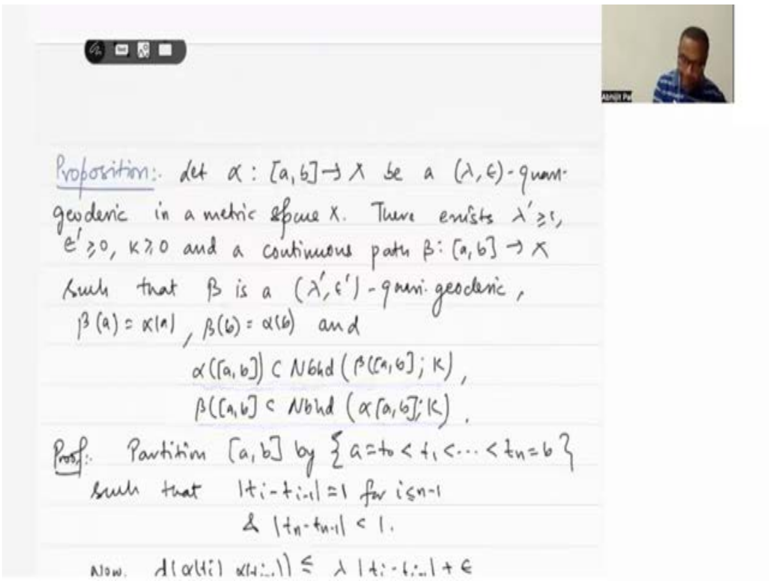

Importantly, β is a λ', ϵ'-quasi-geodesic, and the endpoints of β coincide with the endpoints of α. Furthermore, α resides within a K-neighborhood of β, while β is also situated within a K-neighborhood of α. It’s crucial to note that the parameters λ', ϵ', and K depend solely on the original values of λ and ϵ. This establishes a robust relationship between the quasi-geodesics and the continuous paths connecting them.

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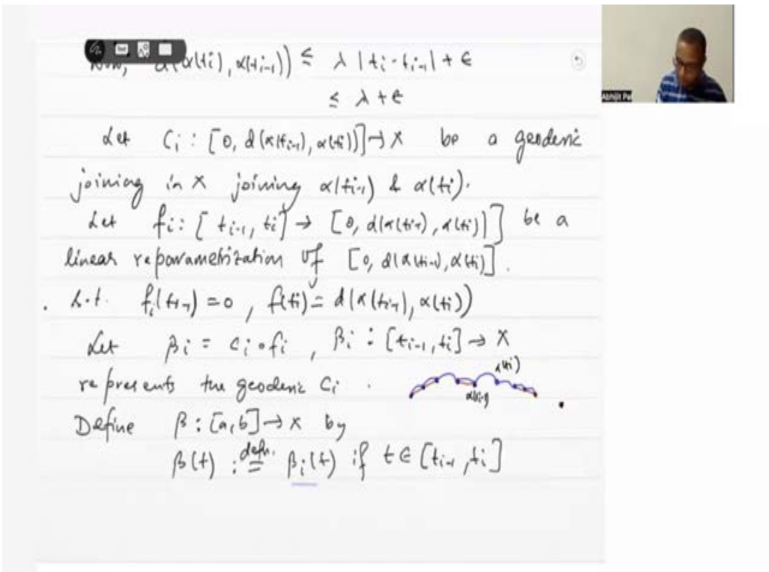

So, how do we go about proving this? Let us begin by taking a partition of the closed interval [a, b] in the following manner. We define the partition points $t_0, t_1, \ldots, t_n$ such that the absolute difference $|t_i - t_{i-1}| = 1$ for $i = 1, \dots, n - 1$, while $t_n - t_{n-1} < 1$.

Now, considering the distance between $\alpha(t_i)$ and $\alpha(t_{i-1})$, we can assert that this distance is less than or equal to $\lambda \cdot |t_i - t_{i-1}| + \epsilon$. This conclusion stems from the fact that α is a λ, ϵ-quasi-geodesic. Given that $|t_i - t_{i-1}| \leq 1$, we can infer that the distance between $\alpha(t_n)$ and $\alpha(t_{i-1})$ is thus bounded by $\lambda + \epsilon$.

Next, we will consider a geodesic in X that connects the points $\alpha(t_{i-1})$ and $\alpha(t_i)$, which we denote as C_i. We define a function f_i from the closed interval $[t_{i-1}, t_i]$ to the closed interval $[0, \text{distance}(\alpha(t_{i-1}), \alpha(t_i))]$. This function serves as a linear reparameterization of the interval, ensuring that $f_i(t_{i-1}) = 0$ and $f_i(t_i)$ equals the distance between $\alpha(t_{i-1})$ and $\alpha(t_i)$.

Now, let's define β_i as the composition $C_i \circ f_i$. Thus, β_i is a path defined on the closed interval $[t_{i-1}, t_i]$ that maps into X. It's essential to note that the image of β_i corresponds exactly to the image of C_i; therefore, β_i represents the geodesic C_i.

To illustrate this, let us visualize the situation. We have established that α is a quasi-geodesic, which means its image may not be continuous. In this diagram, we can see the points $\alpha(t_{i-1})$ and $\alpha(t_i)$ clearly marked. I have connected these two points with a geodesic, represented by the blue paths, which are all C_i geodesics.

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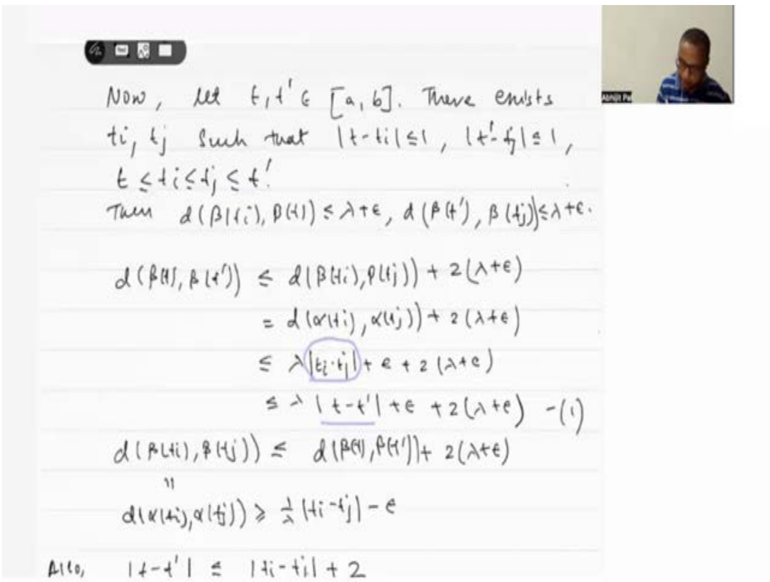

To construct a continuous path β, we take the concatenation of the paths β_i. The claim here is that this continuous path is indeed a quasi-geodesic and lies within a K-neighborhood of α, where K is determined solely by λ and ϵ. We define β(t) to be $\beta_i(t)$ if t falls within the interval $[t_{i-1}, t_i]$.

Therefore, the path β is constructed as the concatenation of the paths β_i. Let me refer back to the diagram we discussed earlier; this is our path β_i. Now, we aim to demonstrate that β qualifies as a quasi-geodesic. To do this, let's select two points t and t' within the closed interval [a, b].

There will exist points t_i and t_j such that $|t - t_i| \leq 1$ and $|t' - t_j| \leq 1$. We will also assume that $t \leq t_i \leq t_j \leq t'$. Consequently, the distance between $\beta(t_i)$ and β(t) will be less than or equal to λ + ϵ, and similarly, the distance between β(t') and $\beta(t_j)$ will also be less than or equal to λ + ϵ.

Now, what about the distance between β(t) and β(t')? We can apply the triangle inequality here. Thus, we find that the distance is less than or equal to the distance between $\beta(t_i)$ and $\beta(t_j)$ plus twice the quantity (λ + ϵ).

It's important to note that $\beta(t_i)$ corresponds to $\alpha(t_i)$, and $\beta(t_j)$ corresponds to $\alpha(t_j)$. Since α is a λ, ϵ-quasi-geodesic, we can apply this property, leading us to the following inequality:

$$\text{distance}\left(\beta(t_i), \beta(t_j)\right) \leq \lambda \cdot |t_i - t_j| + \epsilon + 2(\lambda + \epsilon).$$

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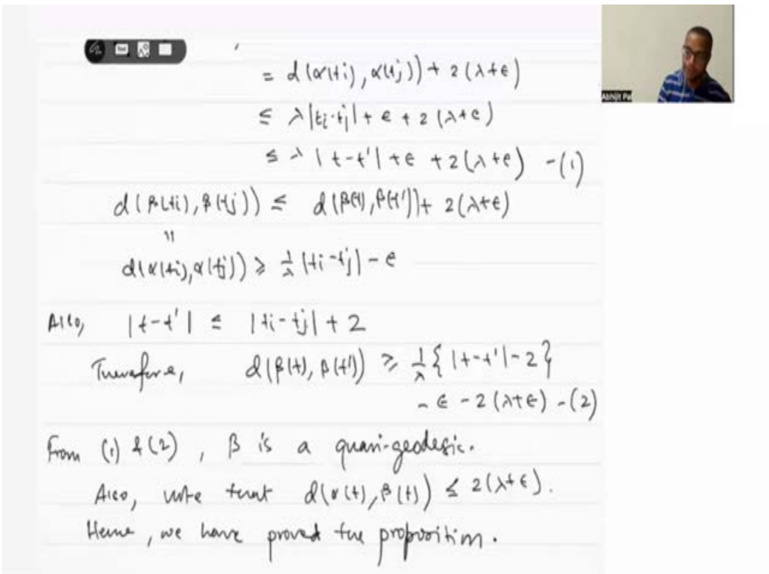

Since we established that $t \leq t_i \leq t_j \leq t'$, the absolute difference $|t_i - t_j|$ is also less than or equal to |t - t'|. Consequently, we can further refine our result to state that

$$\text{distance}\big(\beta(t), \beta(t')\big) \leq \lambda \cdot |t - t'| + \epsilon + 2(\lambda + \epsilon).$$

It's also important to note that the distance between $\beta(t_i)$ and $\beta(t_j)$ is precisely equal to the distance between $\alpha(t_i)$ and $\alpha(t_j)$. Since α is a λ, ϵ-quasi-geodesic, we can assert that this distance is greater than or equal to $\frac{1}{\lambda}|t_i - t_j| - \epsilon$. Additionally, we have the relationship that |t - t'| is less than or equal to $|t_i - t_j| + 2$.

Therefore, we can conclude that the distance between β(t) and β(t') is less than or equal to $\frac{1}{\lambda}|t - t'| - 2 - \epsilon + \lambda + \epsilon$.

Consequently, from the inequalities we've established, we can confidently say that β is indeed a quasi-geodesic. Furthermore, if we take any point t, the distance between α(t) and β(t) is less than or equal to 2λ + ϵ. This means that α and β lie within a k-neighborhood of each other, where k is defined as 2(2λ + ϵ).

Thus, we have successfully proved this proposition. Now, given a quasi-geodesic, we can, without loss of generality, take that quasi-geodesic to be a continuous path, thanks to the results of this proposition.

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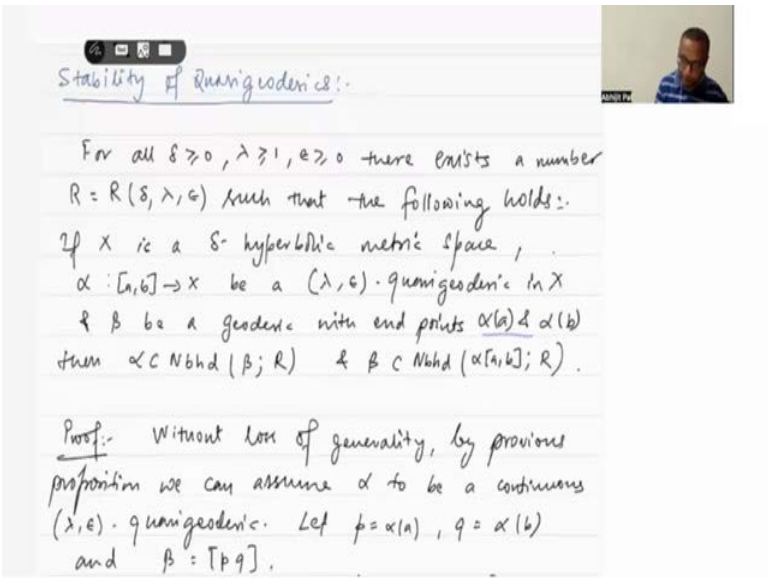

Now, let us proceed to the statement of the stability of quasi-geodesics. We begin by considering a non-negative number δ, a parameter λ that is greater than or equal to 1, and ϵ that is greater than or equal to 0. Under these conditions, there exists a constant R that depends solely on δ, λ, and ϵ, such that the following holds:

If X is a δ-hyperbolic metric space, and α is a λ, ϵ-quasi-geodesic in X, and if β is a geodesic that connects the endpoints of α, then it follows that α lies within an R-neighborhood of β, and conversely, β also lies within an R-neighborhood of α.

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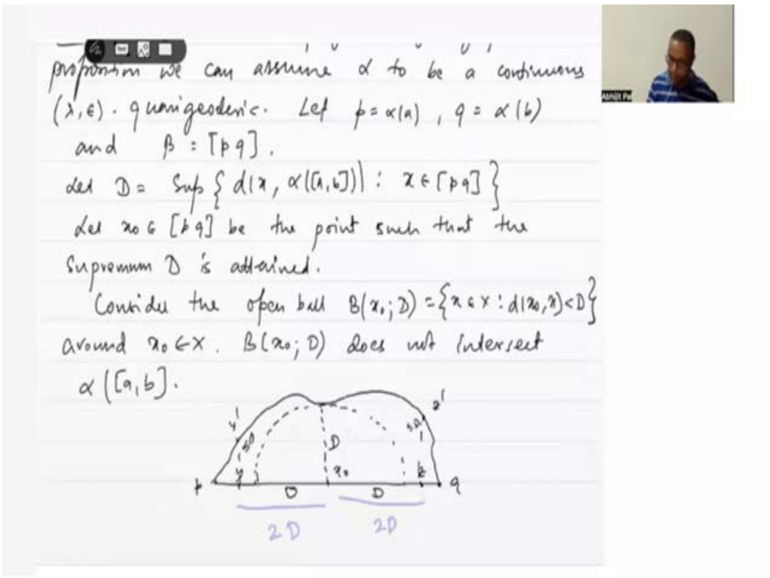

Without loss of generality, based on the previous proposition, we can assume that α is a continuous λ, ϵ-quasi-geodesic. Let us define P to be α(A) and Q to be α(B), and we will denote the geodesic connecting these points as β.

Now, since α is a continuous path that serves as a quasi-geodesic, let D represent the supremum of the distance from any point x on the geodesic joining P and Q to the path α. Because α is continuous, there exists a point x_0 on this geodesic such that the supremum D is attained at this specific point x_0.

Next, consider the open ball centered at x_0 with a radius of t. It is important to note that because we have defined D to be the supremum, this open ball does not intersect with the path α. To illustrate this, we can visualize the situation: here is the geodesic β, and here is the path α, along with the open ball that does not intersect α.

Now, let us consider a point y on the geodesic connecting P and x_0 such that the distance between y and x_0 is equal to 2D. If we are unable to find such a point y, and if the distance between x_0 and P is less than 2D, we will simply set y equal to P. We can apply the same reasoning in the opposite direction.

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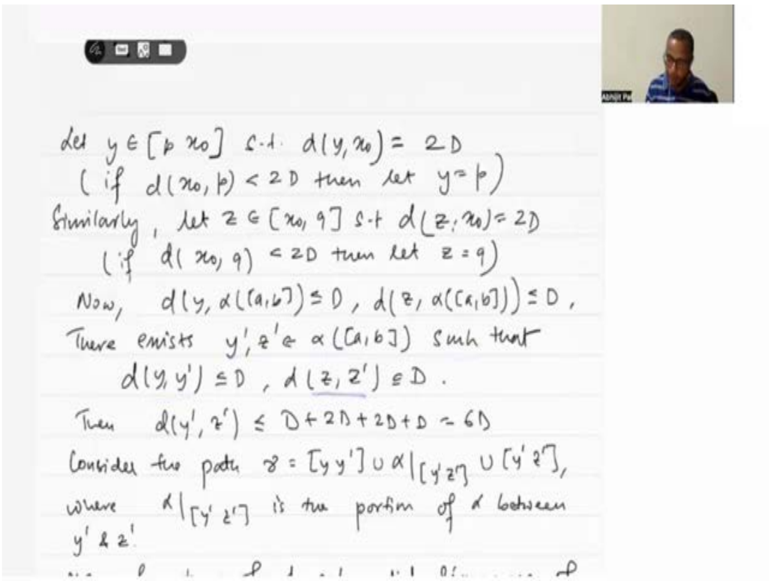

Next, let z be a point on the geodesic joining x_0 and Q, ensuring that the distance between z and x_0 is also equal to 2D. Again, if the distance between x_0 and Q is less than 2D, we will set z equal to Q.

To illustrate this scenario, we can visualize a diagram where y and z lie outside the open ball centered at x_0, each at a distance of 2D from x_0. If the distance between P and x_0 is indeed less than 2D, we will set y equal to P, and similarly for the point z.

Now, considering the distances, the distance between y and α will be less than or equal to D, as D represents the supremum. Likewise, the distance between z and α will also be less than or equal to D. Therefore, we can conclude that there exist points y' and z' on α such that the distance between y and y' is less than or equal to D and the distance between z and z' is also less than or equal to D.

Let us return to our previous illustration. For the point y, we have the corresponding point y' such that the distance between them is less than or equal to D. Similarly, for the point z, we have the point z' also within D of z. Now, let’s define the path γ as the geodesic joining y and y', combined with the segment of α that connects y' and z', and finally the geodesic joining y' and z'. This entire path, denoted as γ, encompasses these segments.

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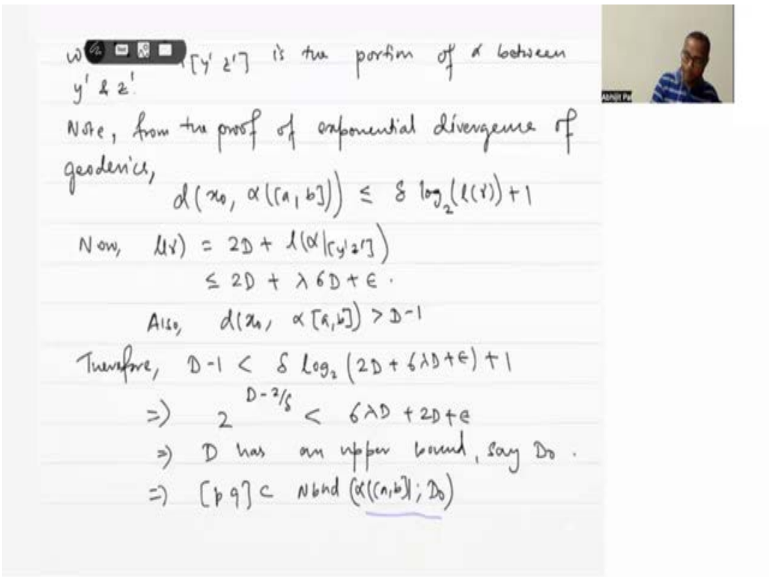

From our earlier discussion on the exponential divergence of geodesics, we can establish that the distance from x_0 to α is less than or equal to δ times the logarithm (base 2) of the length of γ, plus 1. You can verify this relationship using the concept of exponential divergence of geodesics.

Now, when we examine the length of γ, we find that it is bounded above by $2D + \lambda \cdot 6D + \epsilon$. But why is this assertion true? The path γ consists of two geodesics and a quasi-geodesic segment that connects y' and z'. Since α is a λϵ quasi-geodesic, the length of this segment will be less than or equal to λ times the distance between y' and z', plus ϵ. Thus, we can conclude that the overall length of γ satisfies the inequality:

$$\text{Length of } \gamma \leq 2D + \lambda \cdot 6D + \epsilon.$$

Let us analyze the distance between y' and z' as defined by our construction. This distance will be less than or equal to 6D. Consequently, we can conclude that the length of γ satisfies the inequality:

$$\text{Length of } \gamma \leq 2D + \lambda \cdot 6D + \epsilon.$$

Additionally, it is important to note that the open ball of radius t centered around x_0 does not intersect with α. This leads us to the conclusion that the distance between x_0 and α is greater than t - 1.

Moreover, we have previously established that the distance from x_0 to α is less than or equal to $\delta \log_2(\text{Length}) + 1$. Therefore, we can deduce that:

$$t - 1 < \delta \log_2(2D + 6\lambda D + \epsilon + 1).$$

From this, we can further derive that:

$$2^{D-1} < \delta(6\lambda D + 2D + \epsilon).$$

Here, the left-hand side represents an exponential function of D, while the right-hand side is a linear function of D. Consequently, D must have an upper bound, which we will denote as D_0. We have defined D to be the supremum of the distances between x and α, where x lies on the geodesic connecting points p and q. Hence, we find that D is less than or equal to D_0.

Thus, we have demonstrated that the geodesic connecting p and q will indeed reside within the D_0 neighborhood of α.

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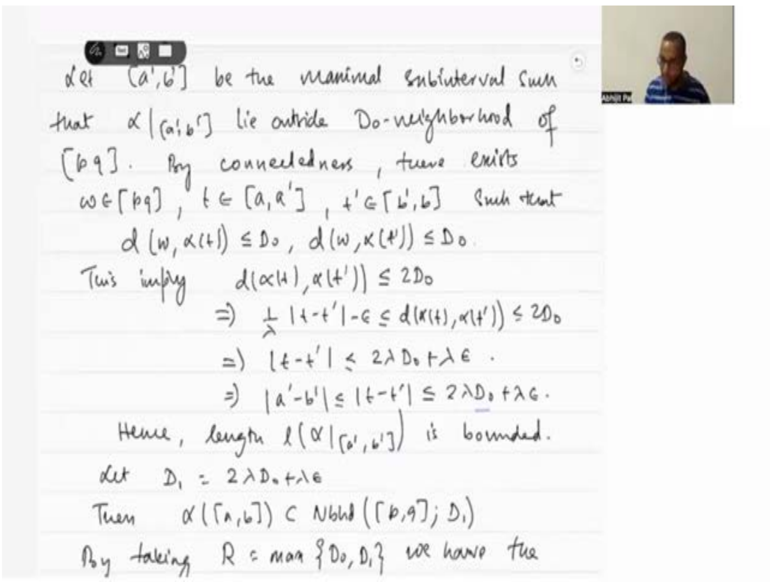

Now, let us move on to the second part, which is considerably easier and does not require the assumption of hyperbolicity. Consider the closed interval [a', b'], which is defined as a maximal subinterval such that the restriction of α to this interval lies outside the D-neighborhood of the geodesic connecting points p and q.

By applying the argument based on connectedness, we conclude that there exists a point w on the geodesic joining p and q. Additionally, we can find points t within the closed interval [a, a'] and t' within the closed interval [p', p] such that the distance between w and α(t) is less than or equal to t_0, and the distance between w and α(t') is less than or equal to D_0.

From these observations, we can infer that the distance between α(t) and α(t') is less than or equal to $2D_0$. Since we know that α is a quasi-geodesic, we can express the distance between α(t) and α(t') as follows:

$$\text{Distance}\big(\alpha(t), \alpha(t')\big) \geq \frac{1}{\lambda}|t - t'| - \epsilon.$$

Given that this distance is less than or equal to $2D_0$, we can derive the following inequality:

$$|a' - p'| \leq |t - t'| \leq 2\lambda t_0 + \lambda\epsilon.$$

If you visualize this scenario, it will become clearer, and I encourage you to verify this assertion for yourself.

Consequently, the length of the path α restricted to the interval [a', b'] is bounded. This is because the length of this interval [p', b'] has already been established as bounded, and this bound is dependent solely on λ, ϵ, and D_0. Furthermore, D_0 itself is influenced by λ, ϵ, and δ.

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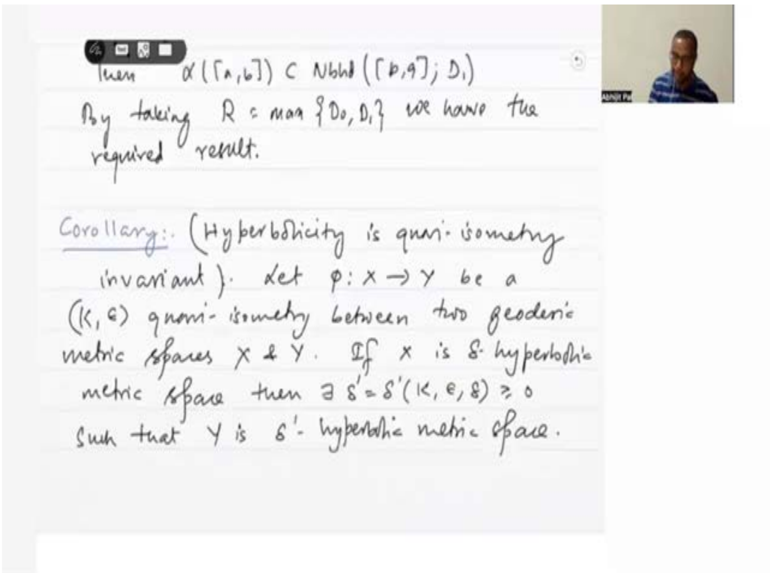

Thus, we have established that the length of α is indeed bounded. Now, let us define D_1 as 2λ

$D_0 + \lambda\, \epsilon$. Consequently, the path α will lie within the D_1 neighborhood of the geodesic connecting points p and q. If we denote R as the maximum of D_0 and D_1, we can conclude that we have achieved the desired result.

As a corollary, we will demonstrate that hyperbolicity is quasi-isometry invariant. To do this, let us consider a $K\epsilon$ quasi-isometry between two metric spaces, X and Y, and we will assume that both X and Y are geodesic metric spaces. If X is a δ-hyperbolic metric space, then there exists a δ' that depends on K, ϵ, and δ, such that Y will also be a δ'-hyperbolic metric space.

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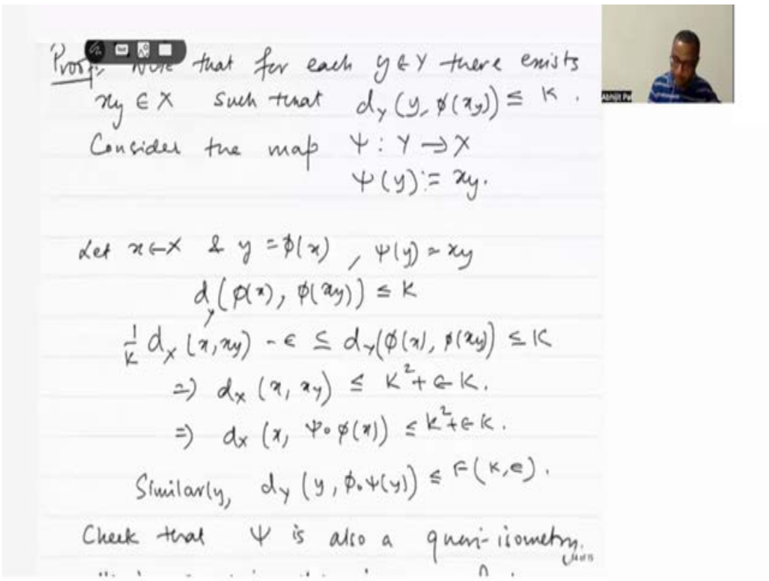

First and foremost, it's important to note that for every element y in the co-domain space, there exists an element x_y in the domain space X such that the distance between y and $\varphi(x, y)$ is less than or equal to K. Now, let's consider the mapping ψ from Y to X, where $\psi(y) = x_y$. I will leave it to you to verify this part.

One can demonstrate that ψ is indeed a quasi-isometry, with the distance between x and $\psi \circ \varphi(x)$ being less than or equal to $K^2 + \epsilon$ K. This bound depends solely on the scale K and ϵ. Moreover, the distance between y and $\varphi \circ \psi(y)$ is also bounded above by some function of K and ϵ. Thus, we can refer to this mapping as the quasi-isometry inverse of φ.

Next, we need to verify that this ψ is a quasi-isometry inverse for Y. Now, let's consider a triangle in Y and establish that Y is indeed a δ'-hyperbolic metric space. To do this, we start with a geodesic triangle in Y formed by the points P, Q, and R.

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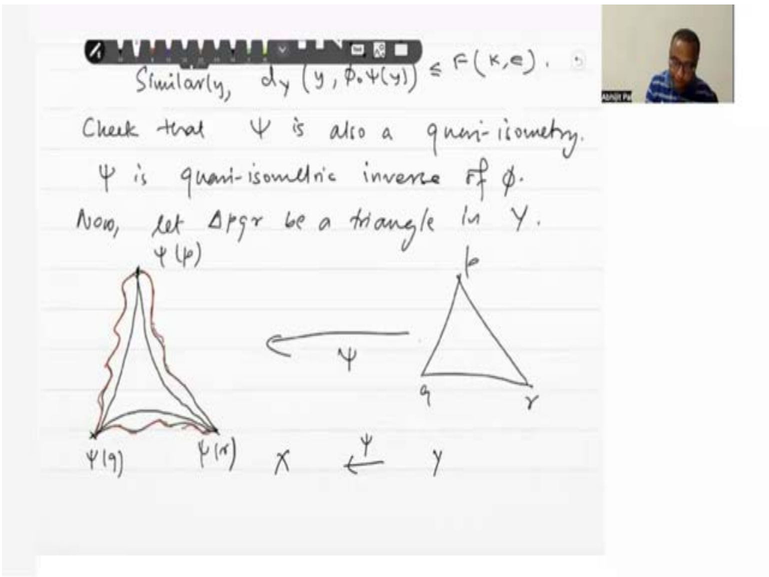

As previously mentioned, we have the mapping ψ from Y to X. Since ψ is a quasi-isometry, it will transform the geodesic connecting P and Q into a quasi-geodesic in X. Consequently, each side of the triangle will be mapped to a quasi-geodesic in X, resulting in a quasi-triangle comprised of the points ψ(P), ψ(Q), and ψ(R). According to earlier propositions, we can consider the quasi-geodesic to be continuous. Now, let's examine the geodesic triangle formed by the points ψ(P), ψ(Q), and ψ(R) in X.

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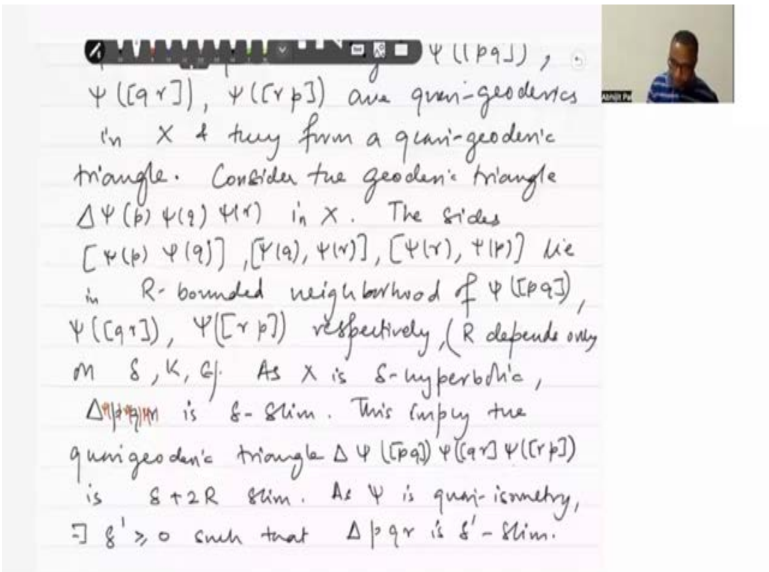

The mapping ψ is asymmetric, meaning that ψ(P, Q), ψ(Q, R), and ψ(R, P) are all quasi-geodesics in X that together form a quasi-geodesic triangle. Now, let's consider the geodesic triangle formed by the points $\psi(P)$, $\psi(Q)$, and $\psi(R)$ in X. Thanks to the stability of quasi-geodesics, the geodesic joining $\psi(P)$ and $\psi(Q)$ will lie within an R-neighborhood of the quasi-geodesic $\psi(P, Q)$, a property that follows directly from the stability characteristics we discussed earlier. Consequently, each side of this triangle will be contained in the R-neighborhood of the corresponding quasi-geodesic sides of the quasi-geodesic triangle.

Now, since X is a δ-hyperbolic metric space, the triangle formed by $\psi(P)$, $\psi(Q)$, and $\psi(R)$ will also exhibit hyperbolic properties. Therefore, this triangle will be classified as a δ-slim triangle. This leads us to conclude that the quasi-geodesic triangle will be $(\delta + 2R)$-slim.

We can assert that there exists some δ' greater than or equal to zero, such that the triangle formed by P, Q, and R is δ'-slim. Thus, we have shown that the quasi-geodesic triangle, the red triangle depicted here, is $(\delta + 2R)$-slim. Since the mapping is a quasi-isometry, we can conclude that the triangle PQR will also be δ'-slim. Importantly, this δ' depends solely on δ, K, and ϵ. I encourage you to explore the details further as an exercise, and I will pause here.