Differential Calculus of Several Variables

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Module 02
Lecture No 08
Chain Rule.

Okay, this lecture I will talk about Chain rule which is very useful for the calculation of derivative.

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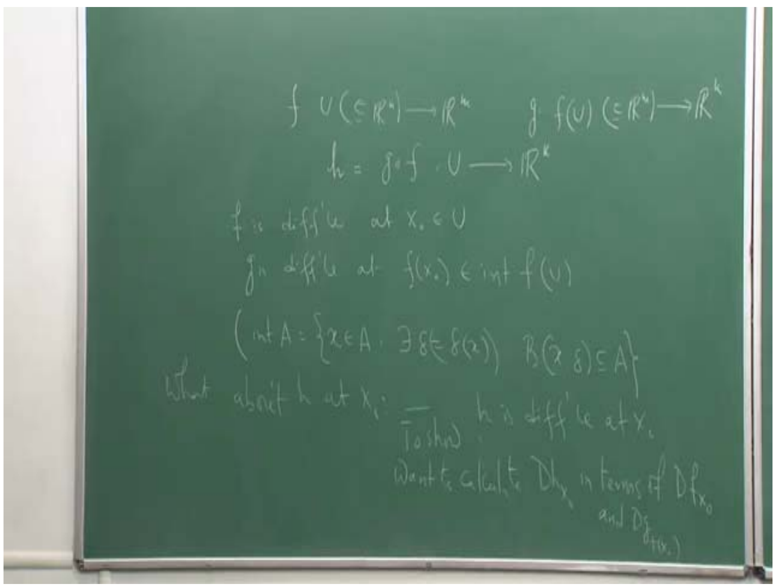

Chain rule says that if I have functions let us say F from U in RN to RM, and G is a function of FU, which is in now RN to RK then I can define this function H which is G compose F this is from U to RK, and suppose F is differentiable at X not in U, U is an open set, okay. And G is differentiable at F of X not which belongs to interior of FU, interior of FU means that if I have a, U is open but I do not know FU is open or not, so interior FU means the open part of the FU.

That is interior of the set A, means set of all X in A such that there exists a delta depending upon X, such that B X delta belongs to A, okay. So for derivative I need open set, I have told you in the first class because we add and subtract things. So F is differentiable at X not in U, U is already open, so any open set interior is same.

G is differentiable at F X not which is interior of FU, then what about H? That then, from the Chain rule of one variable you know H is differentiable, everything is NMK is 1 then H is differentiable and derivative of H is given by G of FX, F prime of X. Here also you would given by same thing, so actually we want to calculate, we want to show that H is differentiable at X not and also want to calculate DH at X not in terms of D FX not and DG at F X not.

Means what, we want to, this is very useful because sometime you can, even one function you can write composition of 2 functions and which are easy to calculate. I will give 2 – 3 examples in the assignments and then you will see how can one break one function as composition of 2 functions and then calculate individual derivative and then apply Chain rule to get the actual derivative. Since I have very short time in the lecture I will keep all the practice problems in assignment.

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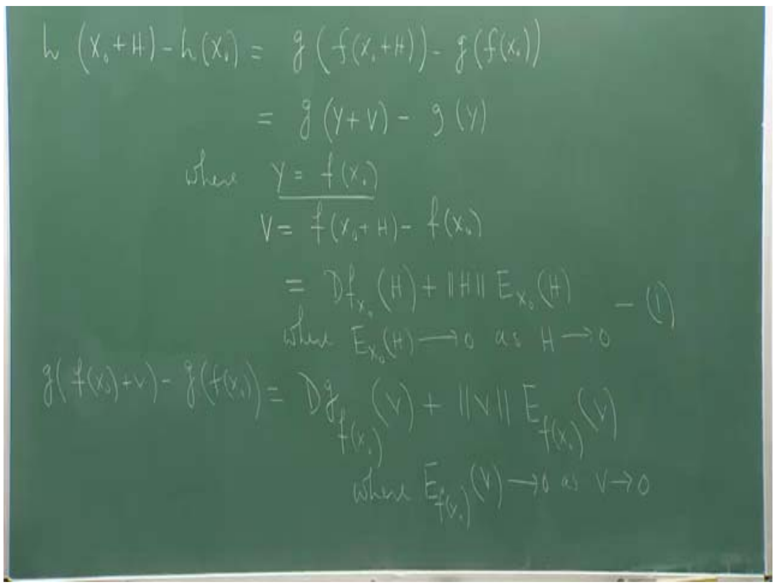

Okay, let us see how it is done. So I have to look at H, H of X not plus H minus H of X not, but write, I write it in the definition, this is G at F X not plus H, correct, H of X not plus H is G of F X not plus H minus G of F X not. Okay, let me write it in this form. G equal to G of some Y plus V minus G of Y where Y is F X not and Y plus V is F X not, so V is, Y plus V is F X not plus H, Y is F X not, so V is F X not plus H minus F X not.

Now I have assumed F is differentiable at X not, correct. So what about V, I can write V as D F X not at H plus norm H E X not H where E X not H goes to 0 as H goes to 0. This is list numbering 1. This I have use differentiability, this is the first assumption F is differentiable at X not. Well, thik hai.

Now G of Y plus V minus GY, Y is F X not, G is differentiable at F X not, so why not write F X not plus V, F of X not plus V minus G of F X not, this equal to, G is differentiable at F X not, so that is G at F X not acting at V plus norm V E F X not V where E F X not H, sorry V goes to 0 as V goes to 0. Look at it I mean, let us number is 2. I have just used my assumption, F is differentiable X not that gives me 1, G is differentiable at F X not that gives me 2. Then I will put it together 2 definition of H.

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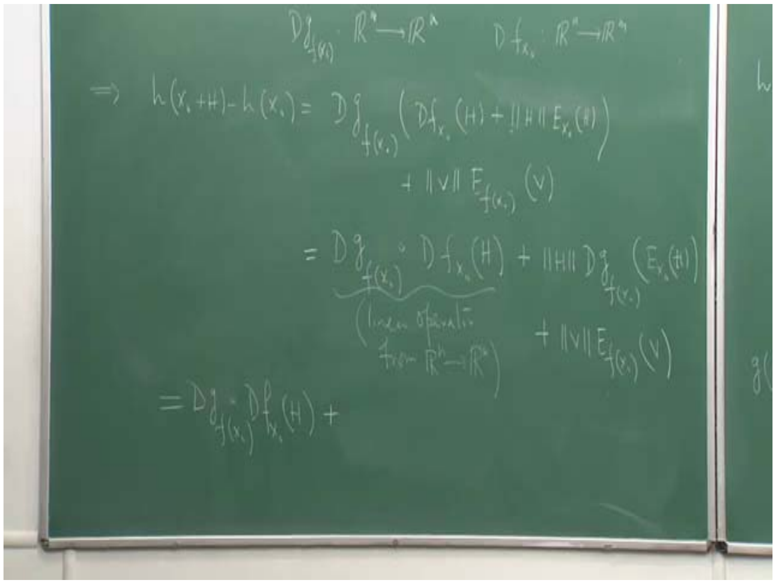

What is it? G F X not plus V, G F X not so that is DG F X not at V, V is, where is V, here, correct. I have just put V from here, 1. The remaining term remains same. Now look DF X not is what, DG F X not is what, according to my definition derivative is a, so DG F X not is, is a linear operator from RN to RK, right.

There is a linear, and DF X not is a linear operator from RN to RM. So this is a composition of linear operator so that gives me GDGF X not, composition of DF X not H, so if I have operator from RN to, from first RN to RM and RM to RN then this is an operator, linear operator from RN to RM. Okay, this is linear so linear will play here, this is the scale which comes out, so this is DGF X not acting at EX not H plus I have the remaining term, V F X not, look at that expression carefully, if I have done everything correct, I think I have done, right.

Now I will write it, I keep it as it is because this is my probable candidate, plus, plus how do I write it, I write it in a very, write it in a little bit different form so that I write (())(11:43) my purpose. Okay, I have written a sum of two (())(11:53), this time it is okay, so I have to take care of. So this is, these two things I have written together so what is this fellow…

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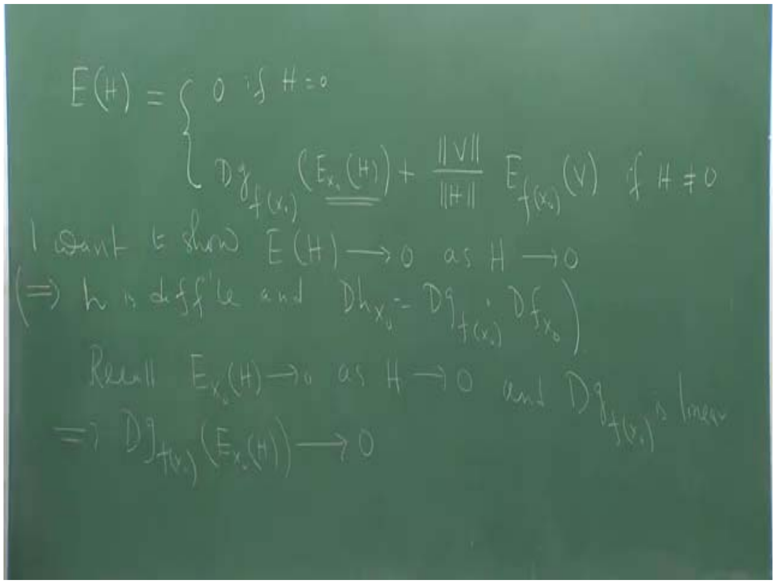

EH, tell me what is EH just you can calculate, EH is, well this seems to be this is equal to if I want to write it I norm H form then this will be, just a minute, yeah. So D G F X not at E X not H plus norm V by norm H, E F X not V, but you see this is divided by norm H, so H is non 0, but what will happen if H is 0, this term is 0, H is 0 then norm V is 0, because V is, recall what is V is, so H is 0, V is also 0, so this is 0, if H equal to 0, and this is if H not equal to 0.

Convince yourself what I have written is correct. I want to show EH goes to 0 as H goes to 0, why is so. If I have achieved this then what I have, I have two of the answers simultaneously then I will that derivative of H that is in this case so we will have, so we will have if I show this, first of all H is differentiable, this will imply, H is differentiable and DH X not is DG F X not, compose with DF X not.

Okay, this is the last part what I have to show. Now what was E X not is, recall EX not H goes to 0 as H goes to 0, look back at your notes at definition of V and DG F X not is a linear operator, any linear operator if a vector goes to 0 then it goes to 0. So that implies DG F X not, E X not H, this goes to 0. That I have used that if A is a linear operator X and (())(15:22) are vector so as in XN goes to 0 then A of XN goes to 0, straight forward linear algebra you have it.

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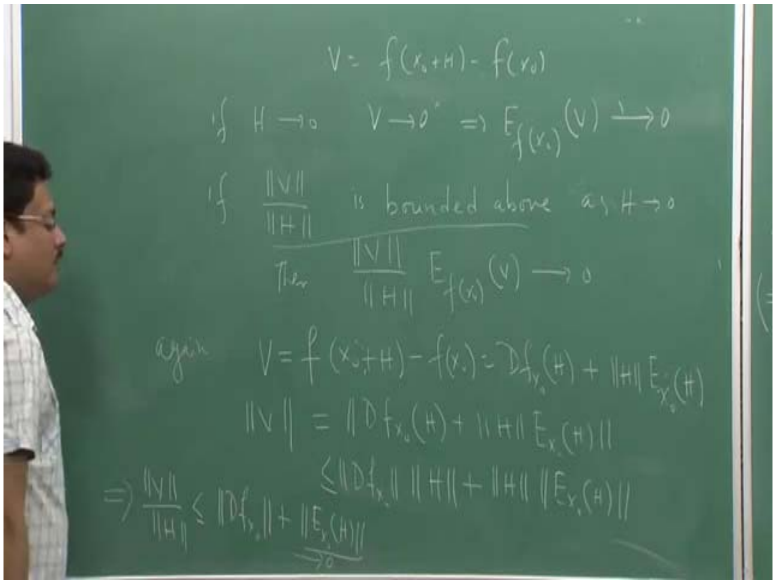

So this part taken care, what bat this part, well obviously V goes to 0 if H goes to 0, V goes to 0 as well because V is again F of X not plus H minus F X not, F is differentiable so continuous, so H goes to 0 V goes to 0, so this implies E F X not at V, this goes to 0 because G is differentiable at this comes from the (())(15:22) part of G. So G is differentiable at F X not so this goes to 0.

This part goes to 0 no problem but what about this part, there is something is multiplied, V goes to 0 H goes to 0, well, here it is. So if this norm V by norm H so something goes to 0 and this is a scalar, if this is bounded above, it is a positive quantity, it is very bounded below by 0, bounded ever then if I multiply, something is bounded ever, bounded ever as H goes to 0 then if I multiply with something goes to 0 that goes to 0, it is very easy to, of course all of you can convince yourself very quickly.

That I have multiplied some sequence going to 0 with something bounded then it goes to 0, so I have to show that this is bounded above. But what is V, again V is F X not plus H minus F X not which is D F X not H plus norm H, E X not H. So norm V is norm of D F X not H plus norm H E X not H lesser equal to norm of D F X not, norm of H we have used it before also plus norm of H plus norm of E X not H that will give us norm V but norm H is less equal to norm of D F X not plus norm of E X not H, correct. Now this quantity goes to 0 as H goes to 0 so it will be less than say, something goes to 0 as H goes to 0 so after certain age I can say this is less than 1.

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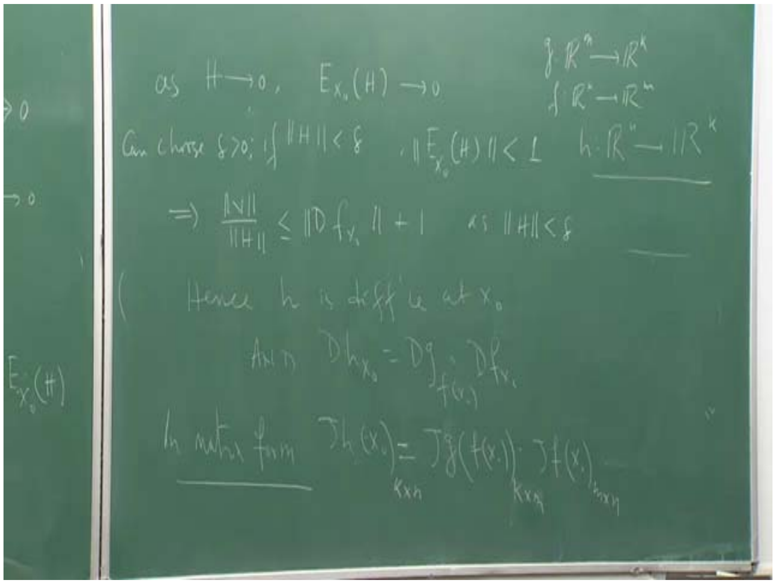

So as H goes to 0, so as, let me write it there, so as H goes to 0 I can say that okay, E X not H goes to 0 so if H is less than some delta then, can choose delta such that if H less than delta E X not H norm will be less than 1 because it goes to 0, could be less than any positive quantity.

So that will show, norm V by norm H is less than D F X not, norm of D F X not plus 1 as norm of H less than delta. So this quantity remains bounded. This is what we wanted to prove is bounded above, so I got this. So hence I got, hence H is again, H is differentiable at X not and D H X not is equal to D G composition of this two operator. If I have composition of two linear operator and if I look at their matrix so in matrix form Jacobian of H at X not, this will be composition of matrix of this and this but matrix composition is simply matrix multiplication that you know from linear algebra.

So that is J F, JG Jacobian of G at F X not, matrix multiplication at F X not, this is a M cross K matrix, and Jacobian of F at X not, this is a, sorry, G is from M to K so this is K cross M matrix, sorry, G was from RN to RK so this is a, JG X not is a K corss M matrix and this F is from RN to RM so this is M cross N matrix so this matrix is K cross N, that is okay, because H is from RN to RK. Okay, that is chain rule as I say it is a very useful one and you will see it in assignment that this helps in calculation a lot.

So basically it finished today's lecture but I want to leave you an exercise which is not a part of chain rule but still you may enjoy doing it and also you will just curiosity what happens, because if real values functions we have seen it, so there is not part of chain rule, the proof is there.

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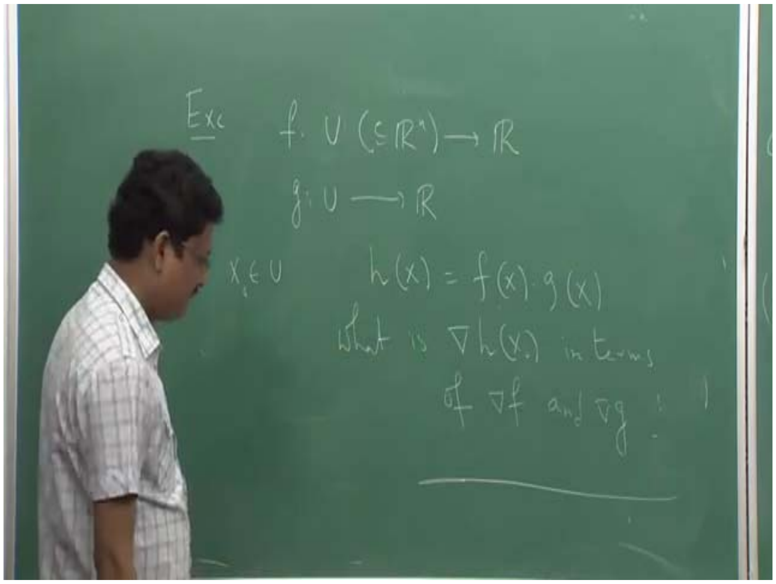

So you try this exercise, suppose I have F from RN to RN and so is G, okay, why RN, some open set U in RN to R real values, so is G from U to R, X in X not in U then I can define this new function H X is equal to F X which is real number into GX. So find out what is Grad H at X not in terms of Grad F and Grad G, so this is the exercise you try to do. This is very easy, just do it from definition, first, but first you must think of what is a candidate and then it is easy. Okay so next class we will do mean value theorem.

Thank you!