Differential Calculus of Several Variables
Professor: Sudipta Dutta
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur
Module 02
Lecture No 07
Sufficient condition of differentiability.

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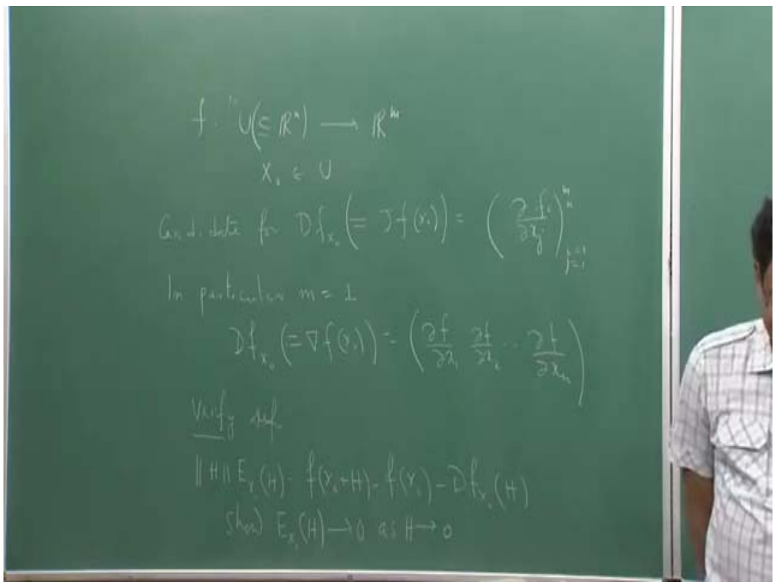

In the last lecture we have seen that if I have function F from a open connected set RN who, sorry, RN to RM and X not is a point then candidate for DF at X not in the matrix form also written as Jacobian matrix is equal to this matrix, right, dell FI dell XJ, I equal to 1 to M, J equal to 1 to N. I mean in particular case M equal to 1, this DF X not, this special notation I have for that that is called Grad F, this is simply this vector dell F, dell X1, dell F dell X2, dell f dell XN and we have worked out two examples for N equal to 2 M equal to 3 and N equal to 2 M equal to 1.

But each time what we have to do, we first come up with this candidate, we calculate these quantities dell FI dell XJ and then actually verify the definition, verify definition that is we put normal H, E X not H equal to FX not plus H minus F X not minus DF X not H and show EX not H goes to 0 as H goes to 0. That is what we do, sorry did.

So each time we come up the candidate we have to check it. Now that’s a lot of work, you must come up with some criteria to check that also we calculate this we are able to say somehow that is differentiable or not instead of checking this because checking means writing down everything here, F of X not plus H, F of X not, D of X not H, and then actually checking this.

So today's lecture I will prove one criteria which is very-very useful and we actually use it every time to check differentiability of the function, from the, from looking at the candidate for Jacobian. How do you do it?

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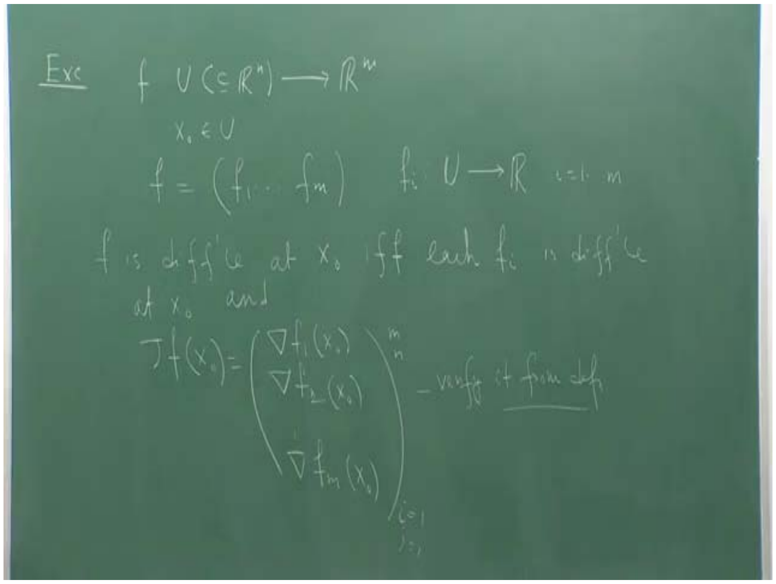

Well first I give you a very easy exercise and you must verify it from the definition and then I will write in the correct way. First exercise is this, suppose if have F, same setup, U from RN to RM, and X not is a point in U, I want to check the differentiability criteria at X not. Well I can write I know, F as M component function where each FI is from U to R, right.

Statement of the exercise says F is differentiable at X not if and only if each FI this U from R, now this real value function is differentiable at X not. And importantly Jacobian matrix of F at X not is precisely this matrix. This is a row vector having N component, each row has N component dell F1 dell X1, dell F1 dell X2 and so on. So this is a M cross N matrix, of course.

So if bother us to checking, so checking differentiability of F from RN to RM bothers us to checking component wise, real valued function FI they are differentiable and this is the relation. Okay, and you see the Jacobian matrix will be from this description, Jacobian matrix will be exactly this, the criteria is each FI is differentiable.

This is a very easy exercise. So you verify it from definition again. Once you verify you can use it everywhere. Now let me state the criteria for differentiability so I actually have to state the criteria for differentiability of real valued function. So let me write the statement clearly and then do the derivation.

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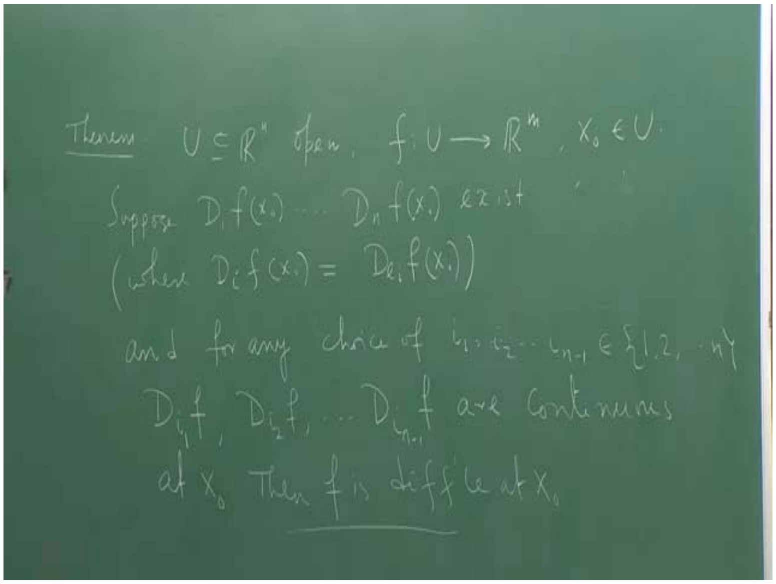

Let me put it as a theorem so U in RN, let me write it in theorem format, open I don't need connectedness here actually. Open, F from U to RM and X not is in U. Suppose D1 F X not, DN F X not exist, okay where DI F X not is a directional derivative irrespective to, let us say Canonical bases that is DEI F at X not, directional derivative is a direction EI. And okay, there are N many, they must exist first of all, if they do not exist there will be no question of F being differentiable.

And for any choice of I1, I2, IN minus 1 from 1 to N, so you have N many, you choose any N minus 1, and I have for this DI1F, DI2F, DIN minus 1F are continuous at X not. Then F is differentiable at X not, so let me explain it here suppose I have F from U to RN X not is a point, F I write this way, okay, fine.

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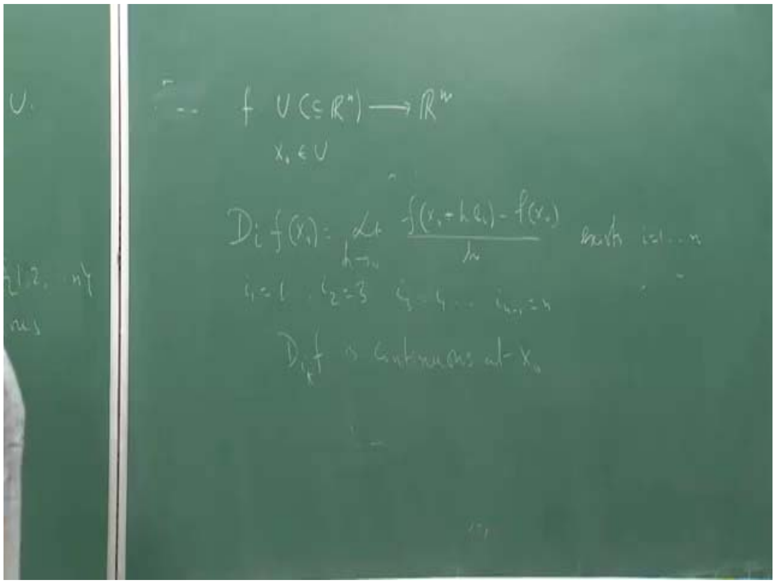

Now what I first need, what does the theorem say that if I look at DIF X not which is Limit H going to 0, F of X not plus H EI minus F of X not divide by H this exist from I equal to 1 to N and choose I1, I2, IN minus 1, any between 1 to N minus 1, so let us say I1 equal to 2, or say I1 equal to 1, I2 equal to 3, I3 equal to 4 and IN minus 1 equal to N or any other choice 2, 1, 3, 4, N or 2, 3, 4, N, or 1, 2, 3, N minus 1, any choice.

This is DIK F is continuous at X not then I can conclude F is differentiable. Once I have that then the verification becomes actually easy. This is no long exercise, this I am explaining here. But any way whatever you state as a theorem you need a proof, so for today's class let me proof this theorem. The proof is not that important but it is the prototype of the proof what one does in several variable calculus, so once you know it you can write down yourself but at least see to it once.

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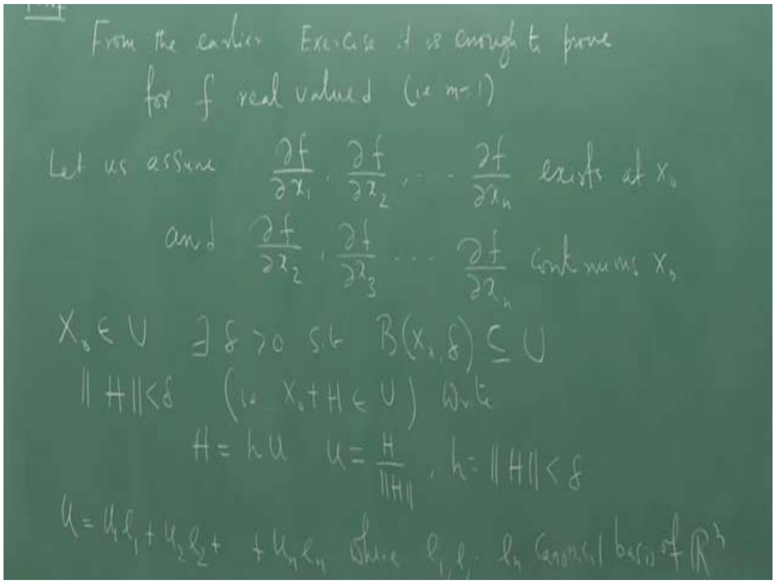

Let's us say proof, first remark, from the earlier exercise it is enough to as prove for F real valued that is M equal to 1. Because after all I can break down in components, say every statement is true component wise and each component is differentiable then the function is differentiable. So this is the first direction I made just to make the proof easy.

So let us assume F is real valued, I have only Grad to consider exists and any N minus 1 of them continuous, so I assume dell F dell X2, dell F dell X 3, dell F dell XN exists at X not, continuous at X not. You may say why I have chosen X2, X3, XN, it could be X1 X3 XN or X1 X2 X N minus 1, well it does not matter, because you see after EIs are coordinate, these are the directional derivative with EIs.

Which one I call U1 which one I call U2 doesn't matter, I can make a rotation and I may call E1 or E2 call E4, sow how you name the coordinates it does not matter. It means to say that okay you have X axis and Y axis like, horizontal axis X and vertical axis is Y, so you just do what, turn the paper 90 degree, the Y axis becomes horizontal and X axis become vertical, so you call X axis Y axis or Y axis X axis, it does not matter.

So I can choose without loss of ()(13:46) that X, dell X2 to dell F X1 continuous. Okay. Now X not is in U, U is open so there exists a delta such that if I take a ball around X not or radius delta this is completely U, correct. Okay, let's take any H such that norm of H is less than delta, that is to say H not plus H, this is also in BX not delta so is in U. So I have to be in domain everywhere to check because F is define only in U.

Of course I can write H equal to H into U where U is H by norm H, you need vector and H is just norm H which is less than detla, so U is a directional vector. And I can write U since I have chosen the basis E1, E2, EN, I can write U as U1E1, U2E2, UNEN where E1 E2 EN just on Canonical bases, this is the basis which respect to the Grad is calculated.

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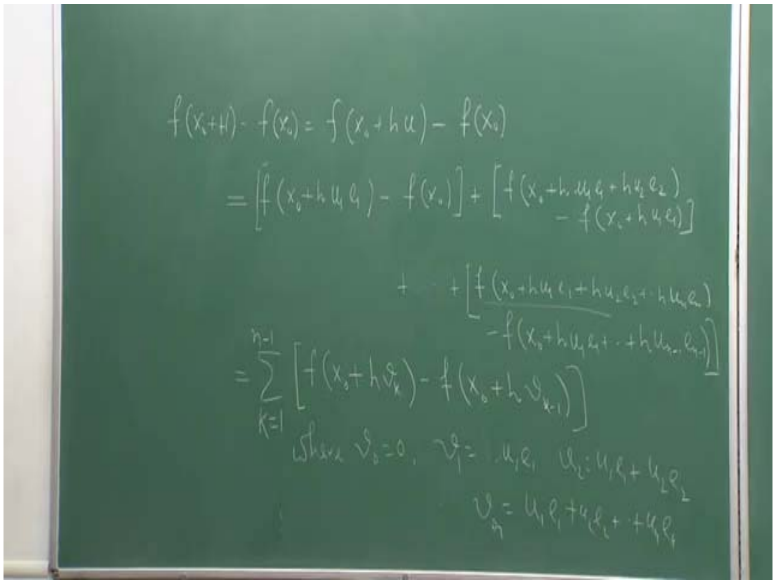

Let us continue on that board, I have to write why is F of X not plus H minus F of X not, this is actually F of X not plus HU minus F of X not, I write it little bit tricky way so that I can use my assumption. I write it in this way, F of X not plus H, U1E1 minus F of X not, okay, plus F of X not plus H, U1E1 plus H, U1E minus f of X not plus H U1E1, so on f of X not plus H U1E1, HU2E2 HUNEN minus f of x not plus H, U1E1 HUN minus 1 EN minus 1.

What I have done? See this term is extra but this will get cancelled here, this term is extra this will cancel in next term, so finally the term remains is this minus this, which is f of X not plus HU minus F of X not. So I have put inside certain terms in a way that in each of the component of this sum only one coordinate varies, here this direction is the direction derivative with, it will come direction derivative with respect to E1, here it will H of U1E1, only U2E2 comes, second derivative will come.

So I can write it in this form, let me write in a compact form, K equal to 0 to N minus 1, F of X not plus H VK plus 1, or K equal to 1 to VK, VK minus F of X not plus H VK minus 1 where VO is 0, V1 is H, not H, U1E1, V2 is U1E1 plus U2E2, so on VK is, VN is U1E1 U2E2 UNEN. Now I will use differentiability and continuity. I will assume that this exists and these things are continuous.

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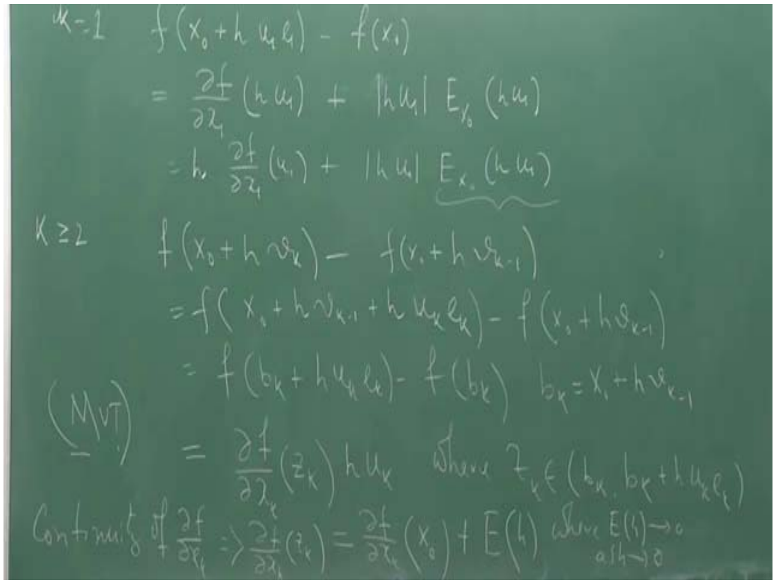

K equals to 0, what is the term, X not plus HU1E1 minus F of X not, which is dell f dell X1 at HU1 plus ()(20:16) H U1 E X not HU1, so this is really H into dell f dell X1 at U1 plus HU1 EX not HU1. This term goes to 0 as H goes to 0. No I should have K equal to 1.

Now for K greater than equal to 2, F of X not plus, HVK minus F of X not plus HVK minus 1, this is write F of X not plus H VK minus 1, plus H UKEK minus F of X not plus H VK minus 1, which is F of sum VK plus H UKEK minus F of BK where BK equals to X not plus H VK minus 1.

Now here I can apply, this is real value and I will prove it do not worry, it is simply Mean Value Theorem. I will prove Mean Value Theorem for functions of taking value in R or in RM but you can think of it as a ()(22:08) theorem because you are just using one variable here. This will become dell F dell X K, add sum ZK with the difference which is HUK.

Where ZK is in DK some number DK plus H UKEK, so this I am applying Mean Value Theorem. Now continuity will apply, I will tell you dell FK dell XK ZK is actually dell F dell XK at X not plus sum EH where EH goes to 0 as H goes to 0. Okay, so finally we put down everything and see what happens.

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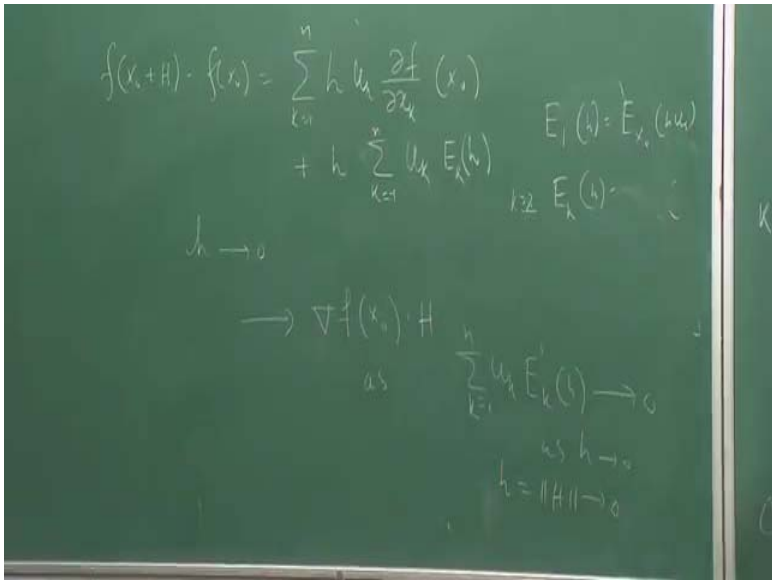

See you write it, so I just jump one step, you verify it yourself that I can actually write F of X not plus H minus F of X not equal to summation H UK dell F dell XK at X not, K equal to 1 to N plus H, U is norm 1, K equal to 1 to N, HK EH. Maybe I should write E prime H or E1H, so EK H, E1H is E X not HU1 and EK, K greater equal to 2, EKH is that, here K should have been added, EKH from that and this as H goes to 0, this goes to which as you see Grad F X not, this is UK sorry, this is UK dot H because the other term will go to 0.

So as EH summation, summation K equal to 1 to N, UK EKH goes to 0 as H goes to 0. H goes to 0, H goes to 0 is H is norm of H. So that is the proof for the theorem and we have a criteria.