Differential Calculus of Several Variables
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Module 02
Lecture No 05
Matrix of a linear transformation.

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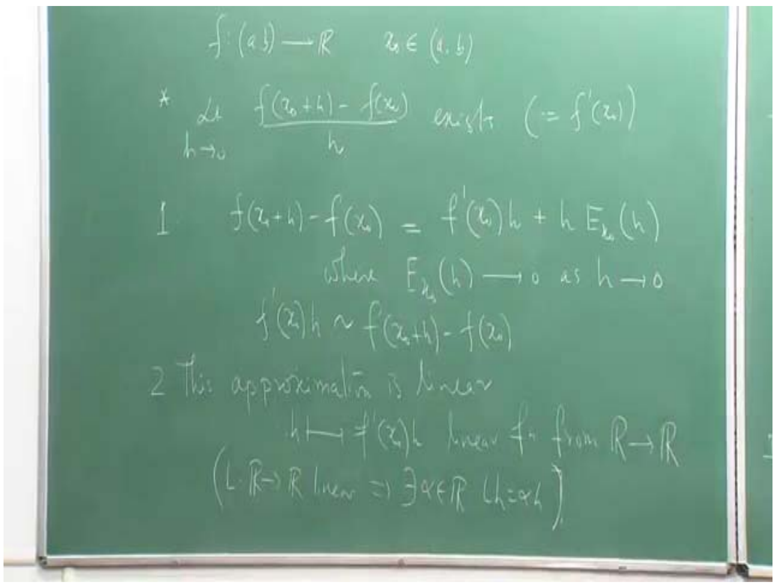

Okay. So let's continue with the fifth lecture. So as you have seen that this two property for function of one variable completely determines as written in other board, the derivative F prime X not that it is, it approximates the increment F X not plus H minus FX not and it approximate linearly. This leads us to a definition of differentiability of function of several variables and that's the correct definition. Now I will write the correct definition. Remember we have discarded the other two, first one completely discarded, second one with the reservation so we just see now what is the correct definition, here it is.

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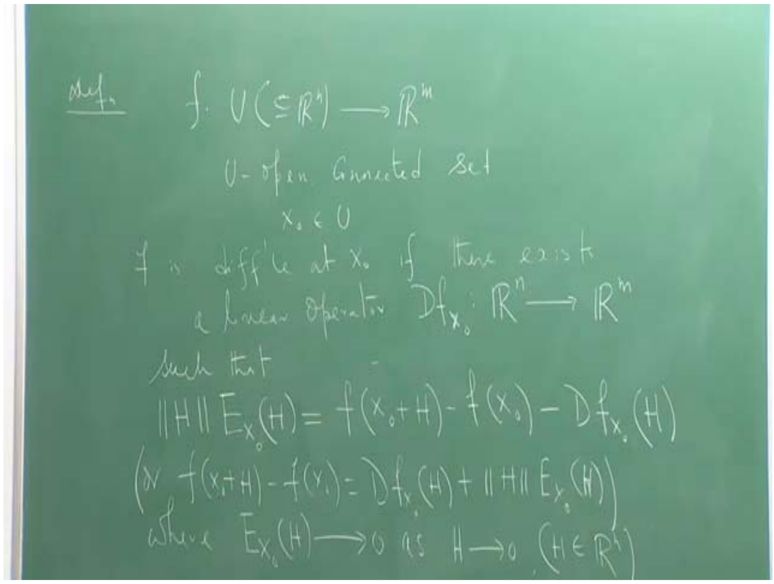

Here is my setup again and U as I said unless otherwise specified, open connected, connected is not that important but sometime it is, open connected set and I have X not in U. F is differentiable at X not if what will happen now, there exist a linear approximation of F for the increment F X not plus H minus F X not, if what will happen now, their exist a linear approximation of F for the increment F X not plus H minus F X not, that is if there exist a linear operator, let us denote is by DF at X not from RL to RM such that I follow the exact notion of, for single variable.

I write this (())(02:46) as F X not plus H minus FX not, the approximation is D FX not H or as you have written before same thing I am writing, D FX not at H where goes to 0 as H goes to 0. Remember H is in RN, right. H goes to 0 means norm of H goes to 0. So this is the exact analog of this property 1 and 2. So DF X not as a linear functional so X on this vector H and approximate FX not plus H up to an (())(04:05) which goes to 0, (())(04:09) norm H and (())(04:11) goes to 0 as H goes to 0. So this is the correct definition you will see right now for derivative of function of several variables and it exactly matches a function upon variable.

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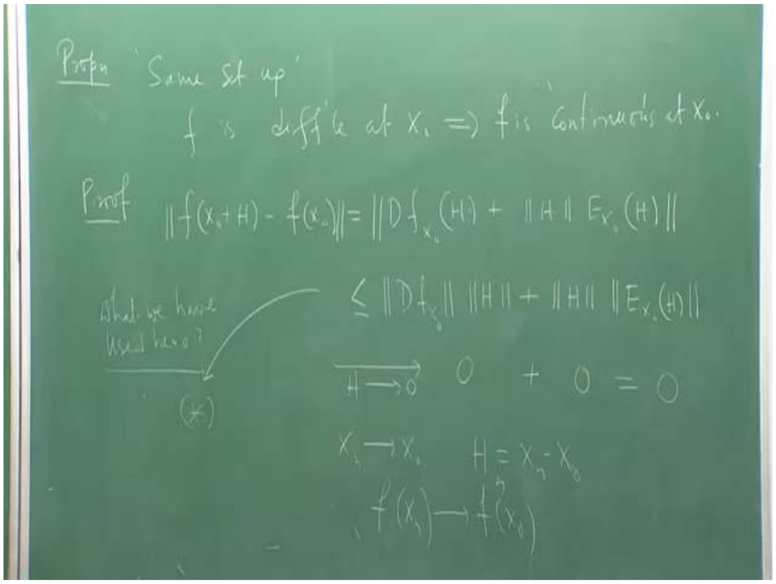

And remember we wanted our differential function to be continuous and we immediately have it. Same setup like that. F is differentiable at X not implies F is continuous at X not. How? It is too easy from the definition. Let me write it again in this form, correct. Now I take mod on both sides. (()(5:43) and equal to, mod (())(5:56) comes out, correct.

Now you put H goes to 0, H goes to 0, norm H goes to 0, so this quantity goes to 0 as well and this norm H goes to 0 and E X not H goes to 0, this norm goes to 0, so what do we get, continuity, because now if I have XN converges to X not, XN X not both are in the domain U then you just put the H to be XN minus X not, apply this you will see F of XN converges to F of X not.

In this equality from first line to second line I have used something here, so what we have used here, okay what we have used here is, let me explain on the other board. So definition is clear, look at the definition again. Look at this proof, module of this any (())(7:30) which I have written which I am going to explain now. So see DF X not is a linear operator from RN to RM, linear operator means all of you know, it takes X plus alpha Y to, so A of X plus alpha Y, AX plus AY, right. If from your linear algebra you know linear operators, along with the linear operators, comes the concept of operate (())(7:58) so what we have used here is this.

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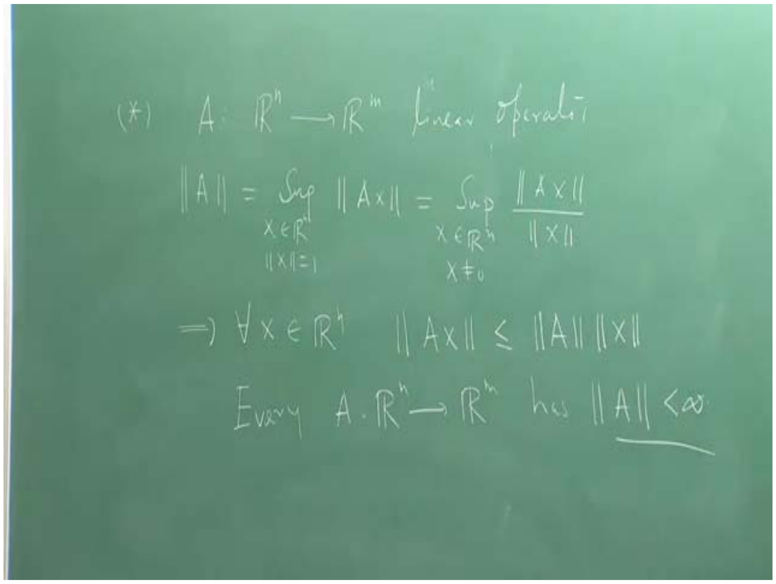

Suppose A, suppose RN to RM is a linear operator, correct. Then we have this notion of norm of A which is supremum of norm X, so X in RN norm X equal to 1, AX which is same as by dividing X in RN, just RN, X not equal to 0, AX by norm X.

So norm of A is a supremum of this quantity, so that will imply for all X in RN is the supremum, bigger than anything, AX is less than equal to norm A mod X, That is what we have used here, DF X not is a linear operator, acting on H, so norm of DF X not H is less than equal to norm of DF X not into norm of H and there is a little fact that you verify from linear algebra that 'Every A RN to RM as norm bounded'.

This is a fixed quantity, so we have it. So this continuity problem is taken care of. Okay, so this notion of derivative is really nice for us and actually what if you, what I said was one possible definition was to fix one direction and then claim that fix one direction, look at this limit along directions and then claim that for every direction it exist and every differentiable we have to discard it for some reason.

But that is sometime, actually we can recover that derivative from this definition so what I mean here, that we will call that second definition as directional derivative and let me show you how to recover the directional derivative from derivative.

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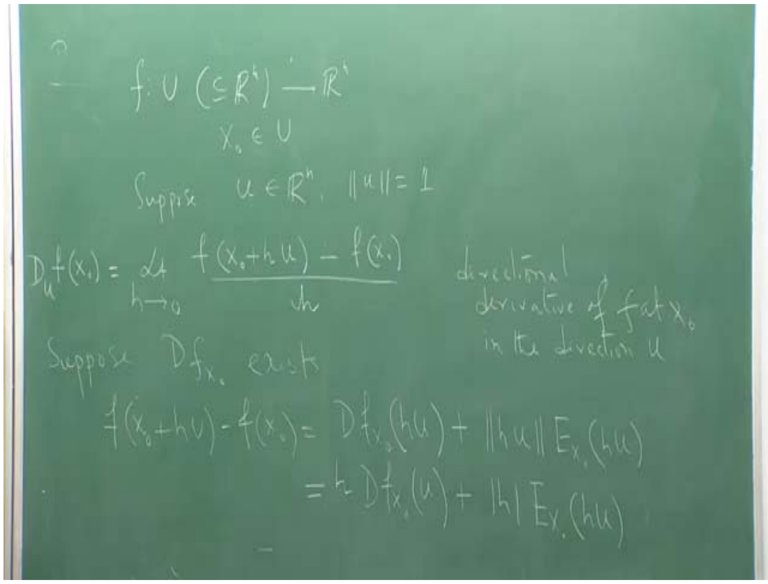

So again, so suppose U is a fixed direction, U in vector that gives a direction. What was our definition, second definition? Limit H goes to 0, F of X not plus HU minus F X not divided by H, okay. Let us give it a name, let us call DU FX not and call it directional derivative of F at X not in the direction U. Suppose DF X not exists that is F is differentiable in the origin of the definition we just made. Then what will happen from here I know from the definition HU minus F X not will be DF X not right acting at HU plus norm of HU EX not HU, correct.

This is equal to, this is linear operator H is a (13:00), HD F X not U plus H into mod of U, norm of U but norm of U is 1, so EX not HU, okay. So I get this.

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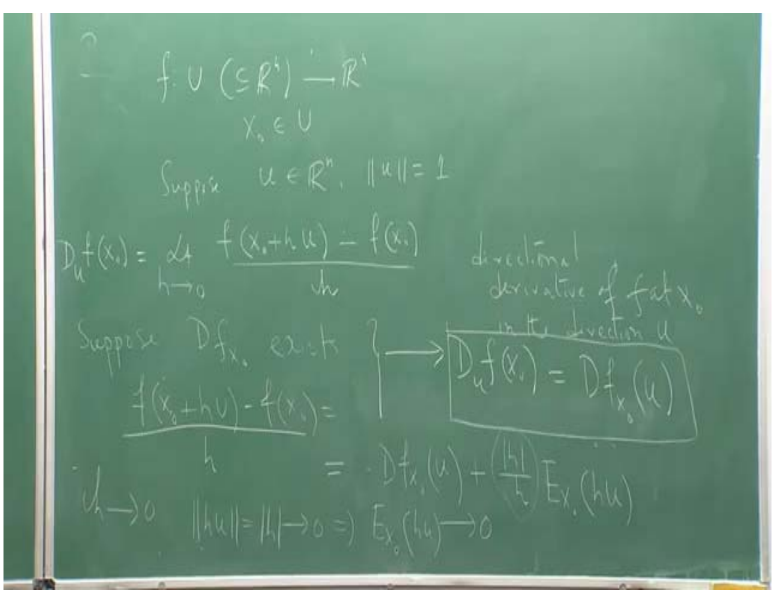

So I divide by H goes away, correct. Now you put X goes to 0, what happens, H goes to 0 implies norm of HU which is mod H, this does to 0 so EX not HU this goes to 0 according to definition of derivative and this quantity is bounded it is plus or minus 1, is bounded by plus 1 or minus 1 according to H, the sign of derivative so what will happen finally we get.

Okay, let me write on this board itself. The entire analysis leads to DUF at X not this is equal to DF X not acting at U and we will use it to actually calculate the derivative of function. Remember this what is there in the box, I will show you in today's class how to actually compute this fellow DF X not with examples, may be next class but we have to use this idea.

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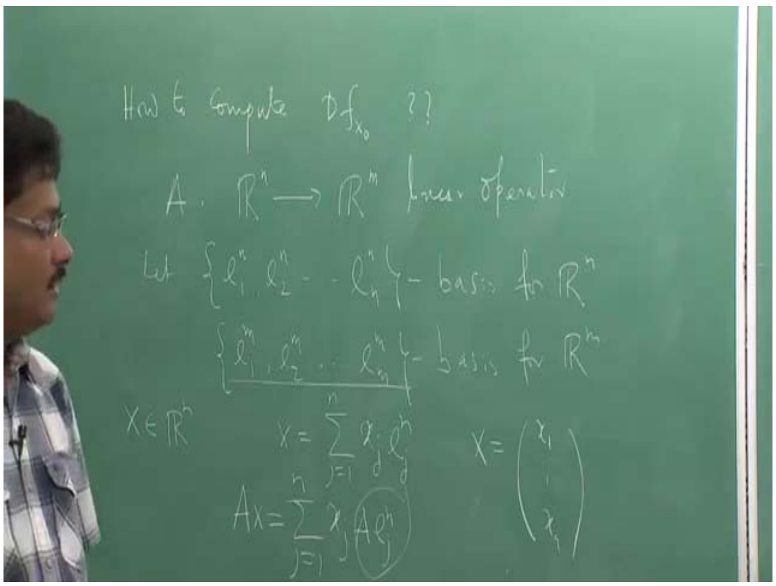

So now let us come to this point. How to compute DF X not, given an F? We have to recall a bit of linear algebra here. How to realize a linear operator? This question is related to how to realize a linear operator? Suppose A is a linear operator from RN to RM. A linear operator to determine a linear operator actually depends or it is completely determined how do you fix basis for RN and RM.

So I will show you or else all of you know it from your linear algebra course but still let me do it, it is very important for us. So let E1N, E2N, ENN, this is the basis for RN. You can take Canonical basis, 011, 100, sorry 100, 011, so all of you know the Canonical basis but it doesn't matter. You take any basis. I will give some example in assignment. And E1N, E2N, and ENN this is the basis for your targets space RM.

Okay, let us take in any X in RN, this fellow being a basis I can write X as summation J equal to 1 to N XJ EJ N, correct. Very good. What is AX?AX is linear operator, so on the sum it goes inside, it goes J equal to 1 to N, XJ, A EJ N. This is by linearity. Now AEJ is where, A is from RN to RM, so AEJ is a fellow in vector in RM. So, now I have a basis for RN.

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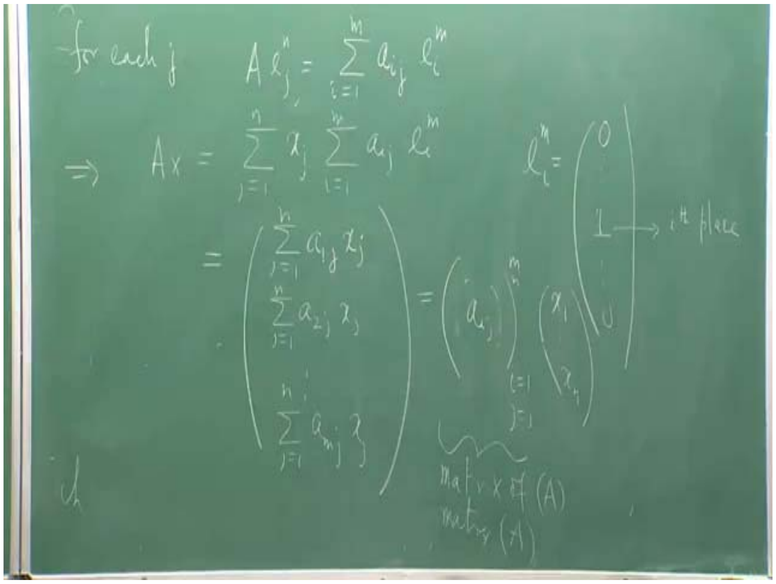

So I can write in terms of the basis or RN, so for each J I can write EAJ N as summation I equal to 1 to M, sum AIJ, J is fixed. I will vary from 1 to M. EIM, correct. I have written each of this fellow in terms of these basis. So what happens to AX, okay. so this you see you can write suppose I fixed my basis EIM equal to 000 at 1 place and 0 that is Canonical basis at I place. You see you can write it as a vector summation J equal to 1 to N, AI1, XJA, sorry A1, XJ, J equal to 1 to N, A to J, XJ and so on, summation J equal to 1 to N, AMJ, XJ.

Simple exercise you can do, but you immediately recognize what is this, this is if you look at this matrix AIJ, which is matrix AIJ I equal to 1 to M, J equal to 1 to N, acting on the vector, X1, X2, XN. So what is says that if X, in the Canonical basis X1, X2, XN and AX equal to this fellow and this, this matrix acting on this and this is called matrix of A. Let us write this as matrix A. So there is a way you calculate matrix o f a linear operator. Let me show you one example and end today's lecture.

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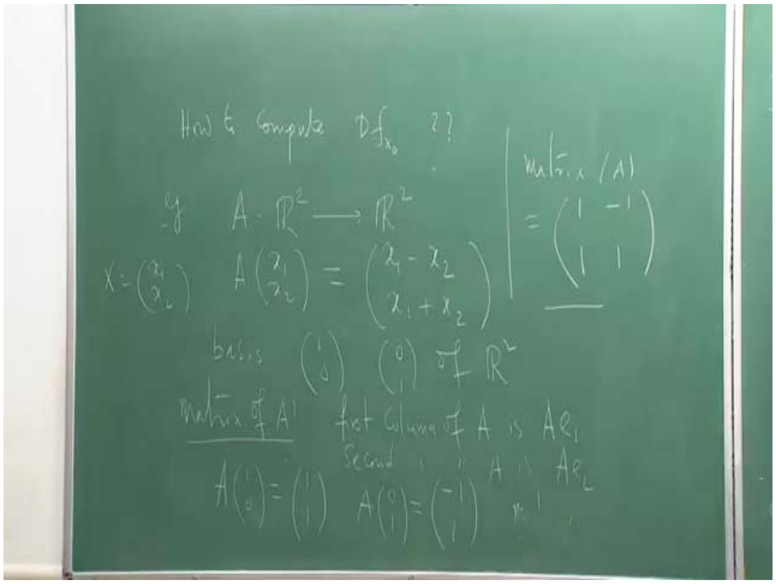

So 1 example, let us take this operator, A from R2 to R2, A of X1 X2, X is a vector in R2 given by X1 X2, let us say X1 minus X2, X1 plus X2. (21:48) basis 1001 or R2 both side, or both are R2, I say this side also this basis, this side also this basis, what is matrix of A. From here you not, from this calculation you note, that first column of A is AE1 and second column of A is AE2. Just look at this calculation you will get it.

So what is AE1, 1011, what is AE2 minus 1 and 1, so matrix (let me write here) matrix A equal to first column is 11 and second column is 1 minus 1. This is one example. Next day we will do this exercise of D of FX not.

Thank you!