Differential Calculus of Several Variables
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Lecture Number21
Module Number 4
Application of IFT: Inverse Function Theorem

Okay, welcome to the last lecture of this course. So today we will complete the proof of Inverse Function Theorem. Now, just let us just recall what we have already seen for a function of one variable. So we had F from interval AB to R, and we assume that F time is continuous, and at some point X not in AB, we had F prime X not, not equal to zero, so from that what we did actually, that we can actually find smaller interval A not B not.

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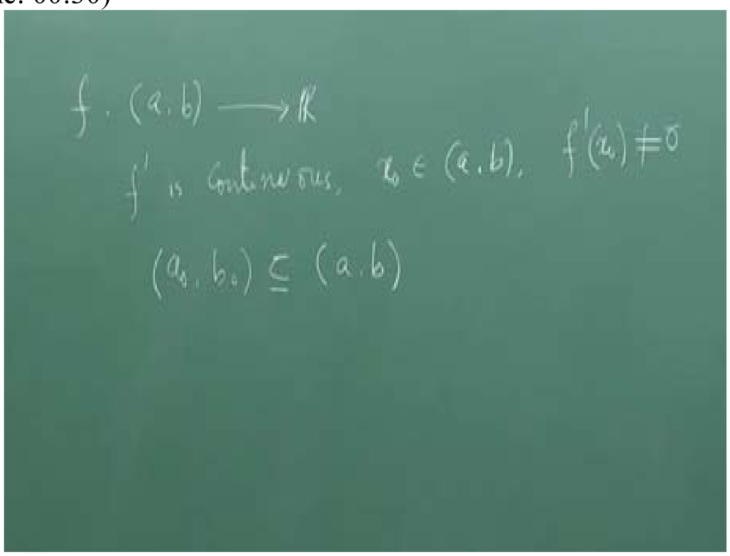

We proved all these things in the last lecture. AB where, F is actually one one and F of A not B not is open. So whatever we proved actually boils down to this thing. If you look at carefully what we did, that we choose some interval where F prime is non-zero, this interval was chosen in such a way that F prime is non-zero, in the entire interval and F prime (a) and F of A not and F of B not are both are defined from F of X not, then we actually saw that one can find given given any Y in F of X not minus M by two.

If (els) if you look at the previous lecture plus M by two, where M was F of A not minus F of X not, which we assume without loss of generality one of them will be (an) call that M, and if we took that then for each Y there, there exists Z at A not B not, such that F of Z equal to Y not, Z not maybe, in the open interval. It's not one of the end points, it means it will work to, (I mean it) just if you look at the proof, you actually see that this Z not will be unique.

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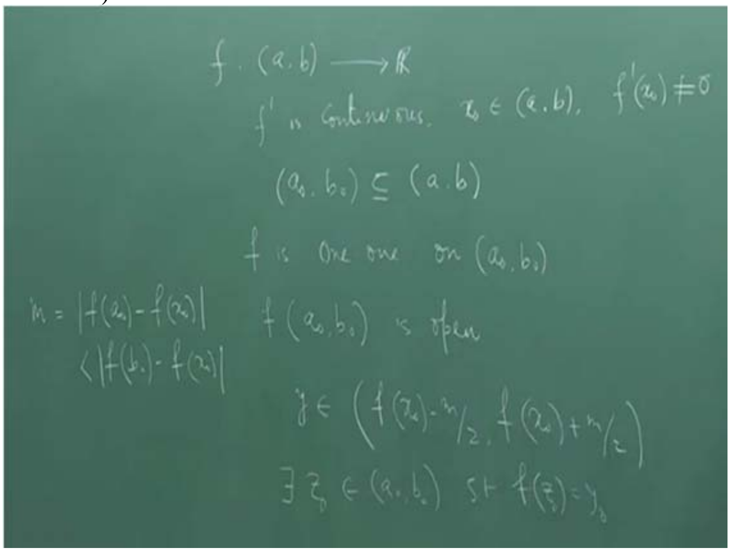

Now, this was the case for F from an interval to getting values in R. Now suppose I have this situation; and F is from U to RN. So instead of F prime continuous I will assume F has all partial derivatives; Del FI Del XJ continuous on U.To call that we can always write F as F one to F N and I have point X not in U where I assume that instead of F prime X not non-zero, exact, I will log off that as the Jacobean has non-zero determinant, that is Jacobean matrix is invertible.

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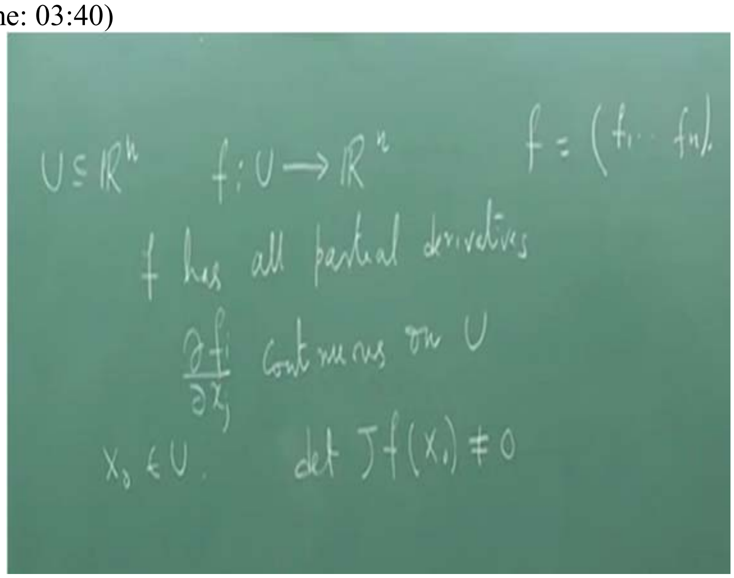

Then, so, this is the same setup and here it was conclusion one two; then there exists a smaller interval. U where F is one one and in that interval, image of that interval is open. So whatever is there, I wrote the same thing here (in) for the case of function from RN to RNand proof here, I write exactly same as in the case of A from AB to R that is whatever we had yesterday or whatever the summary is written here.

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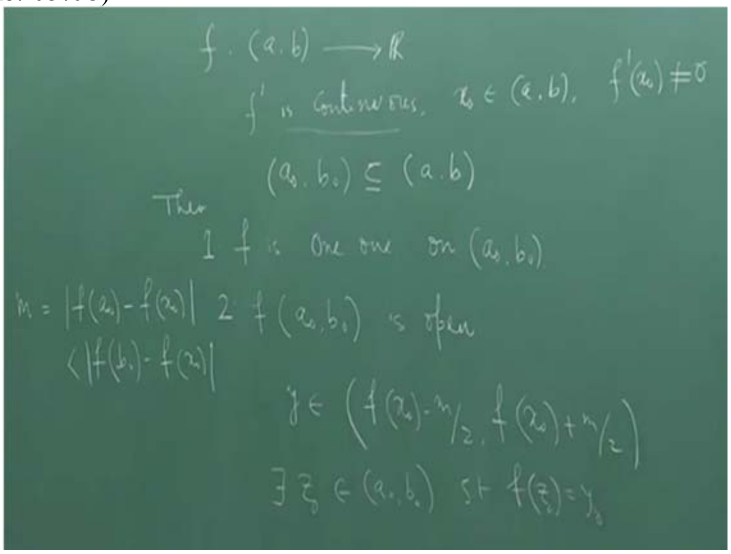

In this case except plus, we needed to find this interval A not B not, so here, and that interval you can find because of continuity of these partial derivatives and where we (fin) define this function you recall HX equal to here after choosing Y we define the function HX equal to FX minus Y.

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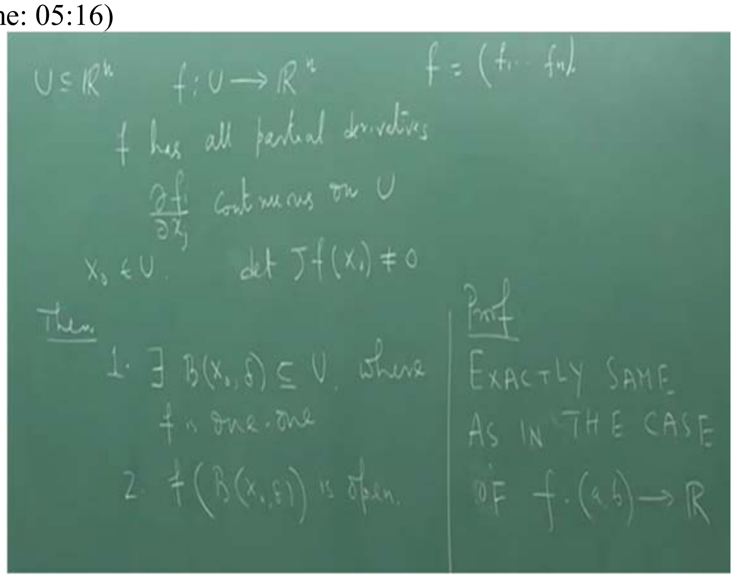

Here you have to observe you will define the same thing, but in this case HX has to be (ep) replaced by mod FX minus Y, Y in this interval and you have to observe this this one and another observation that is determinant JFX is continuous.

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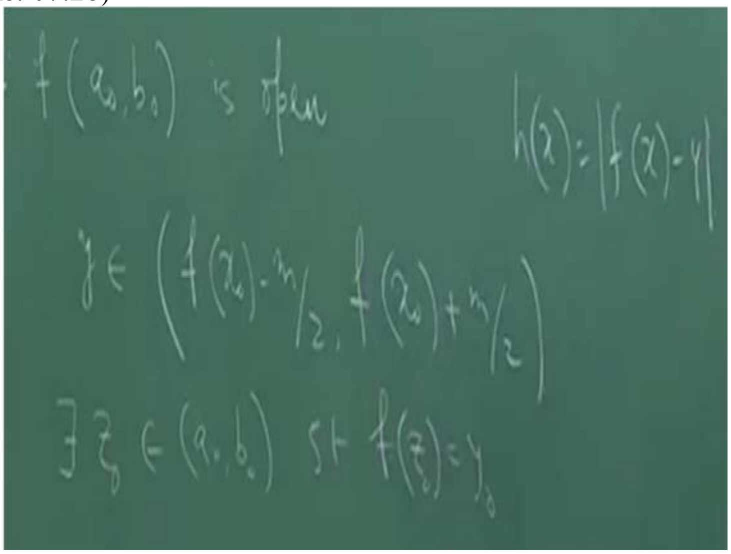

If I give this condition that (they) this Del XJ continuous determinant is also a continuous function and the proof follows exactly same pattern. So let us now, with this observation let us now prove inverse function theorem.Okay, so let me state inverse function theorem first and then prove it. So this proof, this analog, I leave it as in the exercise.

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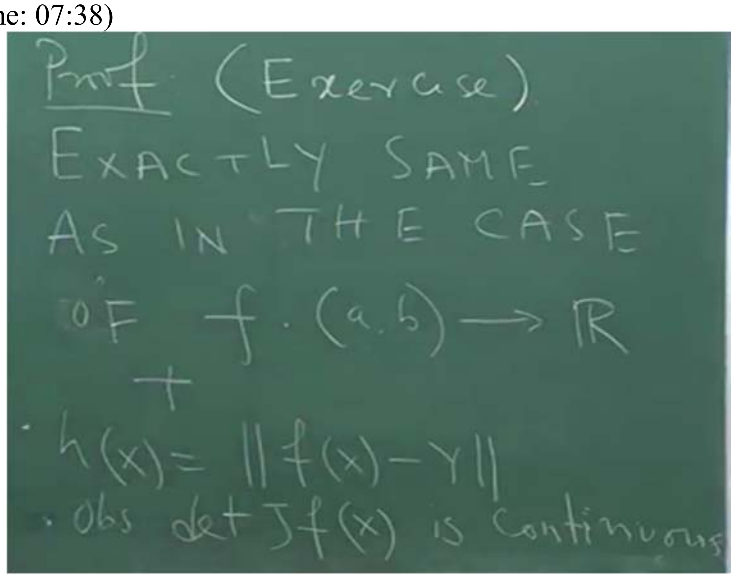

Okay, here goes the statement of inverse function theorem, so let me follow the standard statement (on some) so this statement is, so, while writing the statement of the theorem it’s always better to follow a standard book. In this case I am following the statement from (()) () Mathematical Analysis with the notation on changes maybe.

So, Let, me start with U in RN and F from U to RN, be continuously differentiable, be such that all partial derivatives are continuous. So let, X not is a point in U and determinant of JFX not is not equal to zero, then there exists an open wall or it can be said open set, I can make, let's say theta one inside U and open set theta two in AQ.

X not is in theta one, F of X not is in theta two, and of unique G from theta two to theta one such that G is a inverse that is FGY equal to Y for all Y in theta two and GFX is equal to X for all X in theta one and since I have taken F to be C one I also get G is differentiable, and this fellow is simply JF at GY this becomes Jacob matrix inverse.

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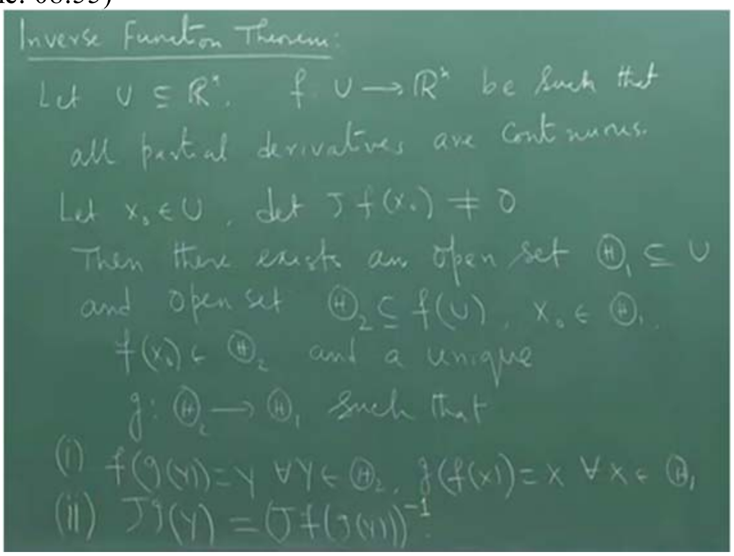

See if you, this is true for any end so if you will look for interval it will look like, so suppose if we define interval to R N is one, and F (eh) F prime is continuous with some X not F prime at X not not equal to zero. Then there exists an open interval inside that interval AB, smaller interval and an open interval in F of AB such that I can write FGY equal to Y and this thing, and so (edge) I can define an inverse and G prime of Y equal to F at GY inverse, that is, one upon F at GY.

So you should be able to interpret it as when N equal to one case. Okay, I will prove this and the proof we will use as I said before implicit function theorem and because of that, as the name suggests Implicit function theorem, actually if I give you an F you actually get that G. This may not be easy, because one may not actually, whatever uses implicit function theorem to get hold of this particular function may not be easy or may not be possible even for some F.

I mean given an expression of F you may not be able to write down the expression for G. So it’s the existent theorem, only, and that is true for anything an implicit function theorem but with the (hel), no, knowledge of existence we can do lot of things when you go ahead particularly when you do Geometry this existence of inverse function and implicit function, that is a great help.

So maybe if you do a course on geometry later on, differential geometry you will see, how it is used left and right. Anyways, remember the statement here, I just erased, so this condition is same, conditions are same here and remember these two observations that I willhave these two, Okay?

So I will use these two observations written on this board, so let’s start the proof. Okay let us get rid of this U fellow, and U theta I mean I mean it may not be one one on enter U but here if we do one one in this thing. So what I (wil) from the previous observation, from previous observation, maybe by shrinking U we may assume or we assume F is one one on U.
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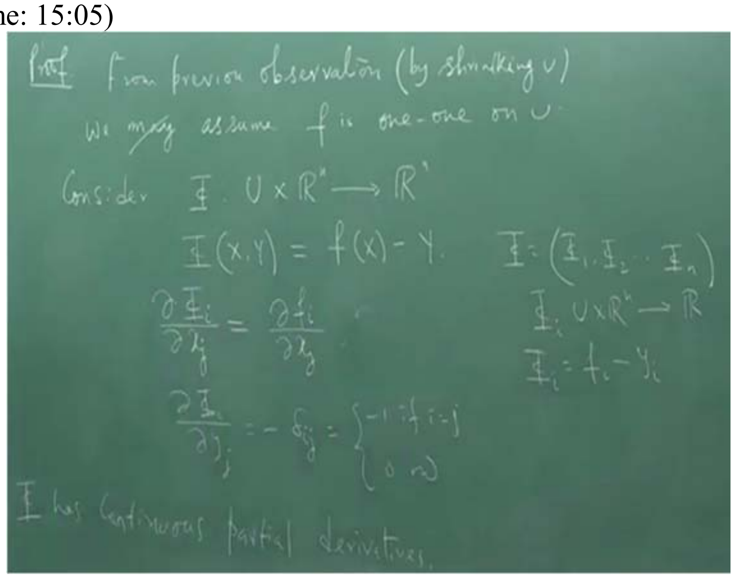

Otherwise go to some ball around X not because after we look other statement, so go to some ball inside U around X not where F is one one. Now consider this function, just a capital phi, from U cross RN to RN phi XY equal to FX minus Y, okay? So what is del phi-i phi J del XJ so phi I can write, phi one phi two phi N, each phi-i from U cross RN to R.

So then I have X variable this is this side FX minus Y, so this is precisely del FY minus so look at that this phi-I will be a F I minus phi-I, so this will be del FY del XJ. And if I have delphi-i, if you look at del phi-i del YJ, this will be simply minus one minus delta IJ, right? Delta IJmatlab(( )) (18.01) that is minus one if i equal to one, I equal to J, zero otherwise.

So phi has continuous partial derivatives, correct? Now, see I have actually come to the setup of implicit function theorem. Phase from U cross RN to RN, del phi-i del FJ del Fi is del FXJ, therefore, given this assumption, what I have? The determinant of J at FX not is non-zero. I get this matrix, I equal to one to N, J equal to one to N, determinant of this fellow is not zero.

So I have two components of phi, U and RN, on the U component this del phi del XJ is same as the determinant of del FI del XJ because this is actually, this matrix is precisely (eh) Jacobean matrix. Okay? At a point, so this is at the point X not, FX not. Okay, now I can use implicit function theorem, which will say that, there exists theta two in RN and G from theta two to U, or you take, G of theta (one) two, which is theta one, such that, what happens?

That the, first component is U, second component is this thing, so phi of GYY is equal to FGY minus Y, which is actually zero, okay, so that gives , FGY Y equal to sorry, this will be here, this GY equal to Y. How did I use implicit function theorem?See.
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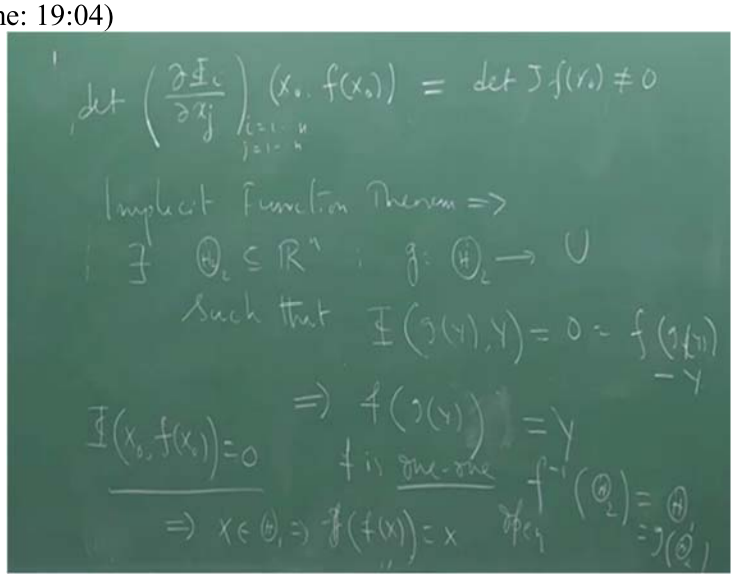

I had phi of X not FX not, this is zero, right? So I am applying to this thing, phi of X not if it is not zero then there exists a theta two where FGY Y will be equal to zero, but FGYY is this fellow, by definition FGYY, maybe I should have written it. This is equal to zero, equal to by definition FGY minus Y. So that gives you FGY equal to Y, and that is the inverse of F.

And here actually if you do not like this let it be in U and since F is one one, this F inverse of theta one theta two, call it theta one, this (is actual) will be zero theta two. So that is your existence of theta one, so this is an open set. If theta two is open, F is continuous if theta two is open, so, and being (o) open and one one, see if it is one one this also gives, what, X in theta one implies G of FX equal to X. This is by one oneness(())(23.23) again, okay?

So that gives the existence of G which is actually the inverse of F, and what about the derivative? It is very easy, you see GFX is X, so if I apply chain rule, then what do we get? That, so apply chain rule, to take derivative that will give you JG at FX compose with JFX is the (deter) Jacobean determinant of the function X going to X which is RNDT. Function X going to X, if I call this function H, then JH of X is simply identity, right?

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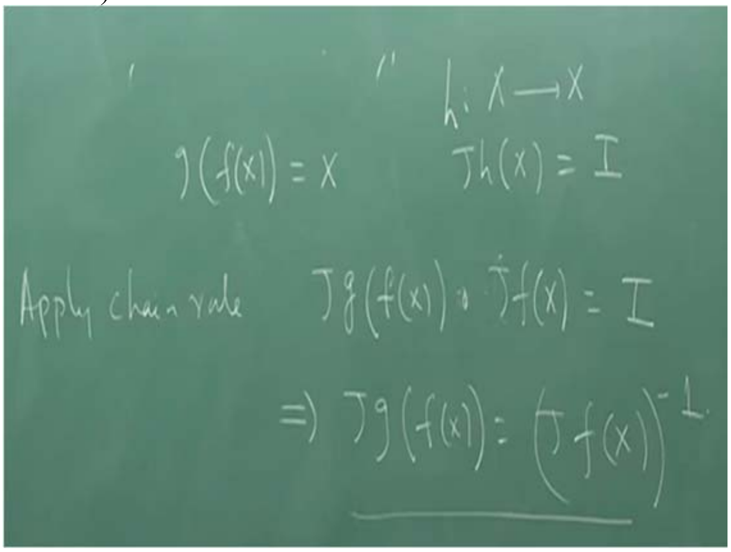

So that shows, that JG at FX which is Y, equal to JFX inverse and that is the proof of inverse function theorem which directly uses implicit function theorem here. So (these this this) again you see this G the derivative we get, sorry, inverse we get it comes through implicit function theorem. So that's what I said, that its very may be difficult to calculate given an F, becauseimplicit function theorem gives existence only, it doesn't give construction.

So inverse function theorem also gives existence only, doesn't give construction of G, but still it is very useful. So that will be the end of the course.

From here you cango to two teachings, one is, you can go to the Multivariate Integral Calculus since you now know (de to de) Jacobean (determ) determinant everything you can go MultivariateIntegral Calculus or you can go to Differential Geometry where you need less integral calculus it may need very less you actually need the whatever background we have done here in this course. So some elementary course on differential geometry you can take up from (the) here. Okay, that's it, thank you everybody.