Differential Calculus of Several Variables
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Module 1
Lecture No 2
Continuity and Compactness.

Welcome you all. In this lecture we look deeper into properties of continuous functions. Now to understand anything in mathematics is always better to start with a counter example that we start with a discontinuous function. So let’s start with an example of a discontinuous function.

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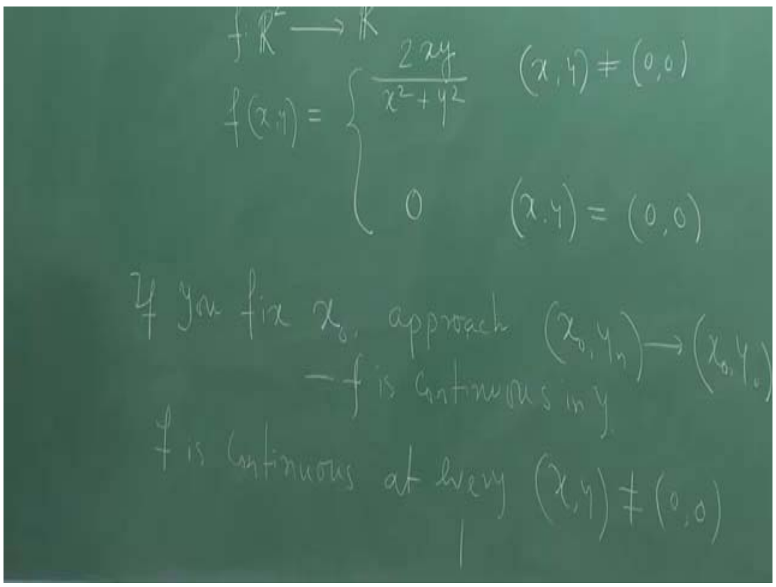

A function defines from R2 to R, very simple one. Let us say, is if you work out this problem you will say there is nothing special about this 2XY, and XY both non zero, not the origin and at the origin I define it to be 0. This is discontinuous, but very interestingly discontinuous. In the sense that if you fix X, take some X and approach, fix some X not, and approach X not Y not only through Y, even 0 0 or even this point X not or any point here, X not Y not, you approach only through Y that is this way or this way, this is continuous.

Similarly if you fix some Y not and approach through X this will be continuous, I don’t write but you can check. Any other point except 0 0, so you must verify everything here when you sit down with your notes you must verify all these statements, F is continuous at every XY not equal to 0 0. Problem point is a 0 0, how? Well, what does continuity mean? If I approach continuity at 0 0 mean, if I approach 0 0 then value of F must converse to value of F at 0 0 which is 0. But I may allow to approach through any path.

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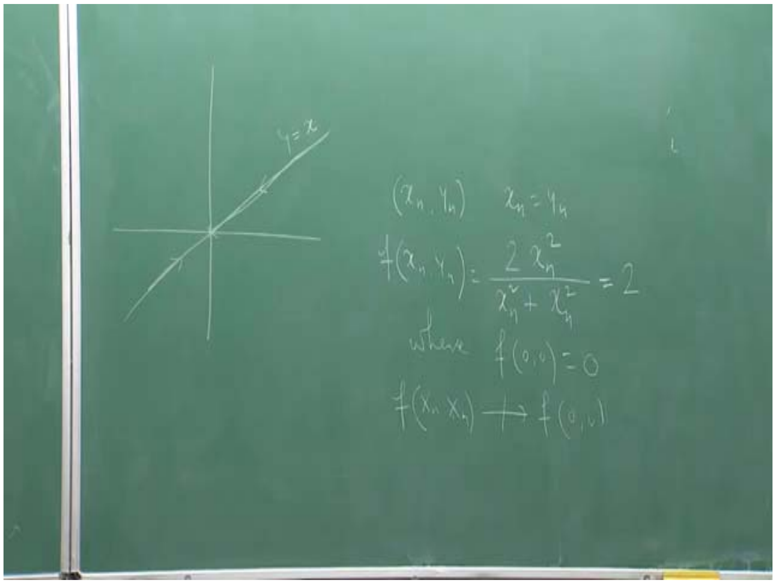

So let's me approach through the path. So this is the point 0 0, let me approach through the path the line Y equal to X. What happens? If I have XN YN and XN equal to YN and F of XN YN equal to 2 XN square divide by XN square plus YN square which is again XN square, so which is 2. From the entire line except at 0 0 the value of F is 2, where F of 0 0 is 0. So F of XN YN does not approach F of 0 0 or F of XN XN, let us say does not approach F of 0 0.

So while our notion of discontinuity so it is a discontinuous function, but very interesting way, right. If you fix X is continuous in Y, if you fix Y is continuous in X, is continuous in every other point but along this path you can find many other path. I have just given a simple example, it is not continuous. So remember while we talk about continuity, a function of several variables we should be able to approach to any path and reach at the desired value of F.

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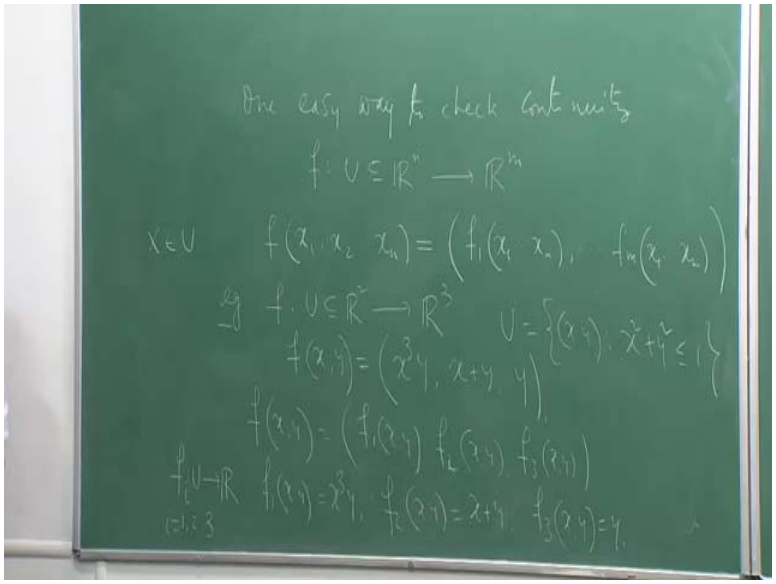

Now if I have a function from several variable, one easy way to check, so (5:26) this example. To check continuity of F from U let us say in RN to RM, okay. F, so X is in U, X has N coordinates, F of X1 X2 XN, this is a point in RM that means it has M coordinates, those M coordinates I write as F1, these coordinates depends on the value X1 X2 XN and I write in this way.

For example, let us say F from U in R2 to R3, let us say U is this region, I mean arbitrarily I am taking. F of XY equal to, let us say X's cube Y, X plus Y and simply Y then F is actually F1XY F2XY F3XY where F1 is from U to R, F1XY is equal to X's cube Y, F2XY, so each FY, I equal to 1, 2, 3 is from U to R, F1XY X cube Y, F2XY is X plus Y and F3XY is Y.

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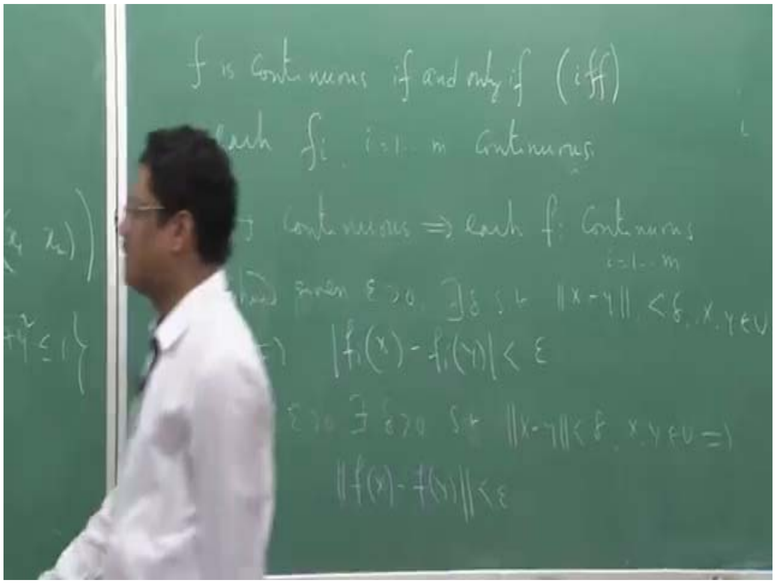

So I can write the function F which takes value in RM as component wise M fucntion each taking value in R and the easy way to check continuity, is F is continuous if and only F, this is abbreviated as (iff) this means this sentence if and only if, this phrase 'if and only if', so I will use this onwards, from now onwards. If and only if each FI, I equal to 1 to M, continuous. It is really very easy to prove but I want to write down the proof.

Because just to show you, I will write it very quickly but I want to write down the proof, just to show you how to write a formal proof in calculus, so sometime I will avoid proofs in this course, not avoid proofs, I mean write it very informally, but here I want to just write down the formal proof. Will illustrate you how to write a proof? Writing a proof is very important in mathematics so let us write this proof. So I have if and only, so I have two statements, so first I have to proof F continuous imply each FI continuous, okay.

Let us go by the definition, what I have to show for each FI continuous, for each I given epsilon their existed delta such that, X minus Y less than delta implies FIX minus FIY is less than epsilon, okay. Now so that is to show given epsilon greater than 0, their exist delta such that X minus Y, XY in U less than delta XY in U implies FI, FI is real value so just FIX minus FIY is less than epsilon. Now let us use the hypothesis F is continuous, now given epsilon greater than 0, their exist delta greater than 0 such that X minus Y less than delta, of course XY in U implies FX minus FY is less than epsilon.

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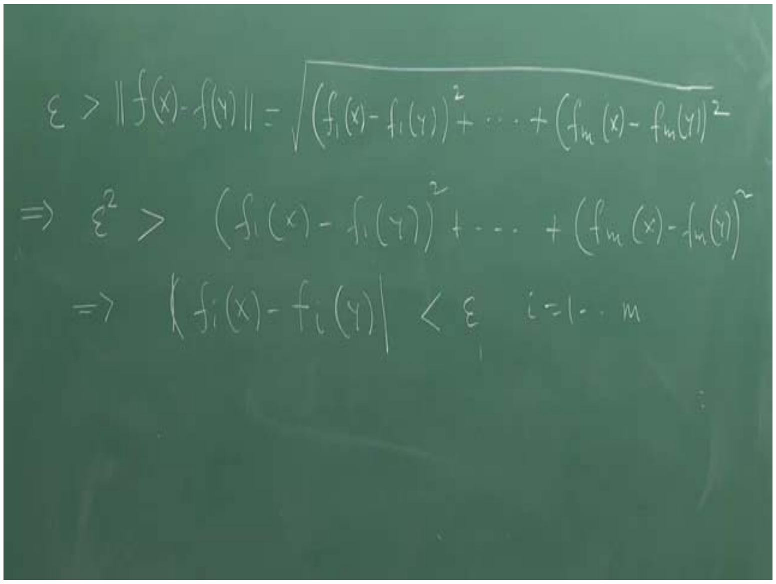

Now let us see the definition of FX minus FY, what is FX minus FY, this norm, so this is less than epsilon. So, okay, so sum of squares of this number, sum of positive numbers is less than epsilon square, so each of them has to be less than epsilon and I got it. Given epsilon I got the same delta who works for it, works for each FI. So this is one way, F continuous each FI is continuous. Otherwise so this is very easy but I write it because this is the way to write formal proof.

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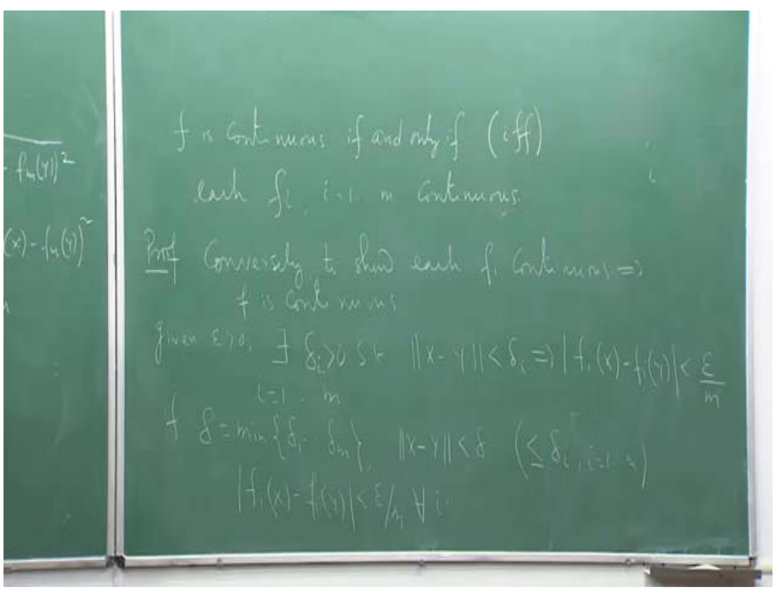

Now I have to write the converse. Converse I will tell you and you will write for yourself. Conversely to show each FI continuous implies F is continuous. Now what is Idea, again given epsilon greater than 0 there exist a delta I because I am looking at FI such that X minus Y less than delta I implies FIX minus FIY is less than epsilon I want at the final so let's put is epsilon YM because there are M many.

So this is for true for each I, I equal to 1 to M, point here is this delta I depends on I, FI. So for each I delta I may be different, but this delta I are all positive number, M many positive number. Now if delta equal to minimum of delta 1, delta 2, delta M then X minus Y less than delta, then is particular it is less than all delta I, less than equal to all delta I for all I because it is a minimum. Hence FI of X minus FI of Y is less than epsilon by M for all I.

Now you calculate this, each of them less than epsilon by M, so epsilon square by M square, this will come M many are there, actually instead of M I will get more, I will actually get here it is less than epsilon by M square so M will come out so epsilon by M and this (())(15:55) epsilon and had I started with root M then I could get rid of this M.

Look at this proof again because you have the facility to run the video back and try to understand that what I am trying to emphasize that delta is depends on FI but there are finitely meaning, so I take minimum and minimum was for each I. There is the way to write a formal proof. I will not do such a thing in future, okay let us go ahead.

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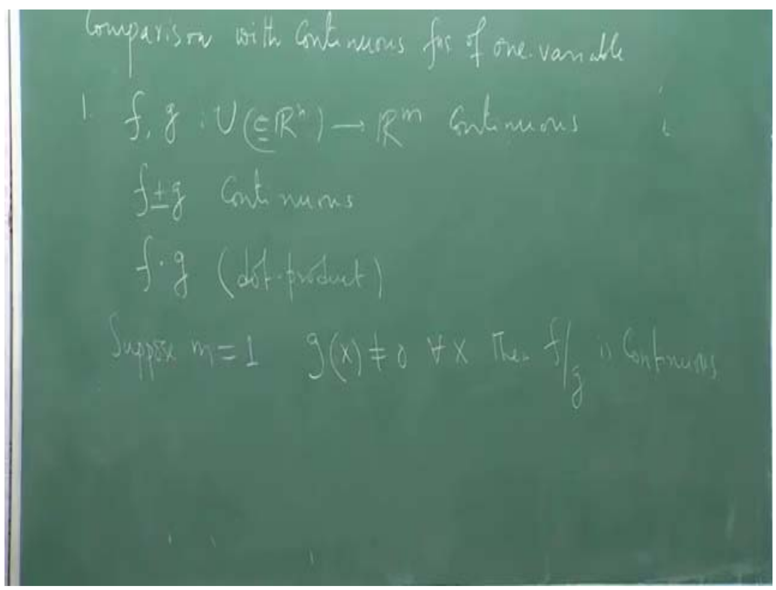

Now next is let say comparison with continuous function one variable. What are the properties which goes through? First F and G continuous, F and G both from U in RN to RM continuous. What happens in case of function of one variable, F plus G continuous, same proof will go through, okay, F plus minus G. F dot G, for function of real variable I can multiply to continuous function to get a continuous function but here F dot G, here it means dot product.

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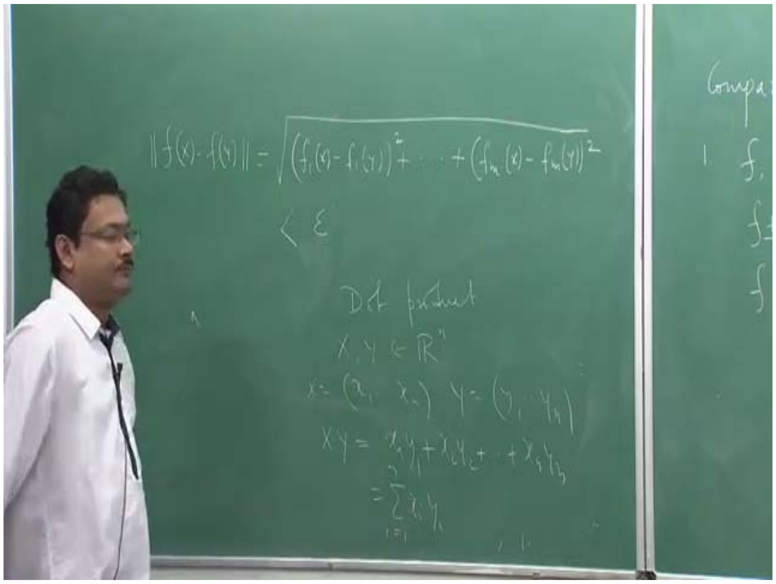

What is dot product? Here is dot product, I hope you know. X and Y in RN, let’s say X equal to X1 to XN, Y equal to Y1 to YN, then X dot Y is dot product which is geometrically the angle between, cos of the angle between X and Y, this is, all it is written.

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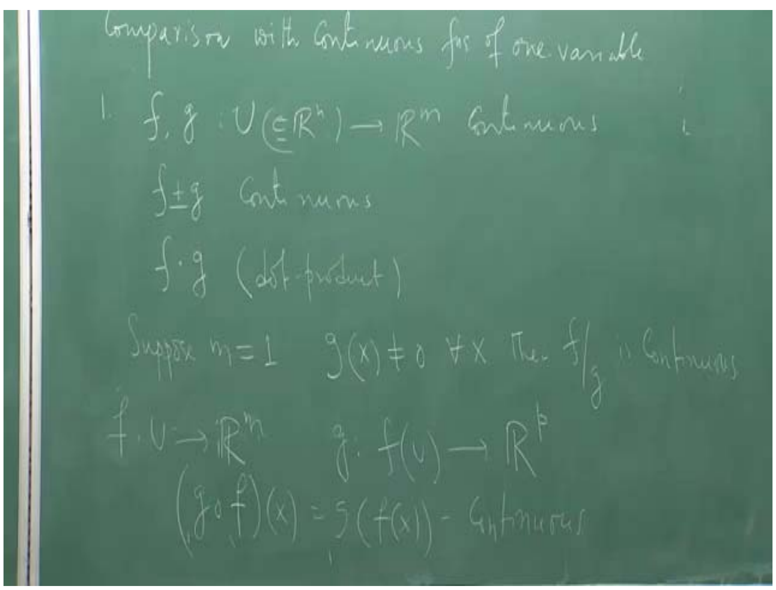

Here by G doesn’t make sense as in the case of but real value it may make sense. That is suppose M equal to 1 and G X not equal to 0 for all X, then F by G is continuous; it makes sense now because they are real values, okay?

And suppose F is from U to RM, sum set RM and G is from FU to sum RP then G compose F which is by definition is G of FX, this is also continuous. So these are called typical routine exercise which you must solve. But more interesting properties of continuous function in R, more interesting property is this.

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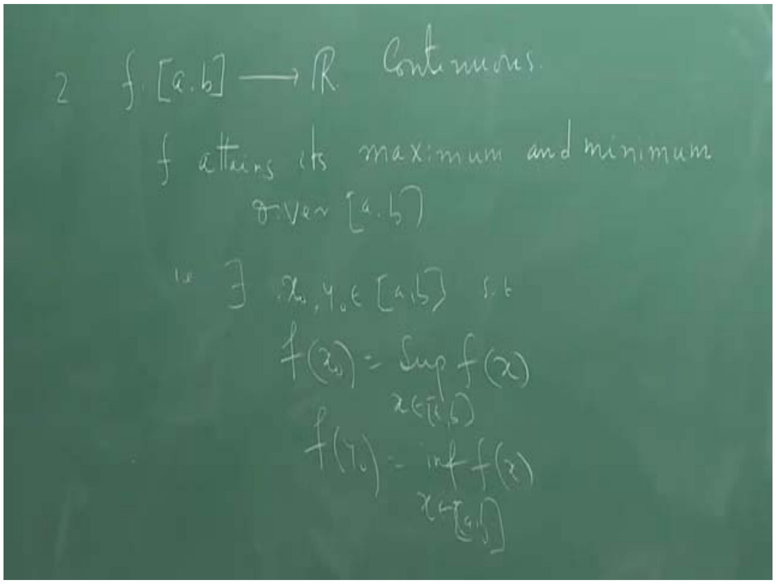

Suppose F is defined on a close interval A B, continuous then certain interesting thing happens. First thing you are going to, address is that F attains its maximum and minimum over A B i.e. there exist Y not Y not in A B such that F of X not is supremum over FX X in A B and F of Y not is equal to infimum over FX X in AB. Why such a thing happens? And is it possible to lift this property for fucntion of several variables?

Such a thing happens because of one very fundamental and very easy result in calculus, one variable calculus in fact this is a result in several variables calculus Bolzano Weierstrass Theorem.

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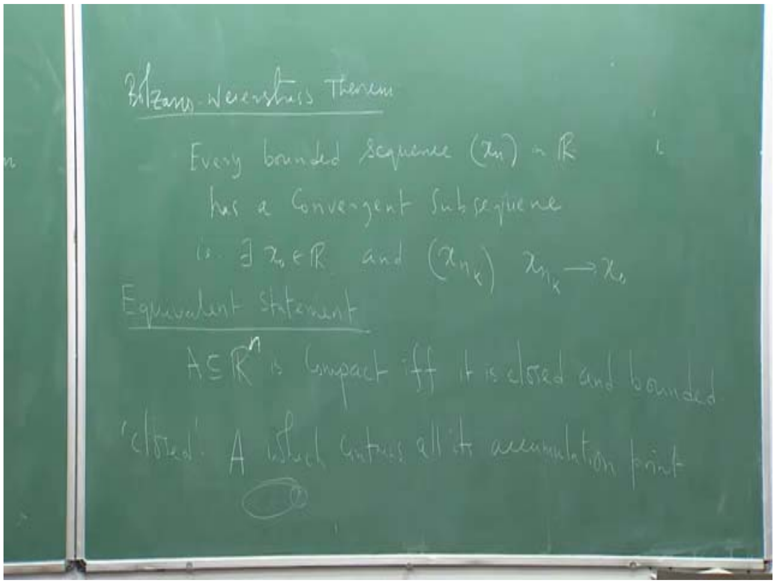

Bolzano Weierstrass Theorem says the version elaborate is this, every bounded sequence XN in R has a convergence of sequence, i.e. to say there exist X not in R and sub sequence of XN, XNK so as that XNK convergence to XNK.

An equivalent statement to Bolano Weierstrass is sometime referred as Heine Borel Theorem is saying that the sub set A in R is compact if and only if (I am using this notation I have already defined) it is closed and bounded. Bounded all of you understand that it is upper bound and it is lower bound, what is a ‘closed’, ‘closed’ means this compliment is open or except A which contains all its accumulation points.

Accumulation points means what say we has a set here, the point is accumulation point. So I am just recalling you can just check your note back in for variable calculus. If I draw a ball around that it has to contain some point other than that from the set A, so I hope you know what is a close set, all of you. Otherwise look at the definition or you can ask, you can put your question through the portal and bounded. What is ‘compact’?

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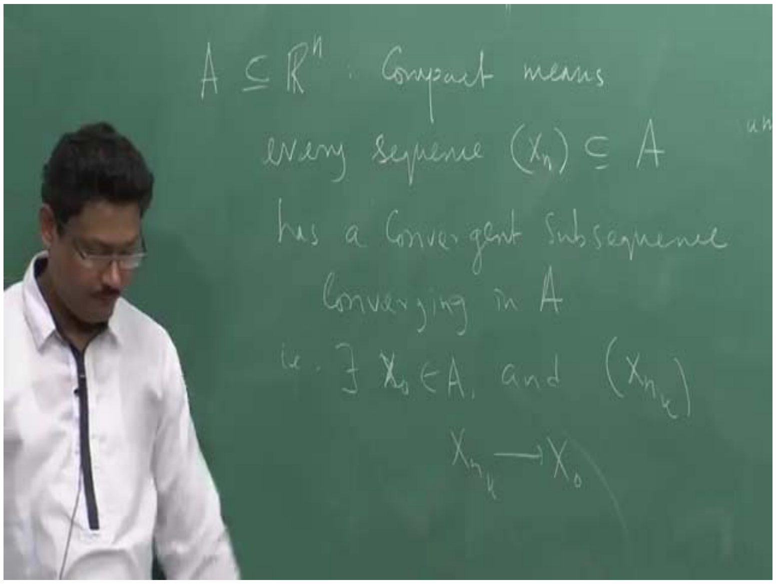

Okay, compact has many equivalent definition and (())(24:54) other than RN but for us 'compact' means A in RN or N equal to 1 is compact means simply the Bolzano's property. Every sequence XN in A has a convergence of sequence, converging in A itself that is there exist X not in A and a sub sequence (okay I am using RN so let me use bold letter) XNK converges to X not, important point is that X not must be in A and this equivalent statement is equally valid if I replace A in RN. If set in RN in compact if it is only closed and bounded.

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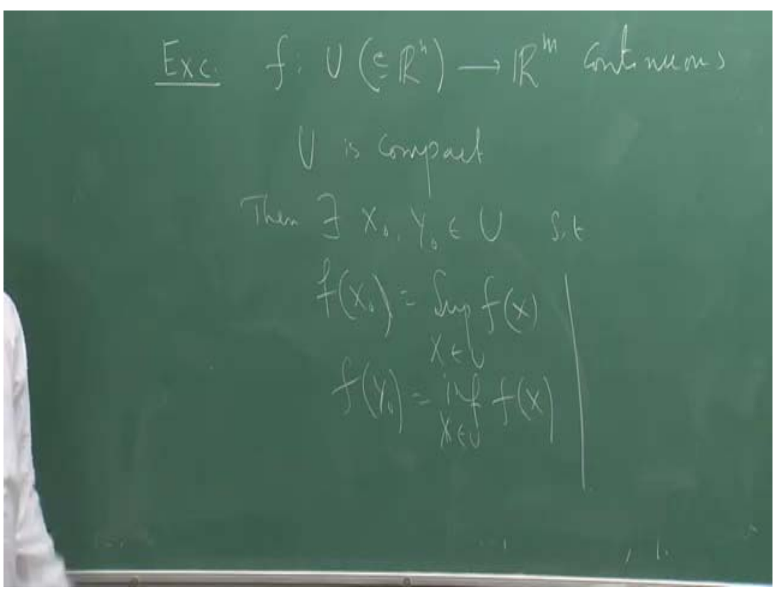

I will not prove this statement Bolzano Weierstrass statement but at this point using this I will leave you to next exercise which is very important and I want all of you do this because otherwise there will be no meaning of the course in the future. F from U in RN to RM continuous and U is compact then there exist X not (same statement) Y not in U such that F of X not is equal to supremum over FX X in U, F of Y not is equal to infimum of Y X in U, this property goes through. Maximum and minimum exactly leave through.

Next time we will see what happens to other properties like most importantly like intermediate value property. Okay, that's the end of second lecture.

Thank you!