Differential Calculus of Several Variables

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Lecture Number 19

Application of IFT: Lagrange's Multipliers Method

Okay! So let us continue with this, problem of 'Lagrange Multiplier Methods'. So, we have to find the extremumum value, maximum or minimum, subjected to, some surface jx equal to c; and we're particularly looking at a problem in R2. So what happen? We have, ba, used basic function theorem with the assumption that f and g, has continuous partial derivative. And in a neighbourhood too have

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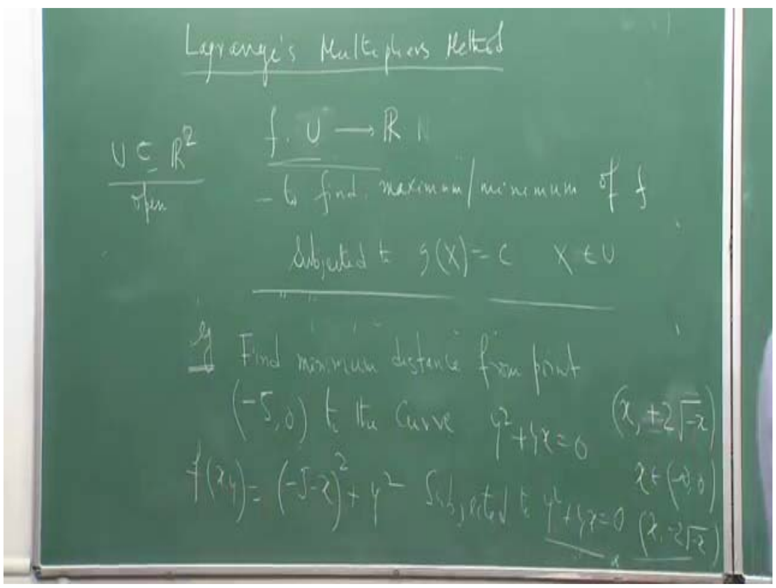

assumed greater than zero. I want long zero, but long zero means in the neighbourhood I can take less than zero against this zero. So see it doesn't matter. So, the problem by 'Implicit Function Theorem' use is that to, that I can write in that neighbourhood, y equal to gx, in the neighbourhood of x naught.

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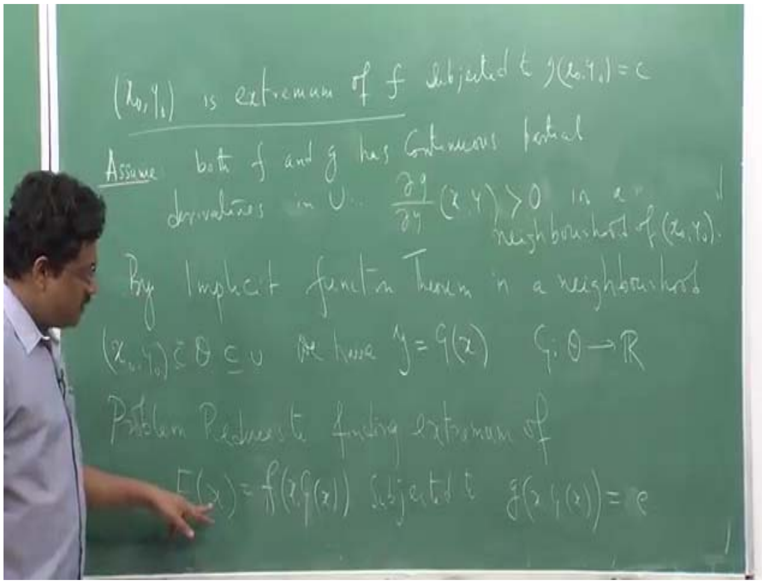

Here I'm important, using Implicit Function Theorem very heavily. The, full force of it, that I can write y equal to gx, and the problem is now, becomes to, function of one variable only. That you, find the extremumum of this function fx, which is fx gx, subjected to x gx, g gx gx gx equal to c. So I've assed there is some extremum. So what will happen? Function of one variable, if it has extremum, then it's surely a settle point, right? So, I must have, I should first look for, this is a extremum, so I should look for, this equation, f prime x equal to zero.

Remember, I'm still in the neighbourhood. I want to locate this point. I know that, suppose I'm assuming the extremum, I want to locate this point. With this assumption that okay, g is fine enough, nice enough. f and g, f is also nice enough. So I should look for f prime x equal to g, f prime x equal to zero because, it's a settle point. Extremum, at extremum, f prime will be, according as the function, the derivative f prime x will be zero. But what is f prime x? f is, little fx gx. So if I apply chain rule, I'll get it del f del x plus, at (x,y). So now I'm trying to look at this point x naught, y naught, h? Plus, del f del y, g prime x equal to zero. This is chain rule.

Let me put a tag here, one. I also have that I'm on the surface x g , small g of x, capital gx equal to c. So, g prime x is zero. So if I take derivative, I will have; right hand side is constant, so it's zero. So let's put two. Now I have g prime here, g prime x here. Let me just get it of this g prime x, because g is a implicit function. I do not know what is g, because, Implicit Function Theorem, which has give existence. But I have two equation involving g prime, so I can get rid of g prime.

So this will implies g prime equal to minus, del g del x divide by, I'm not writing (x,y) again, del g del y. Correct?

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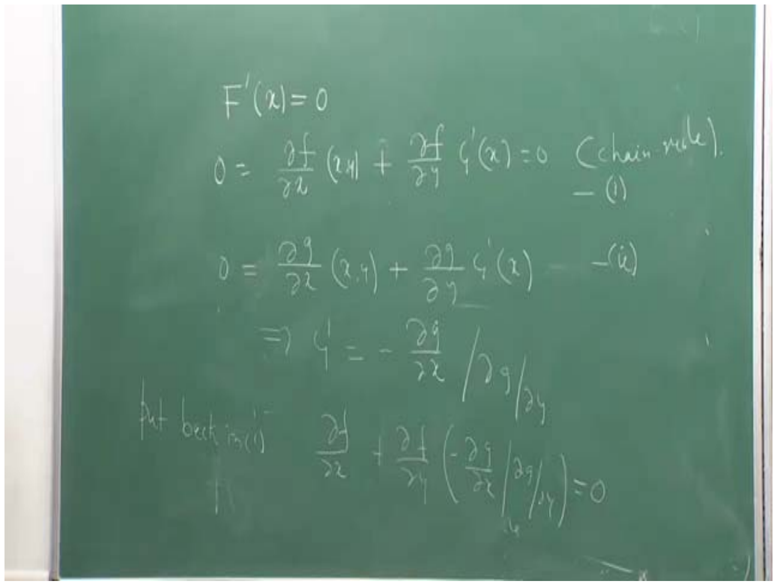

So put back in one, that will give us what? Del f del x, last del f del y into minus del g del x, divide by del g del y. This is equal to zero. Now del g g, del y, I am dividing, but I have assed it greater than zero. So this is this calculation is valid. So what we'll have? From that, I can get rid of the denominator, which will give us del f del x, del g del y, okay? May be minus, plus del f del y, del g del x equal to zero, at the point of extremum. Correct? And I write it; inner products of, inner product. This is precisely writing this line.

So (x,y) is hidded here. I've not retained, I'm continue it. So what does it mean? What is this fellow? This is simply grad f, right? And (x,y). So I've not used this (x,y), it is always there, but now I write, this is grad f, at (x,y). Dot with minus del g del y, del g del x, what is that? Well, grad g is del g del x, del g del y. What is minus del g del y, del g del x? You see, if, this is, I'm looking at surface g(x,y) equal to c. So this is, the gradient. So, this fellow, if you. So this will give us zero. So if this is the gradient, this fellow is orthogonal to the gradient.

Right? So this minus del g del why, del g del x is orthogonal to the gradient. But here I have inner product. This fellow is zero. So del f del x, del f at (x,y) is orthogonal to the gradient, which means, it is parallel to grad g(x,y). Okay? So let me write it here. So grad f at (x,y), x naught y naught is parallel to, because is orthogonal to the vector, which is, orthogonal to grad g(x,y). So that means, this vector is a multiple of, this. Two vectors parallel means they are multiple of each other.

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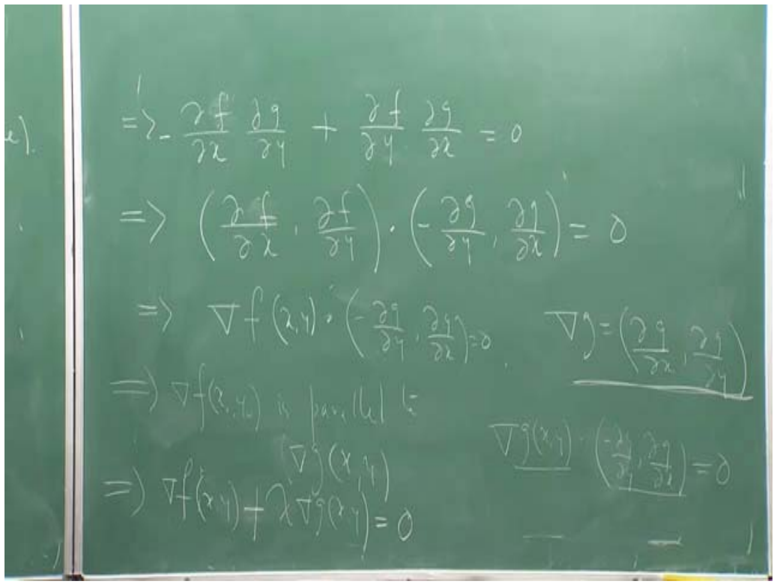

And, instead of writing this way, I can take tis, this side and I can adjust the sign, I can write, this is equal to zero. So what else say? It says, that if we go back to our original problem, that find extremum value of f(x,y), subjected to g(x,y) equal to c, is equivalent to find extremum of, correct? Grad H is zero. I should say ex, not extremum, but sa, should say, they are extremum or not, that will depend 'saddle point', because equating to zero gives you saddle point only. I'll talk about it later, that this method only gives you saddle point.

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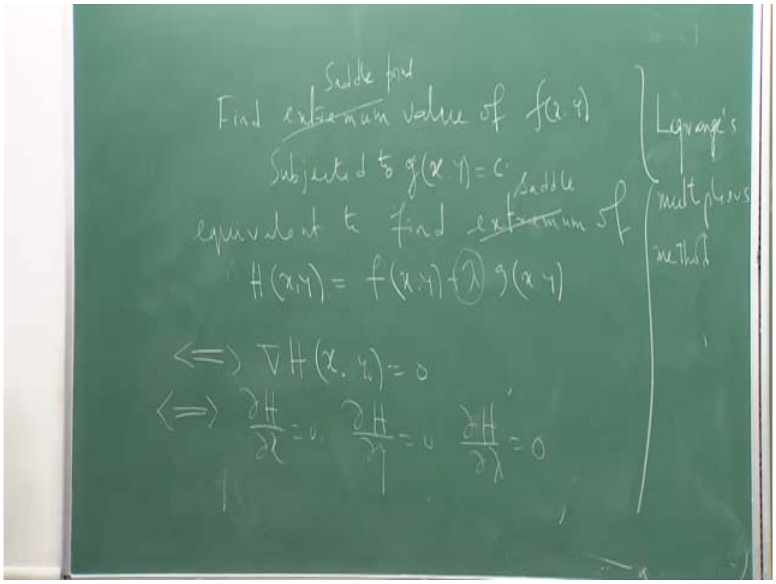

If it is a extremum or minim or extremum or, it is not, that depends on the geometry of the problem. So we have reduced the problem of finding the, atleast the saddle point of f(x,y) subject to g(x,y) to a function H, capital H, f plus lambda g, and you find the extremum. And how do you do it? Well you all, there is three variables here, x, y and lambda. Lambda is not known, right? This, for some lambda. So we have ssssss, same as saying grad H x naught, y naught is zero. So, I have to find lo localty in this point x naught, y naught, I have to put, grad of that fellow is zero, which is, del H del x, zero, del H del y, zero, del h del lambda, zero.

This is, so called, Lagrange's Multiplier method'. This lambda is called the multiplier. Okay? Clear? Let me get back to the example, and show how it is done. And how, we have decided extremum amount, minim maxim everything. That depends on the geometry of the problem, and the just example I wrote down that will illustrate this point very well. So what was the example, recall. So find minim distance from the point minus five zero to the parabola y square plus four x equal to zero.

As I said at the illustration of Implicit Function Theorem, that we can replace y square by minus four x, x negative. But let us apply this Lagrange Multiplier method here. So this is, what is f(x,y), what was f(x,y)? f(x,y) was the distance from (x,y) to this minus five, zero, that is five plus x square plus y square. So find minim, minimize this subjected to g(x,y) is equal to x square, sorry, y square plus four x equal to zero. So I've applied this Lagrange's Multiplier method. So, that by that method, I should look at the function H(x,y) equal to five plus x square y square plus lambda y square plus four x.

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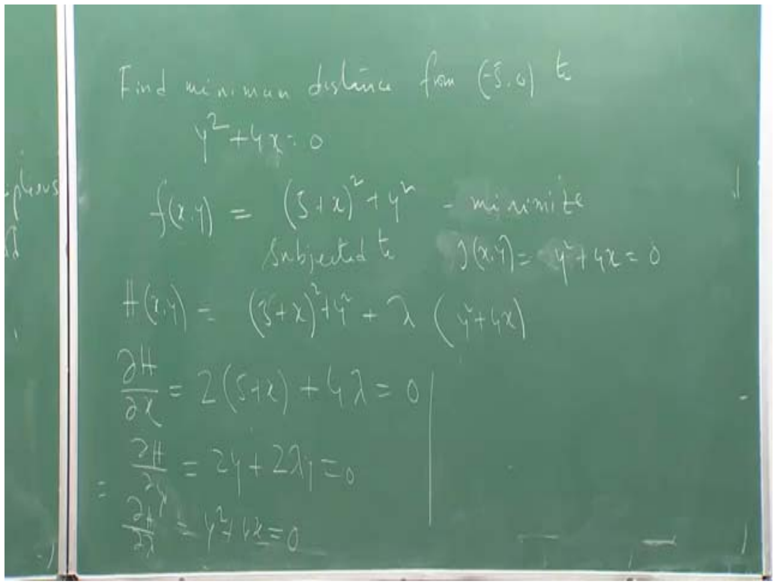

And then what I should row, I should look for the saddle point first. So that is del H del x, how much is that? Two into five plus x, plus four lambda, that I'll put to zero, del H del y, two y plus two lambda y equal to zero. And del H del lambda equal to y square plus four x, so you'll get back the equation with g, zero. From that three equation I've to solve x, y and lambda. Let's see. Now I have to be careful, as I said, this will give us only saddle point, because I just put grad to be zero.

Grad zero doesn't ensure maxima or minima. It may be a maxima, it may be minima, it may be just a saddle point. Well, look at the equation two. Two y plus two lambda y equal to zero, okay? So here was a picture in, so as I said, all this, while solving by Lagrange Multiplier method, always remember the geometry. This was a parabola, and here is the point minus five, zero. Equation two, if y equal to zero, then three will give x equal to zero.

So the point is, the origin zero, zero. And distance, from minus five, zero, to zero, zero, is simply five, right? Well, again look back, look at two, if y not equal to zero, so I get this point, which is a saddle point for H. If, correct? If y not equal to zero, then two will give lam, y cut, I can cancel y here. That will give lambda equal to minus one. Which we'll give by one, x equal to twelve minus one here, minus four. , how much?

Minus four by two, so minus two, that issss, minus three. Yeah. Minus three two, two into two four minus four, zero. And y equal to, from three, or from two, two or three, y equal to. No, from three only. y equal to , so minus three, four into three twelve, so root twelve, which is, four into three two, plus minus, plus minus, three. So, one point here, one point here. And that is expected from the geometry because of symmetry, any point here, that'll be equal distance point in the below.

Now is it a minim or maxim? So the point x naught, y naught now becomes minus three plus minus two root three. And distance, minus five zero to minus three plus minus two root three, how much is that? From this, this is simply twelve plus, so square root of f(x,y), this is four. So earlier, from origin, distance was five, here I get four. So this is a candidate. Now is it minim or maxim? Of course it has to be minim right? Because it is a saddle point, and, it cannot be maxim, because maxim distance I can go to infinity, because this is extended to infinity.

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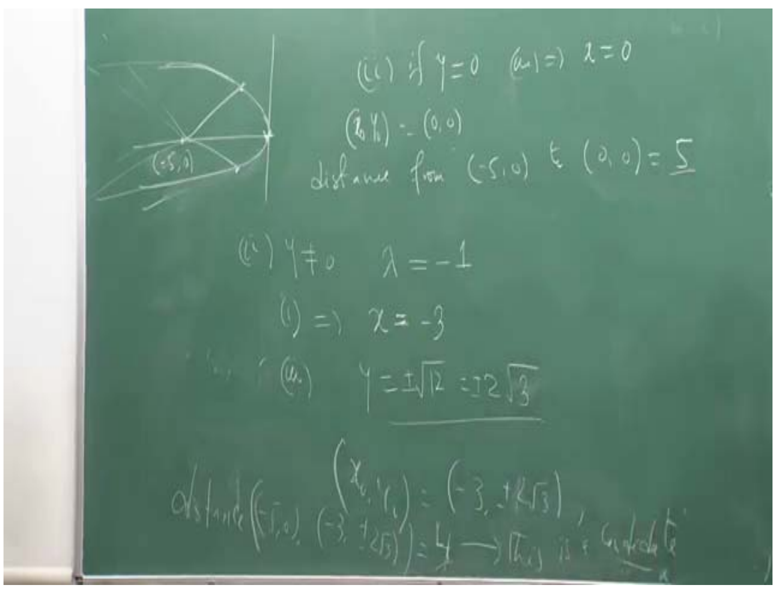

So maxim distance cannot be four, because I can't join a very point at a very far away to minus five, and the distance will not be to be much much, ba, I can make it bi bi big, as big as I want. So it cannot be maxim. But why it is minim? Well, I'm looking for, minim of this fellow, subjected to this thing. y square plus four x equal to zero. That gives me. So what you try to do, here, I mean while solving the problem, you don't have to do it, but actually you can show that you move little bit, and calculate distance from minus five zero to this.

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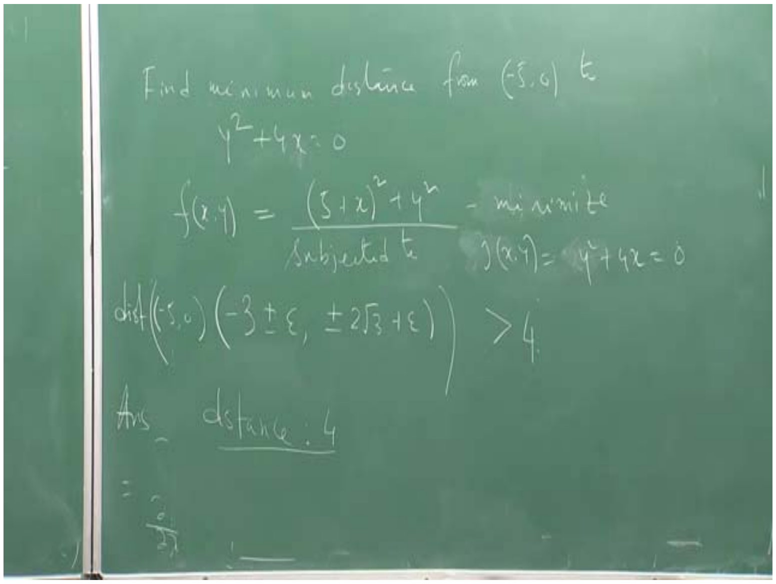

And you show, this is always greater than four. That's very easy to show. Right down the formula, distance is greater than four. So, distance is actually, answer, four. So this is Lagrange Multiplier method. I have the assurity to it fire function from of two variables. Now what we'll do, if I have, function of several more than two variable? Suppose I have the problem, instead of R2 I have R3. So you've to find extremum subjected to a surface, level surface. The problem doesn't change a bit. It's same.

What will happen? I will asse f and g as continuous partial derivative, and suppose x naught, y naught, z naught is a possible candidate, when they are around the neighbourhood, around a neighbourhood theta, what do we have? I will have that there's a function g, such that y equal to g one x, and I will have function g two. Okay, some other neighbourhood, but I can take the minim of that. So this is, Implicit Function Theorem. And then, problem reduces to finding extremum of H(x,y,z), the same derivation, derivation. Do it yourself.

Oh, subjected to two condition actually. Sorry. I need two condition here. See first, when whenever I need one condition, and we have subjected to intersection of those two surfaces, which is a curve. Okay? Now send me the equation, we'll work. I mean you go through the equation, instead of del f del x del and del f del y, here it will be involved del f del x, del f del y, del f del z, and, g one prime, g two prime,

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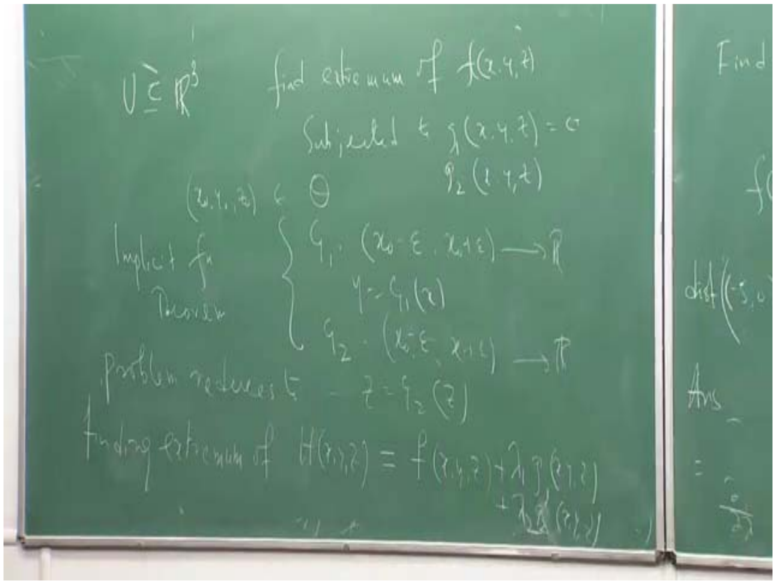

you will have three equations. You have to remove g one and g two from the equation, you'll get back to this one. So do the derivation yourself.

I will put it as an assignment problem also. And now you see, what will happen, if I take, U in Rn? And, instead of two, I have k many condition, where k is less than n. To apply the Implicit Function Theorem. So I'll have a neighbourhood, of x naught, the critical point, where, I will have g one, g two, so on upto g k.

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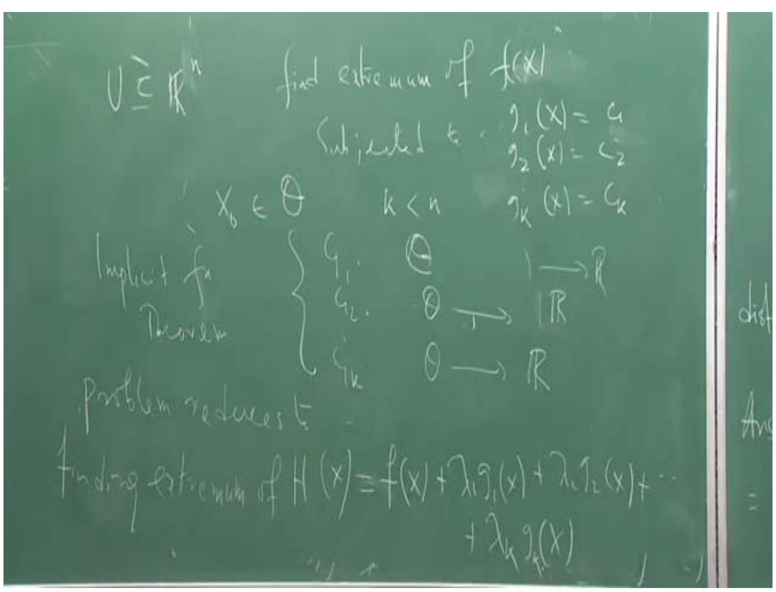

Same derivation will tell us that it reduces to finding extremum of Hx equal to. And that's it. That's Lagrange Multiplier method. And direct application of Implicit Function Theorem. In the next set of last, , that'll be last two lectures, we will discuss another application, and there, very important application of Implicit Function theorem, which is Inverse Function Theorem. Thank you.