**Differential Calculus of Several Variables**
**Professor Sudipta Dutta**
**Department of Mathematics and Statistics**
**Indian Institute of Technology, Kanpur**
**Mod 04 Lecture Number 17**

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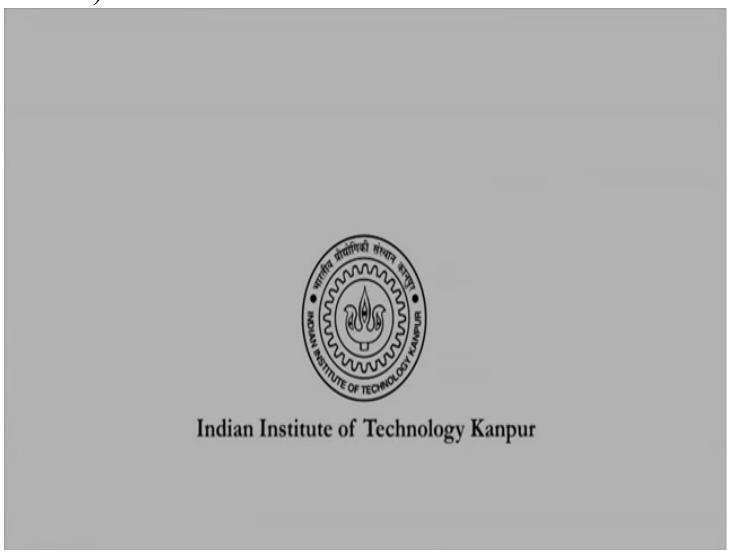


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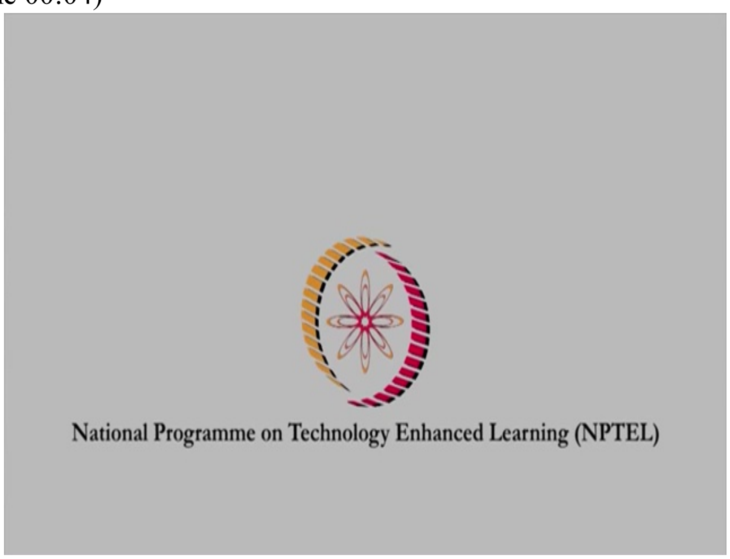

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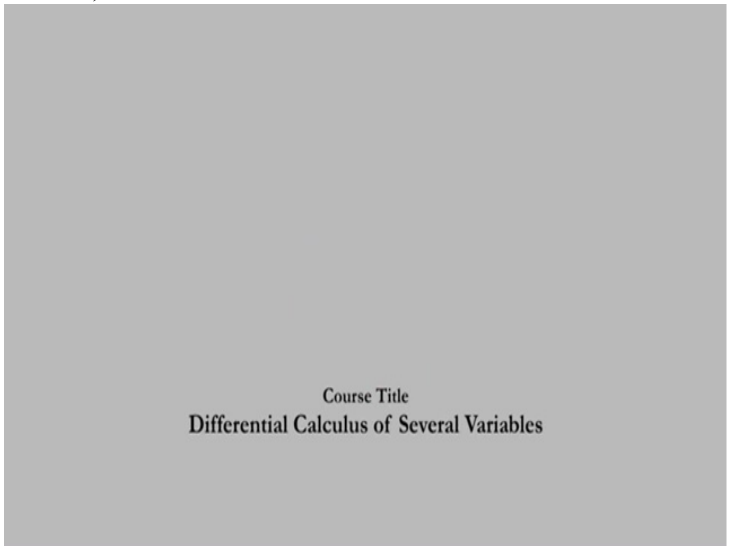


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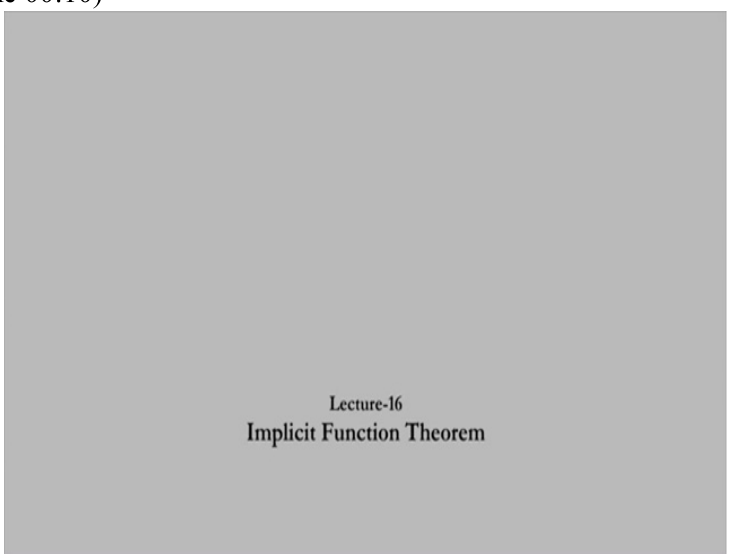

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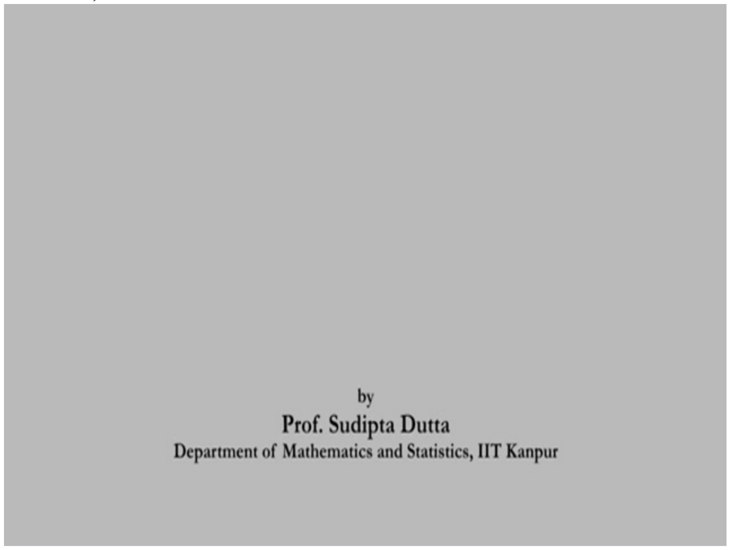


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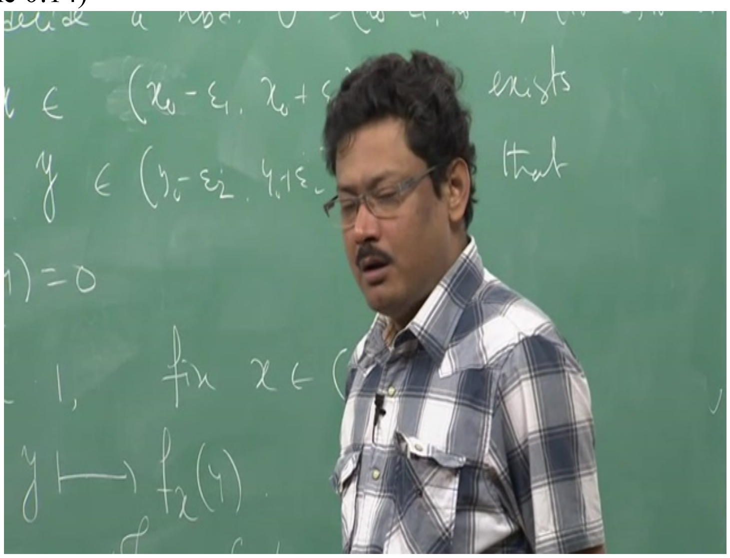

So, Ok let us come back and continue with the implicit function theorem.

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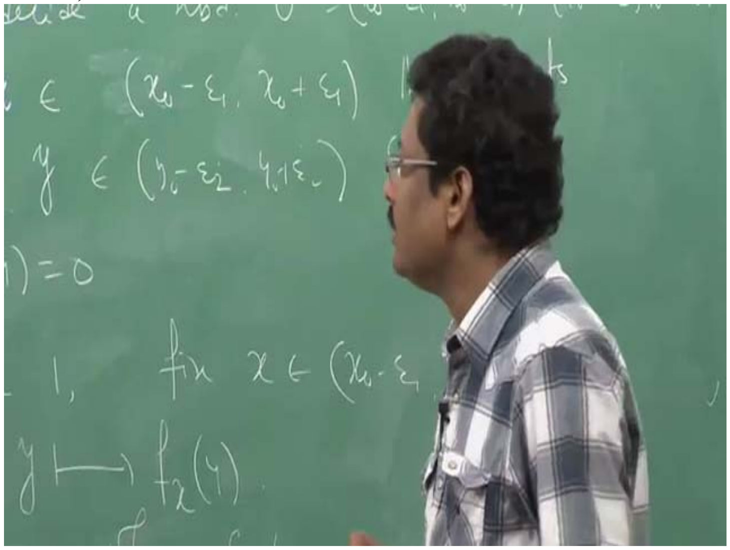

What we observed, that we could achieve our goal to have every level curve locally as a graph, that if I have this for every fixed x  in the neighborhood, what we have to decide

I can have a neighborhood of y such that y going to f (x, y),

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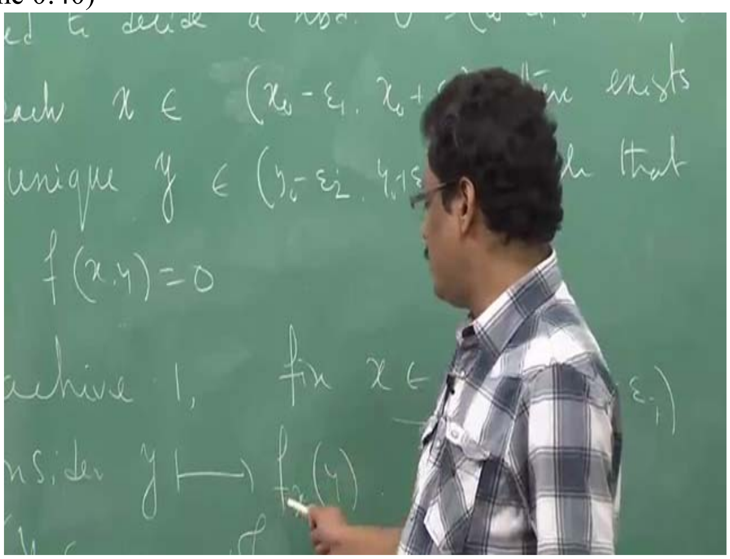


this function it is as zero here, zero here means it changes sign, they are the two end points and the uniqueness will be guaranteed if this is strictly increasing. This is one of the sufficient conditions.

There could be many but we have to attack

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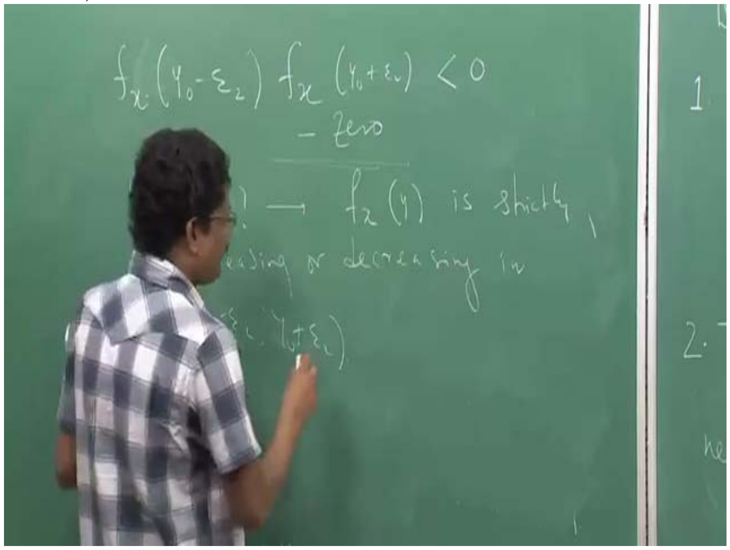


whatever we have in position, so if it is strictly decreasing or strictly increasing then I will have unique zero

And what does it mean by strictly decreasing or strictly increasing? That means I should have “del f x” so I should have “d f x” “d y”

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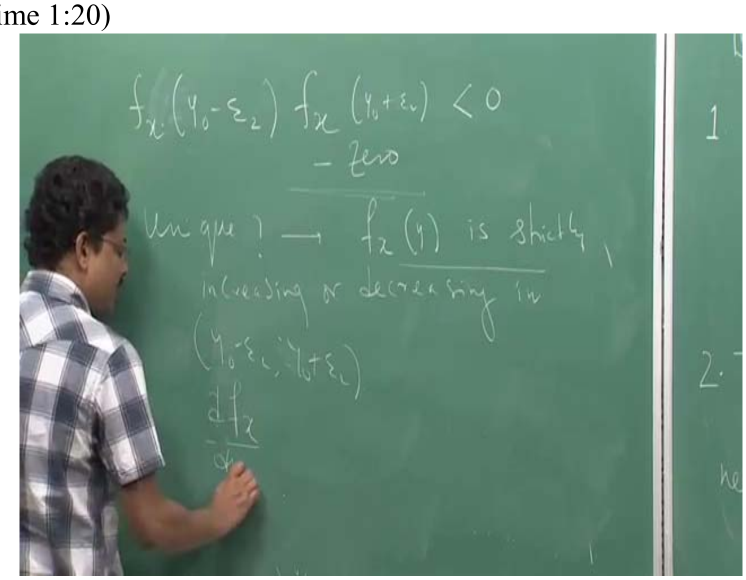


this is greater than zero or strictly less than zero, here it is strictly increasing or strictly decreasing for each fixed x in this interval, so that is to say,

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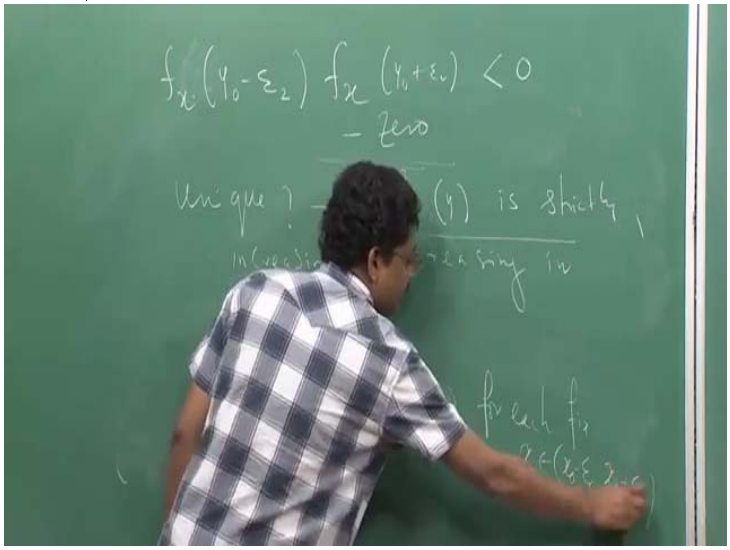


in terms of f, “del f” “del y” (x, y) is strictly greater than zero or strictly less than zero for (x,y) in the interval U

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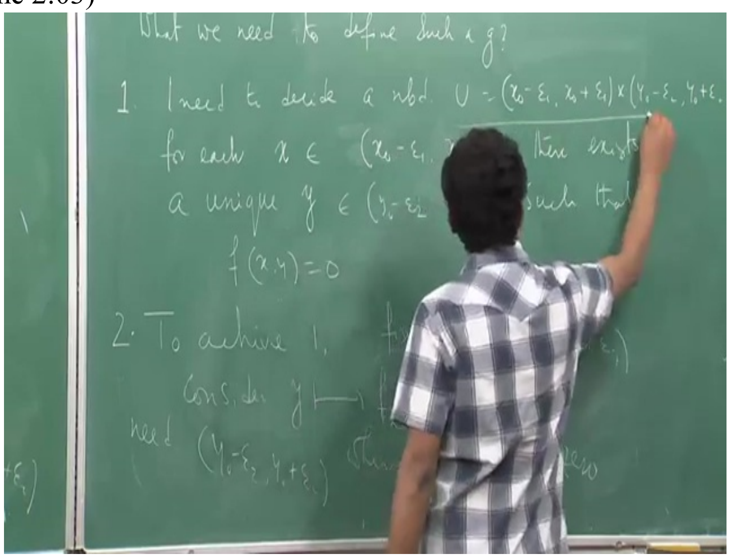


Ok

Well what I am trying to do, given each x in this interval, we can try to find the unique y. This is sufficient condition

You understand very well, Ok that we could have,

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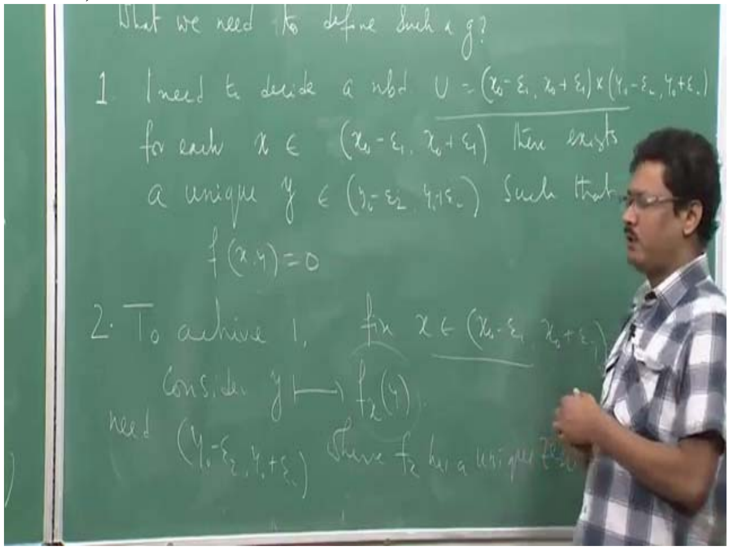


instead of defining function of x, you can define function of y; that will also do. So the role of x and y could be interchanged. So first I...depending on the situation...that suppose I do not achieve this, but I have"del f" "del x" (x, y) is greater than zero in some interval

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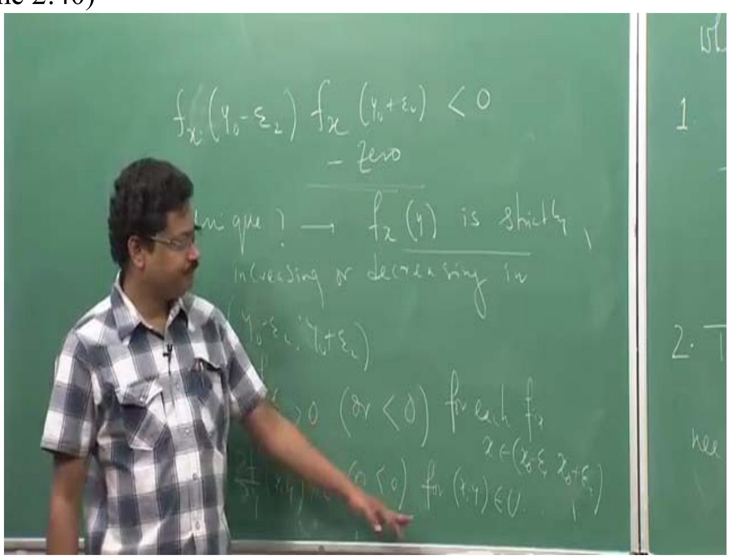


and for fixed y,f (x,y) changes sign in the interval of "x naught" minus "epsilon" to x plus "epsilon", then I could have achieved this. So this is

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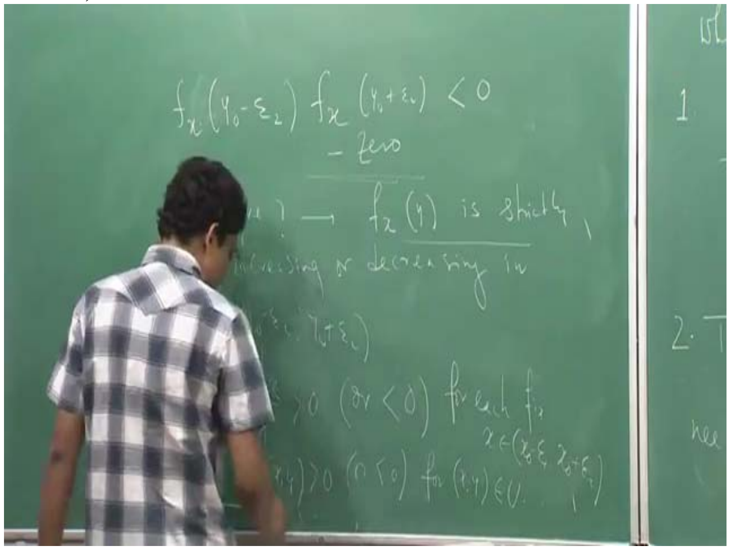

for writing function of x Instead of that we could try to write function of y as well.

So role of x and y could have interchanged

SO what we actually mean here is that, conclusion is that I need one of this, either”del f” “del x” or ”del f” “del y” and (x, y) in this interval

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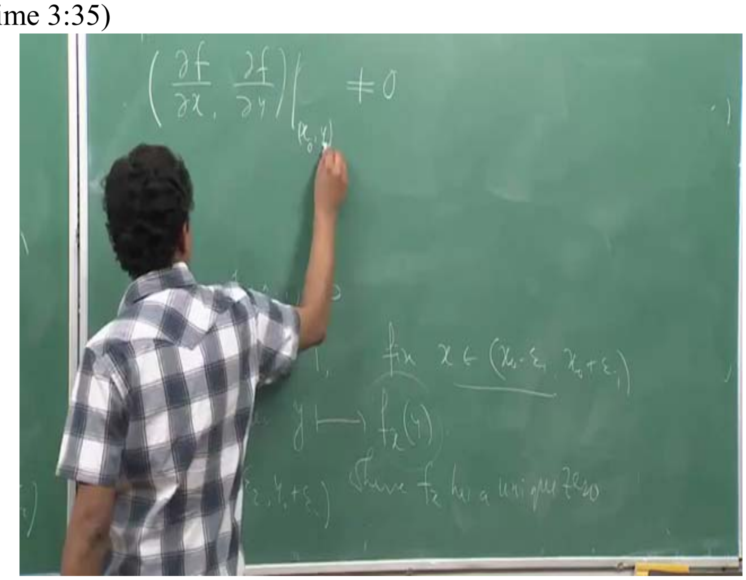

is non-zero

Or “x naught”  “y naught” point, non-zero and ”del f” “del x”,” del f” “del y” continuous at “x naught”  “y naught”

Then what will happen, it is non-zero,

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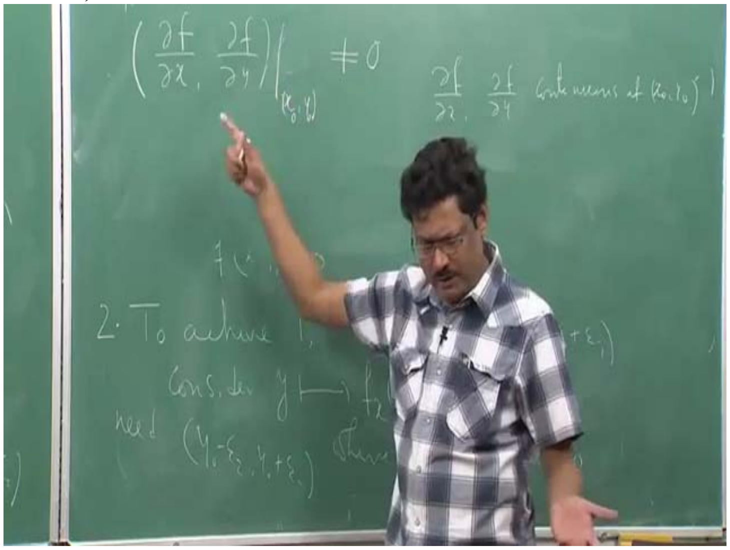

so one of them is non-zero So, let's say positive or negative; If it is positive or negative then continuous and I can decide the interval around where it remains the same sign.

Ok

So I need continuous partial derivatives of f and "del f" del x", "del f" "del y" at that point

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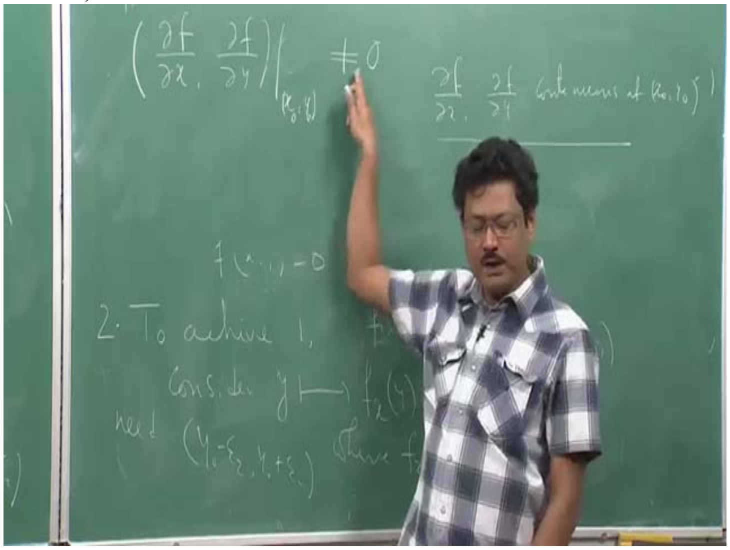

non-zero

So if you look at it geometrically, that is a very nice interpretation.

So what we...so this is...this fellow is what…. this fellow according to our notation is "grad f" "x naught" "y naught"

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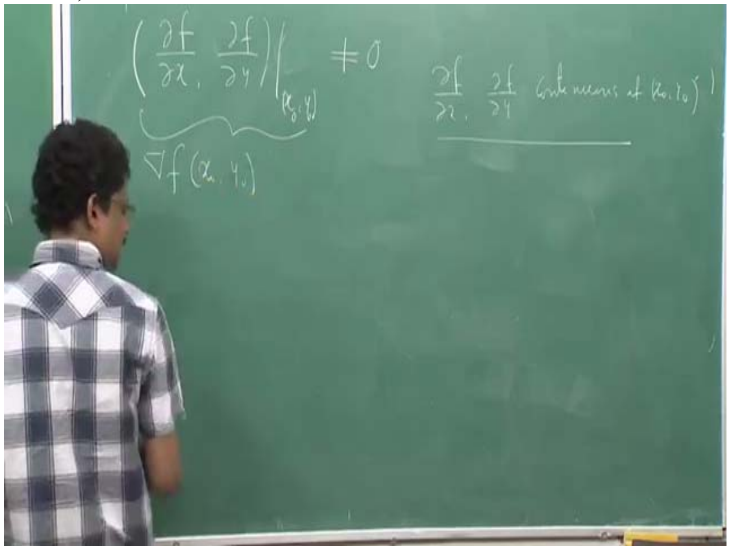

Now what does “x naught” “y naught” belongs? This belongs to this level curve f (x, y) equal to some constant of (x, y) equal to zero.

If I have a level curve, let us say this

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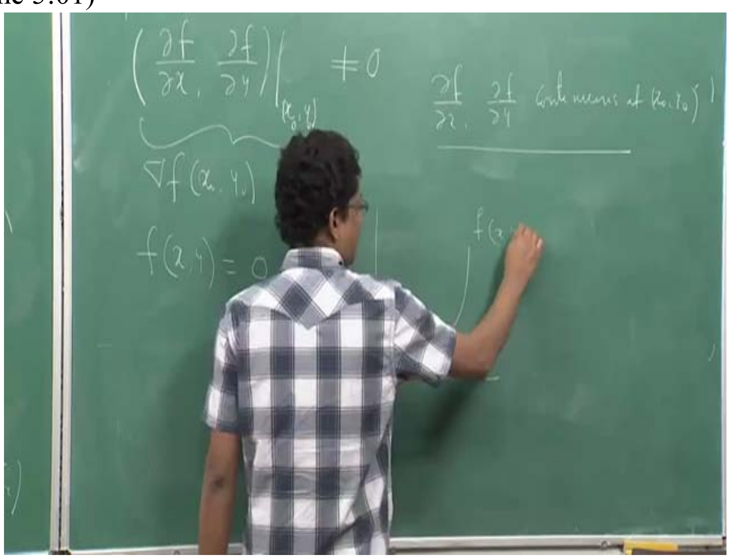

and I fix the point “x naught” “y naught” here, then you can check very easily that

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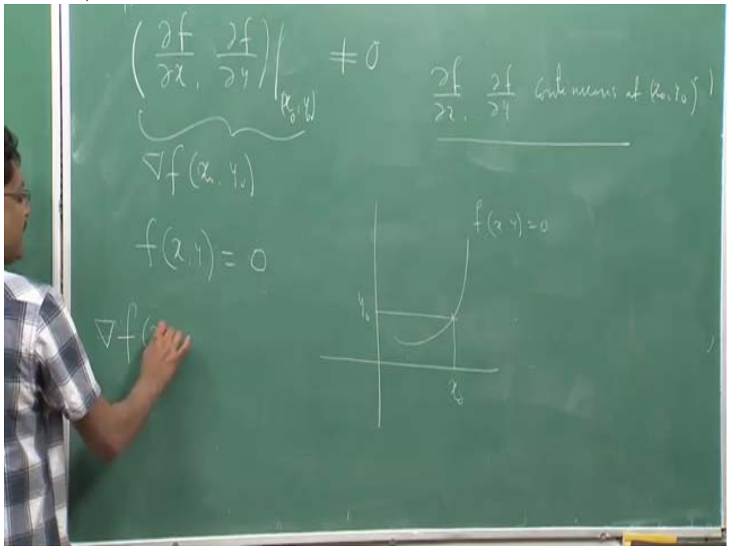


this fellow....this is precisely the normal at “x naught” “y naught” .

For a level curve, grad at some point is the geometrically, is the normal to that point.

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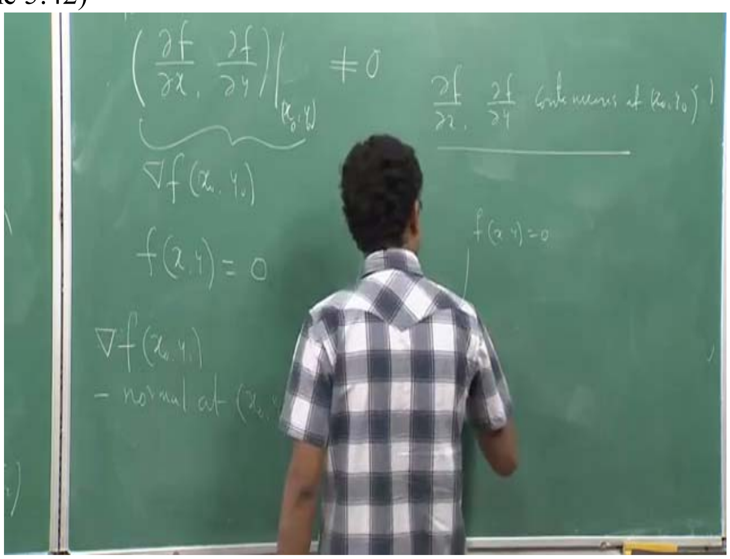


And this condition means....normal is non-zero; normal non-zero means it is not parallel to x axis.

Ok, so this condition “grad f” “x naught” “y naught” or let’s say

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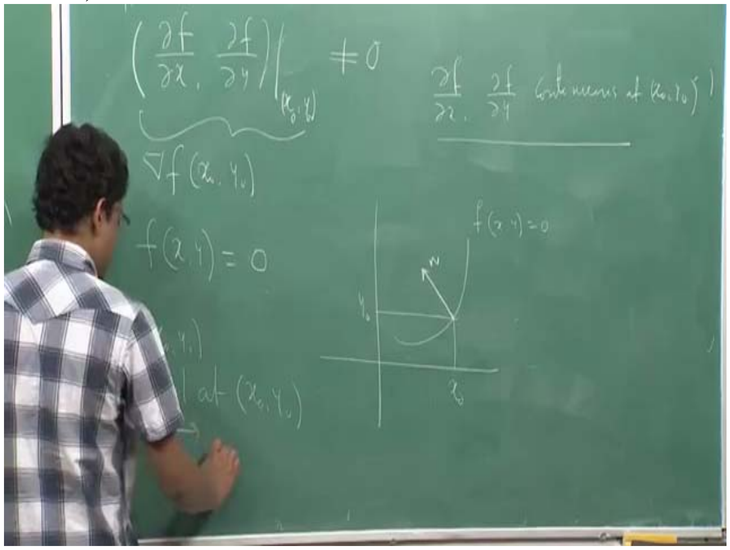


n not equal to zero, that is n is not parallel to x axis. This is what geometrically, the condition we are imposing.

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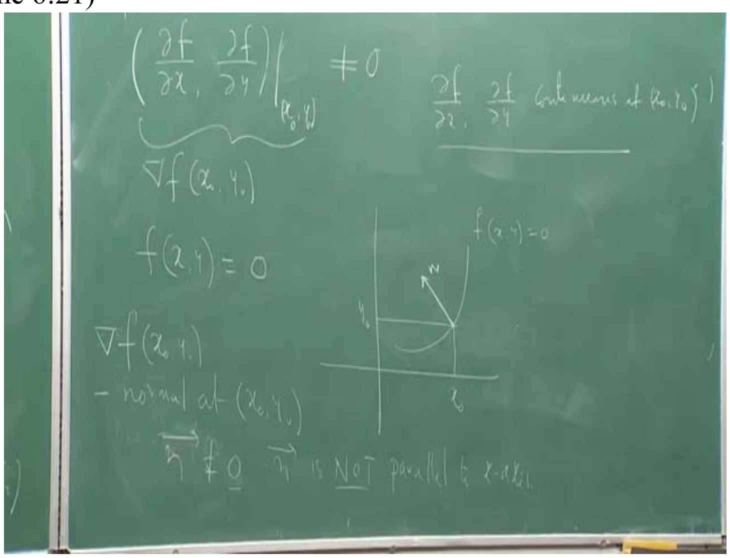


And what it means...and also normal is changing continuously

Now if normal is not parallel to x-axis, why we achieve it? You forget everything else. We have this curve and normal is not parallel to x-axis, then the tangent…

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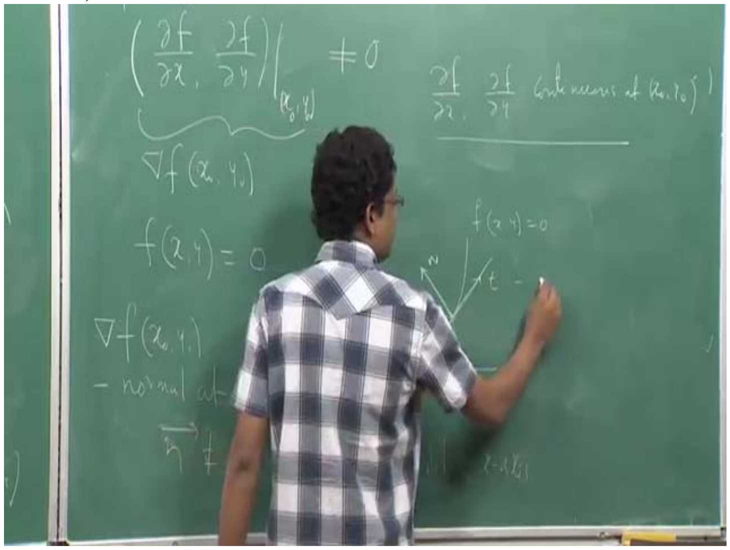


this is not parallel to y axis; because normal and tangent are perpendicular to each other. If at a point, tangent is not parallel to y axis, what does it mean?

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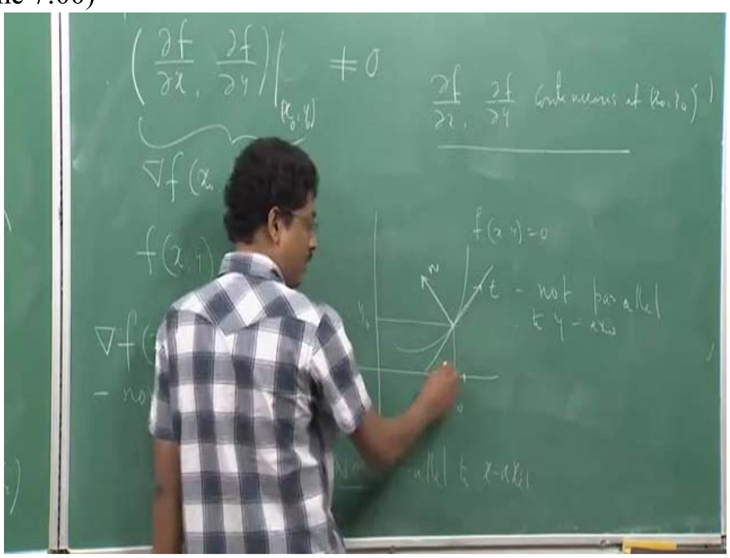


If I move little bit from here, this side or this side and fix x here, say “x naught” plus something, some little bit I move from “x naught”, so in that neighborhood, I have to decide Then I draw this line; this is the line x equal to “x naught” plus “delta x”

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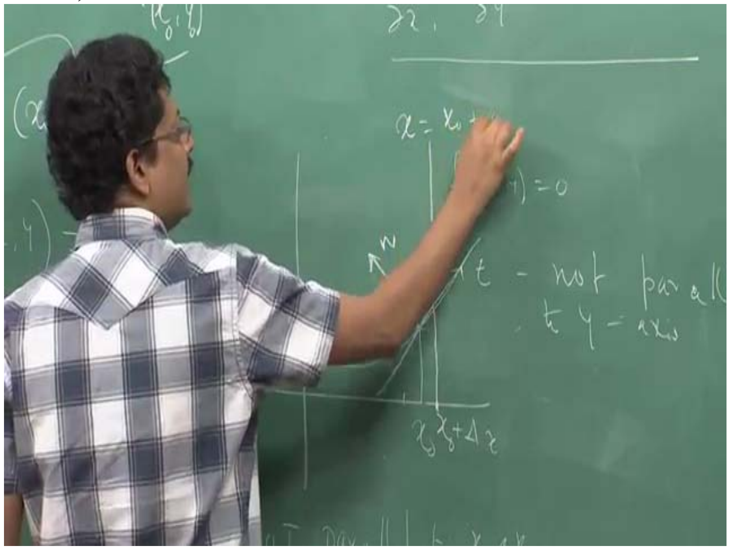

Then this line will intersect the curve at unique point. Similarly here, “x naught” minus “delta x”

So this line, “x naught” minus “delta x”,

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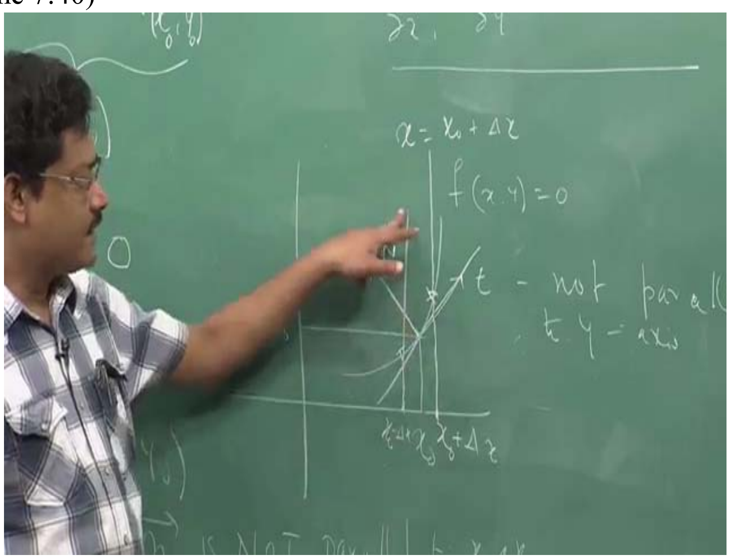


y equal to “x naught” minus “delta x” this line, this will intersect this curve at unique point.

And that is my unique y; that is what we are looking at

So geometrically, that is it, what is going on.

And it is very easy, that is why I said that is very easy to understand in function of 2 variables taking values in R.

What we all demand is, normal exist, non-parallel to x axis

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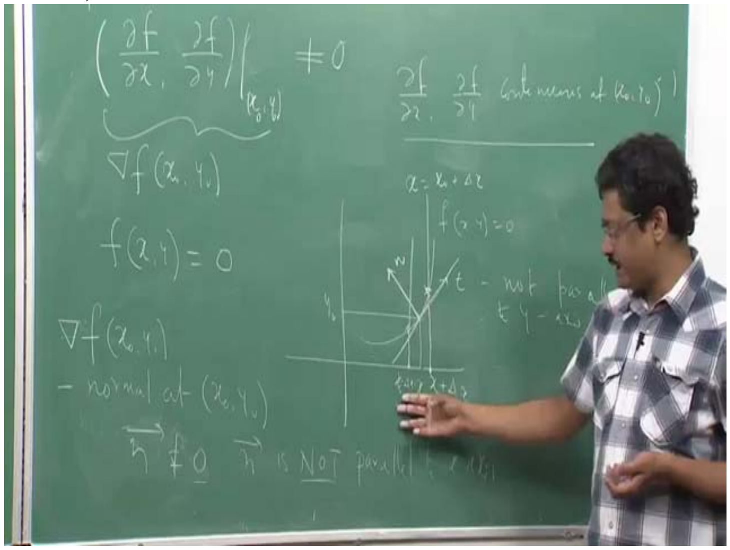


and it is moving continuously. If I have that, we have it.

And of course you can see, instead of this, why we cannot achieve it everywhere, why I need this?

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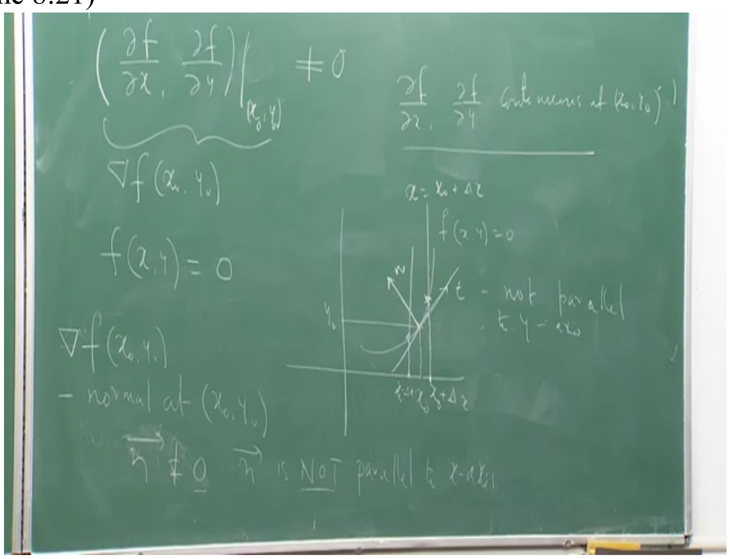


Suppose I have picture like this and

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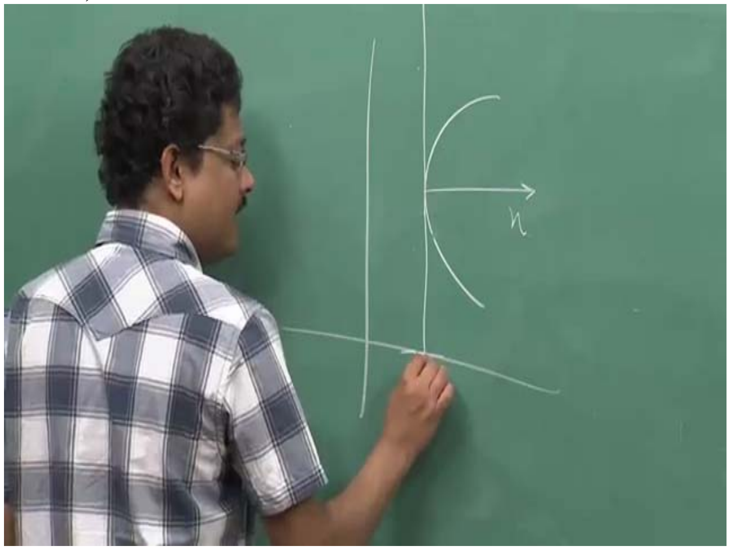


I am looking at this point “x naught”.

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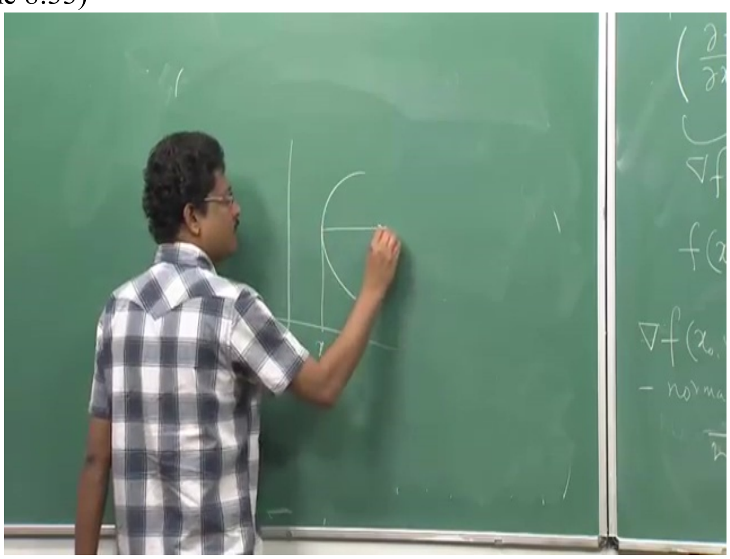

This normal is outward normal, this way.

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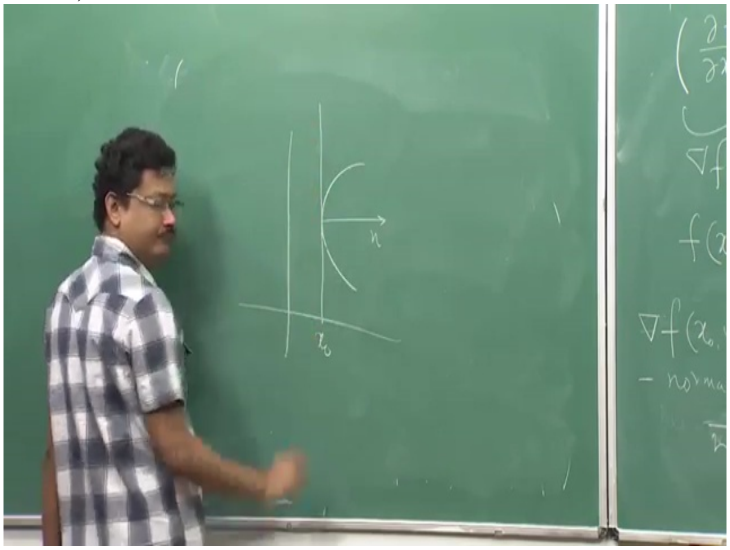

So tangent is this line itself.

Now if move from “x naught”, moving from this side does not make sense, because there is no portion of the curve....if I move this side, any point you move

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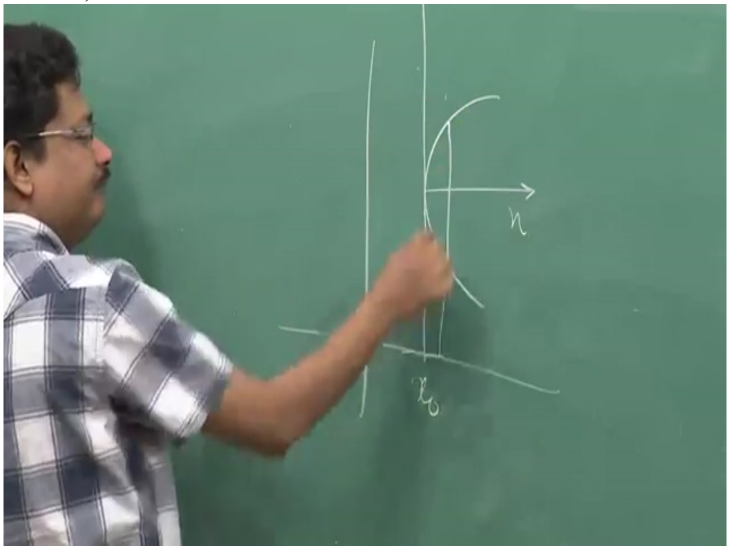

, you will always have two points on the curve.

So you cannot decide y uniquely.

So that is the geometry of implicit function theorem.

Now I hope we have understood what is needed,

Once you have understood we just put them, whatever we need

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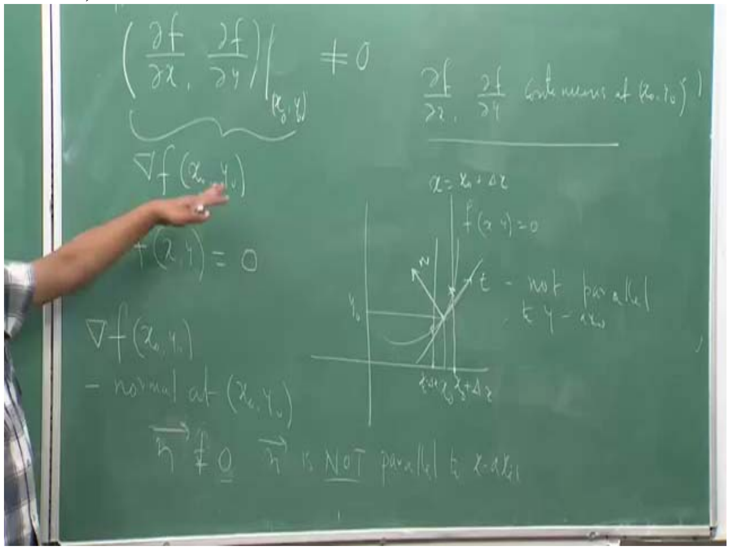

as assumption and prove the theorem.

So let me state implicit function theorem for function of two variables.

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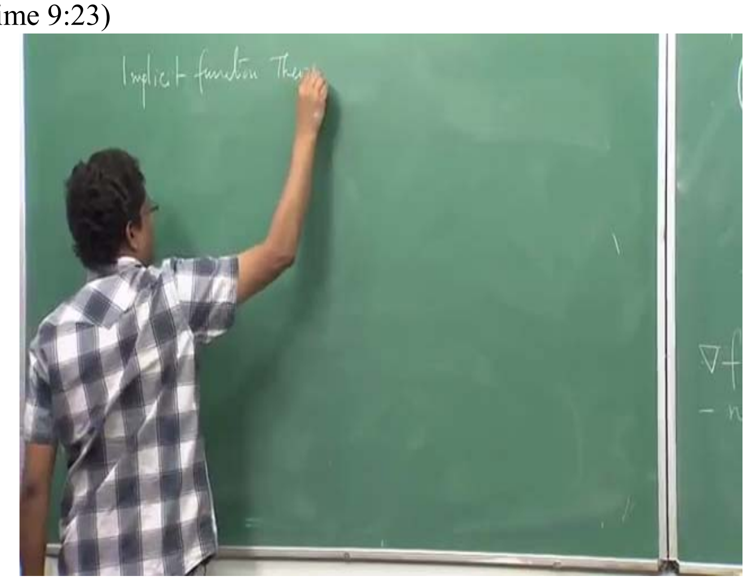

For f from R2 to R

So you state your theorem, you prove your theorem, you have to do it carefully

So far, what we have been doing, we have been doing informally whatever we need

So we put it formally now.

So let us put the set up.

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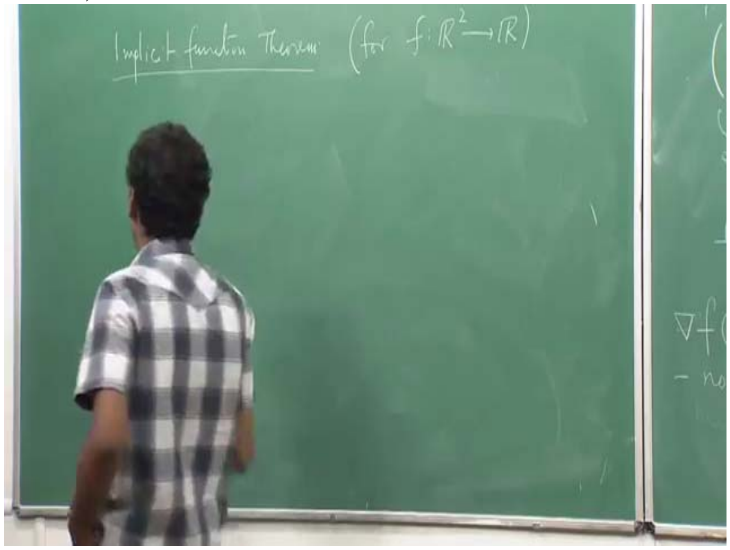


Let U in R2 be an open set.

If from U to R,

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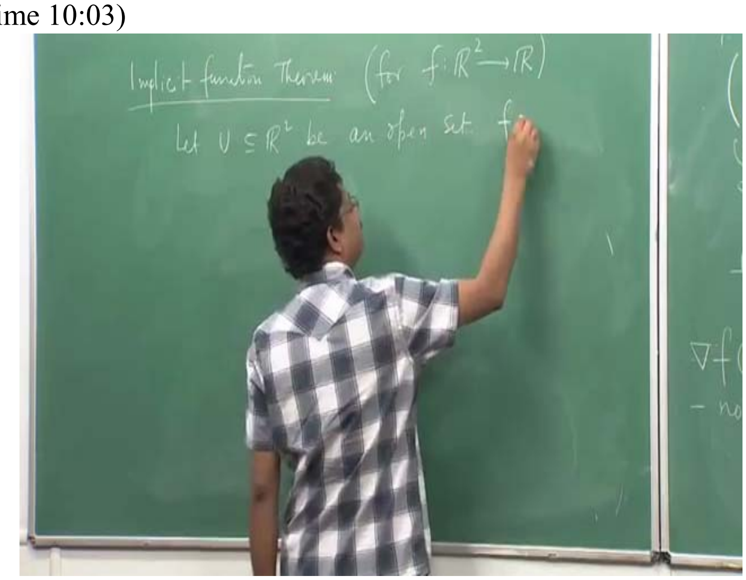


Ok, and “x naught”, “y naught” belongs to U

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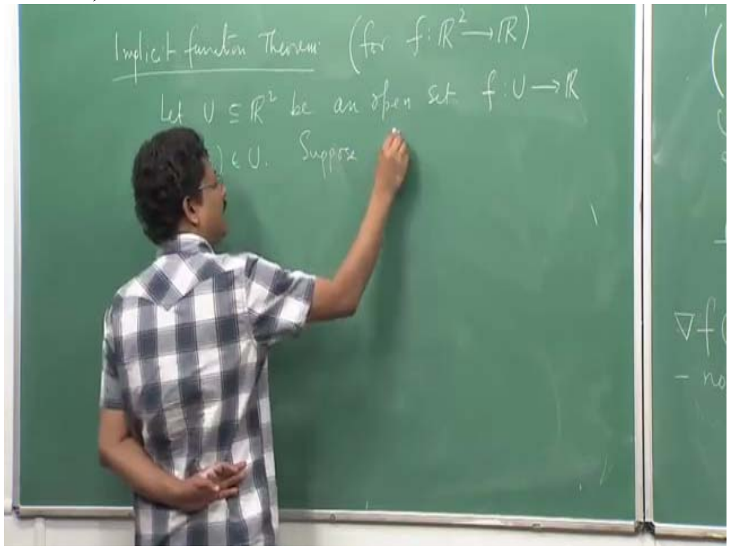


So suppose, f has continuous partial derivative

“del f” “del y”

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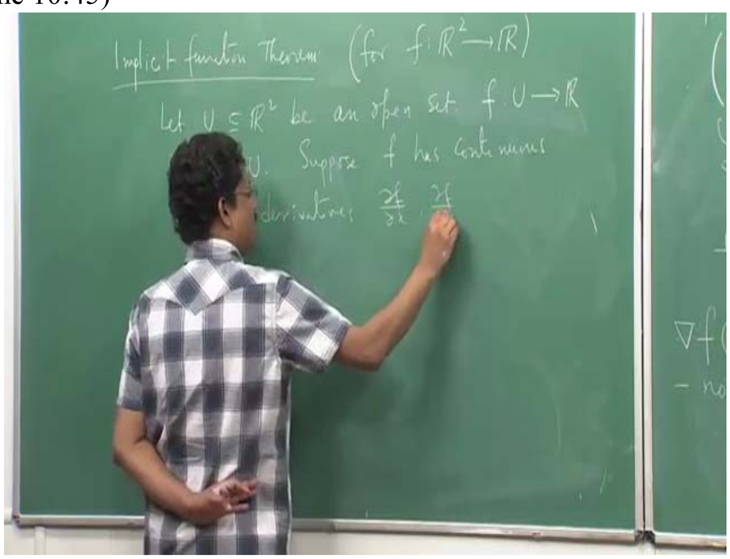


at “x naught” “y naught” and of course “x naught” “y naught” is on the level curve, equal to zero.

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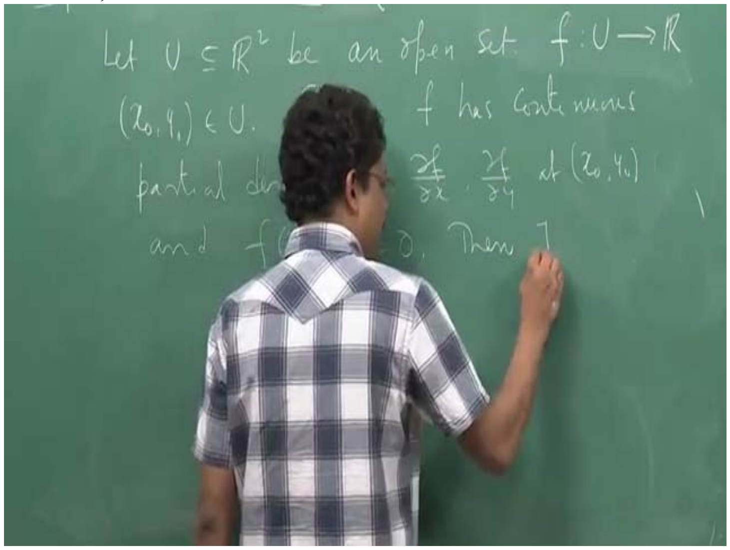


Then there exists a “epsilon” greater than zero and a neighborhood, let’s say theta of “x naught” “y naught” and a function g

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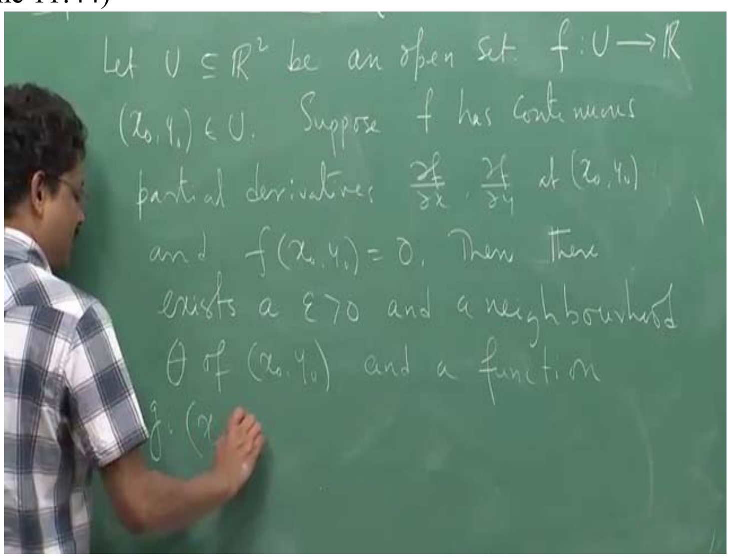


of “x naught” minus “epsilon”, “x naught” plus “epsilon” to R such that

what happens

1 f x g x

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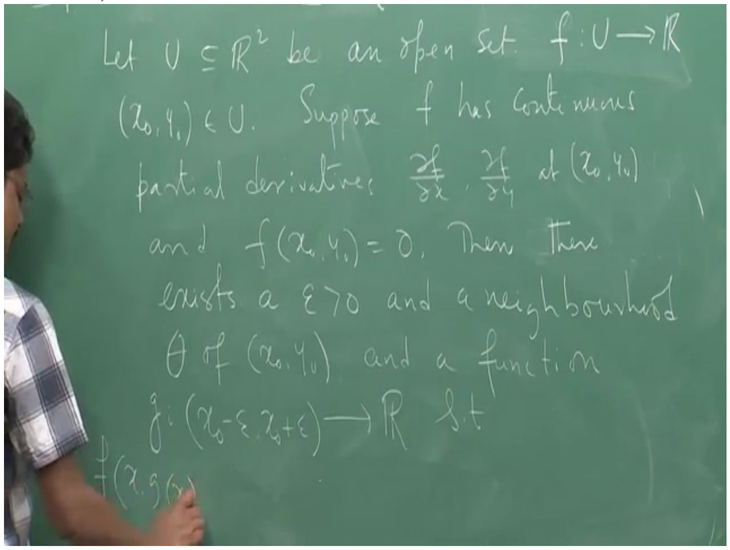


equal to zero for all x in “x naught” minus “epsilon”, “x naught” plus “epsilon”, g of x belongs to theta and of course g of “x naught” equal to “y naught” .

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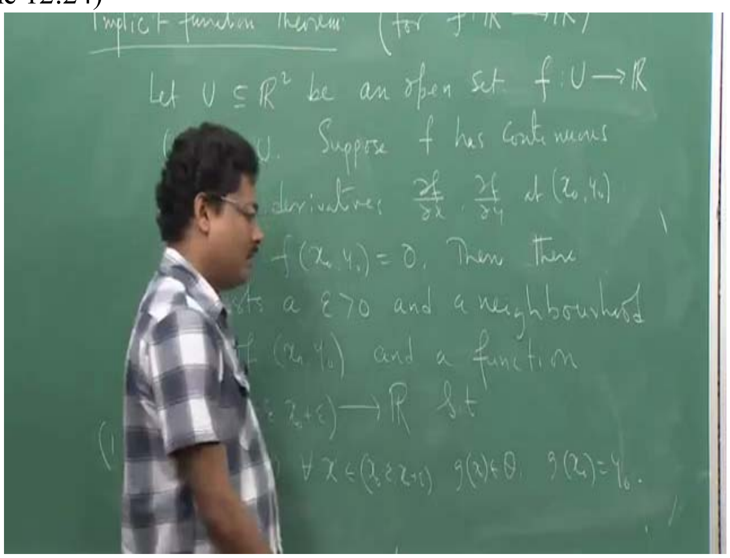


This is what we wanted to achieve

But we can actually achieve something more.

What, that of course...what we achieve, what we are going to write is much desirable

We have taken f is very nice, it is continuously differentiable

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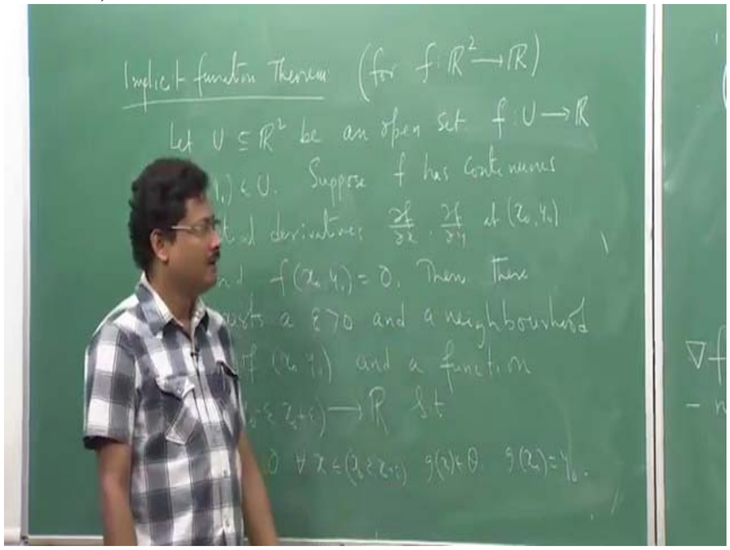


This is C1 function; it is continuous for second derivative.

So f is a smooth level curve.

Whatever...we have now defined what is smooth curve, what you understand in layman's terms that f is nice...it is nice function of 2 variables.

And so we will also get

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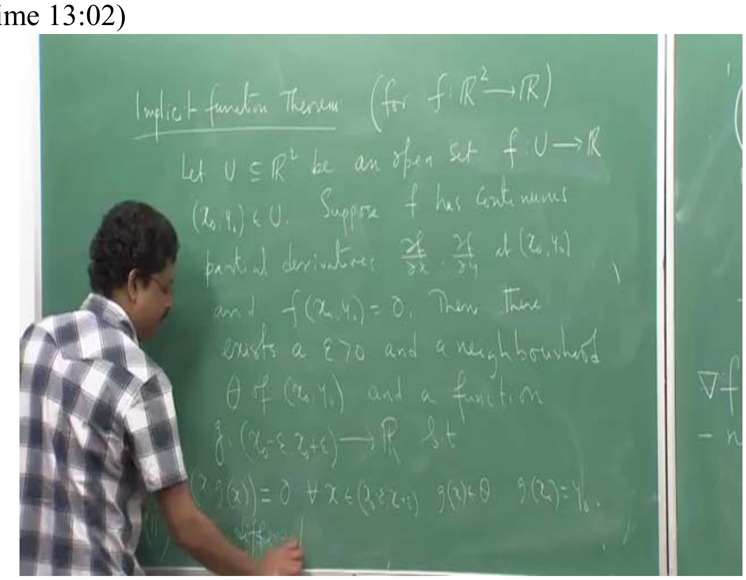


g is differential in the entire U

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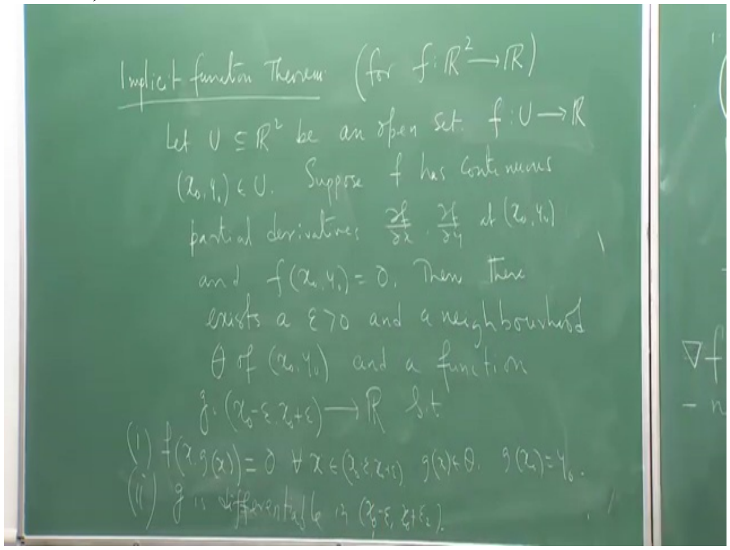


So this is the statement of implicit function from R2 to R

Let us understand it, and remember it, from R2 to R from here, the generality statement is just mere formalities

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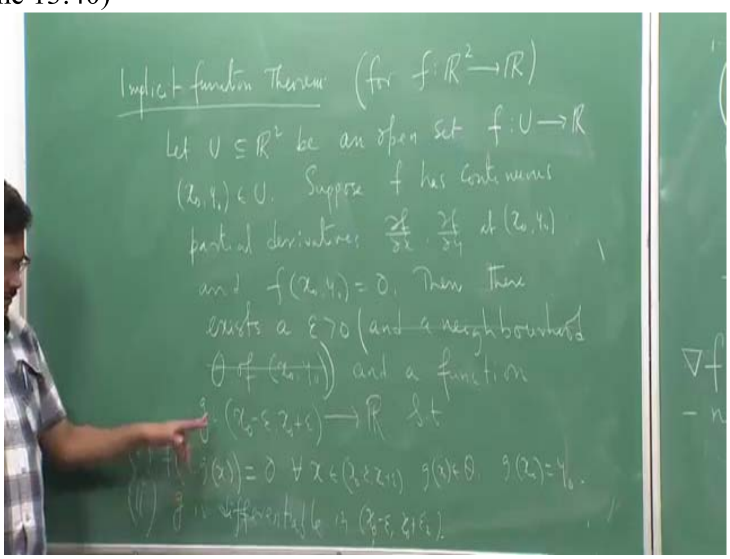


So this part is not really needed in the statement but it is actually part of the proof; part of the proof, I have just included it. it is not needed in the statement.

Whatever be the question is, the question is answered here...without this statement, this neighborhood

But the way I have presented it, I mean we have motivated ourselves, actually this will be true, this theta will come and what will happen is,

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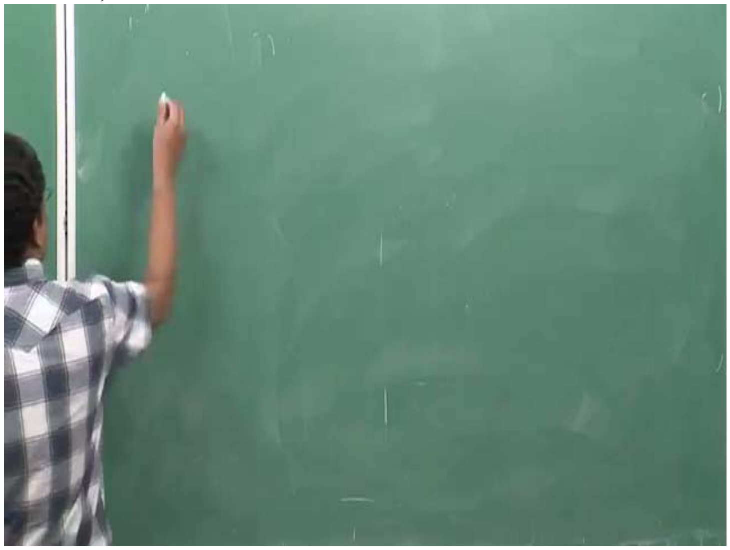

what these true statements means that actually there exists a differentiable g such that f inverse of 0 intersection theta, this is the local part equal to x g x,

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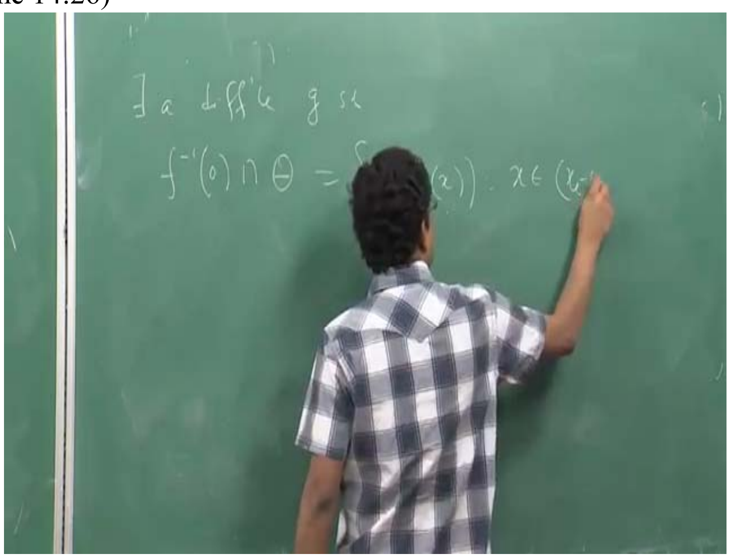

x in “x naught” minus “epsilon”, “x naught” plus “epsilon”

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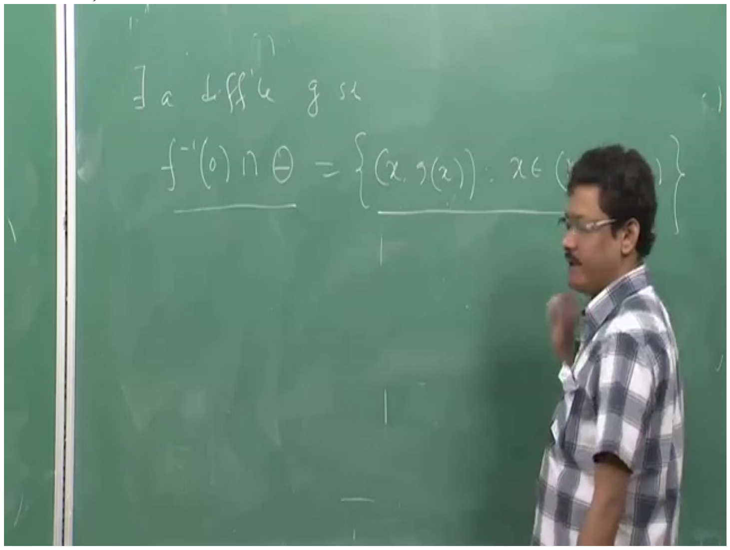

So this part of the curve is the graph of this function

Ok, now we actually know how to prove it

We have actually made the statement other way round, that...we have actually put the assumption what we actually needed, but still we need to write down the proof formally

So we know all the assumption

This continuity is needed at “x naught” “y naught” to maintain the same sign of ”del f” “del x” or ”del f” “del y”

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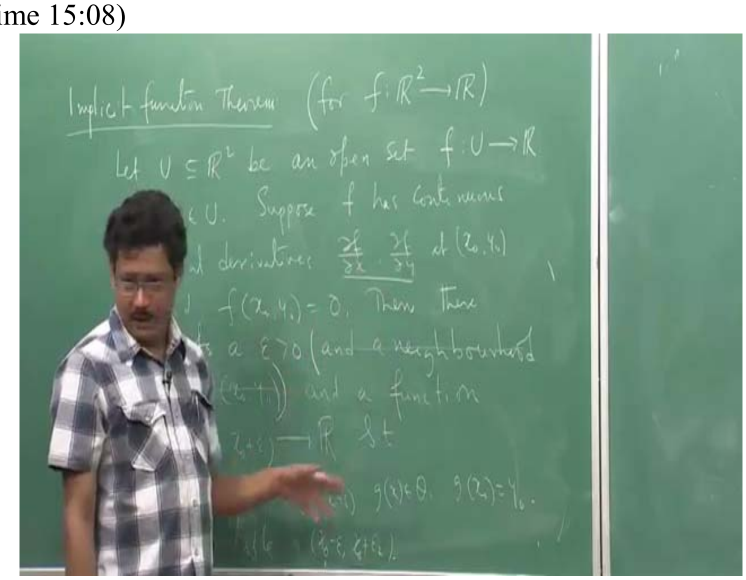


so that in the neighborhood, for a fixed x, if f y is strictly increasing or strictly decreasing function of y, and non-zero...

Oh I have not put this assumption this here, somehow I missed one assumption, right

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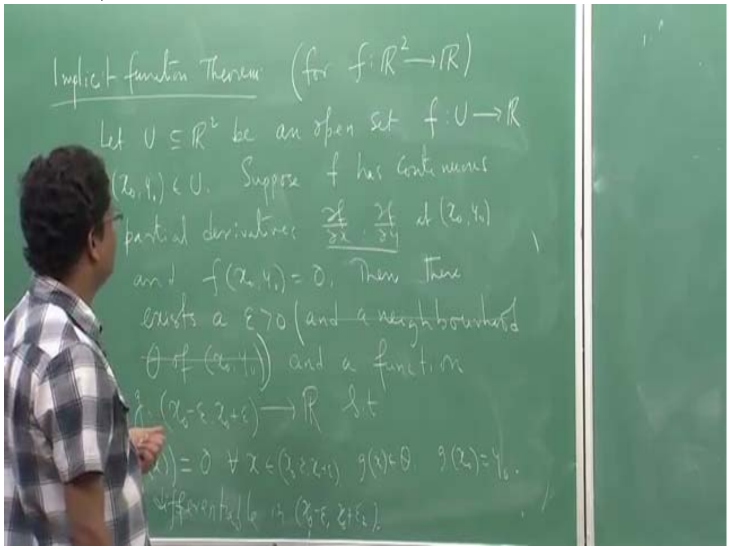


Partial, partial derivative and most important assumption, let us assume”del f” “del y” at “x naught” “y naught” , sorry, sorry, sorry this is greater than 0, or less than 0 does not matter But this was part of our assumption, we need a normal.

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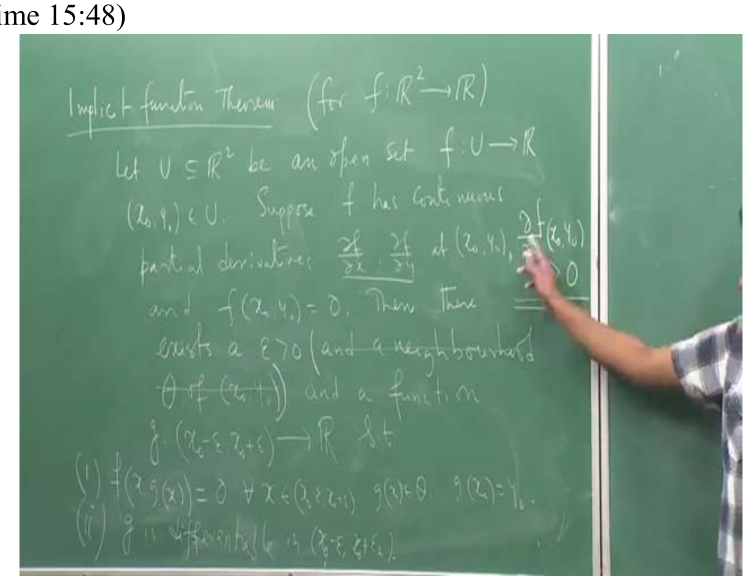


So this is the actual statement

Ok, so we need , since it is greater than 0, it will maintain..continuity will tell you it will maintain the same sign and we know how to,

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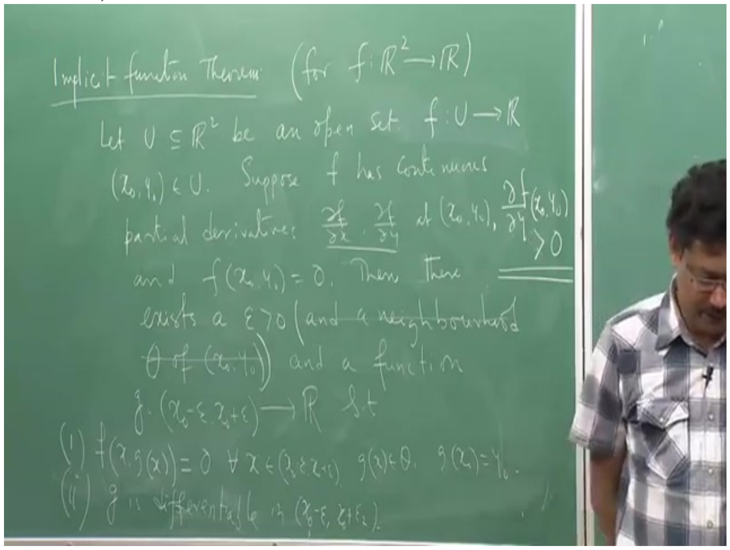

actually go about it.

So let us write down the proof formally

what is the step 1?

Well, what we do,

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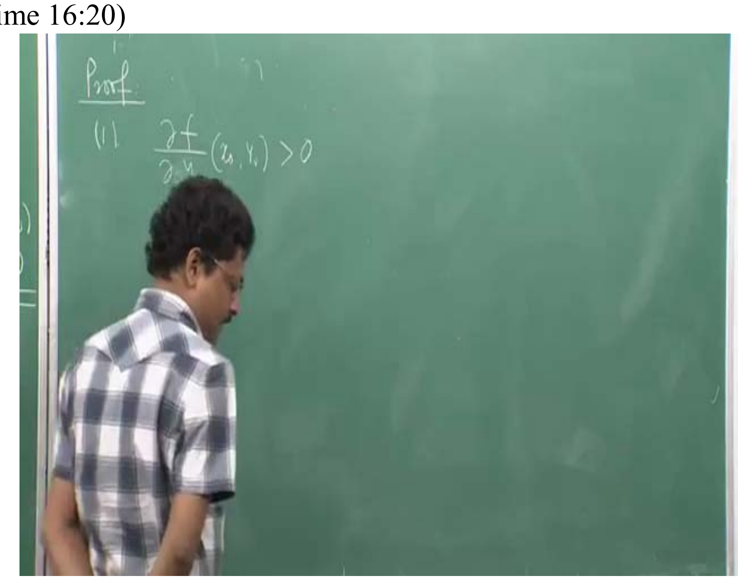

I have ”del f” “del x” “del y”, “x naught” , “y naught” is bigger than 0

So I can choose a close interval.

I can choose an open interval first

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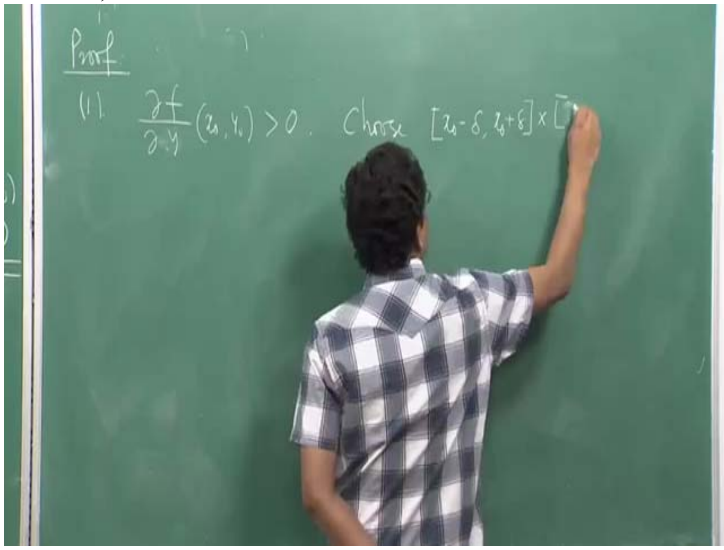


and then inside that, I can choose a closed interval

You will see why I need a closed interval.

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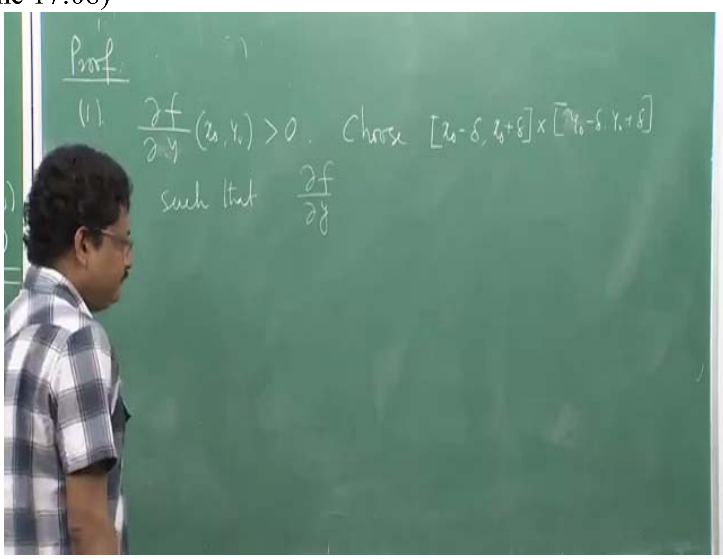


such that"del f" "del y" at x, y is let us say bigger than some number b, Ok for all closed interval, give it a name,

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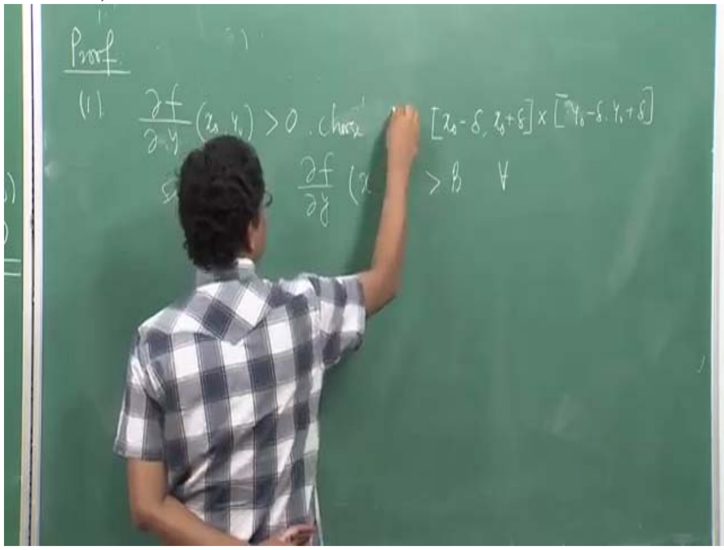

choose this interval, what do you call it, how do you call it D, small b here, for all x, y in D

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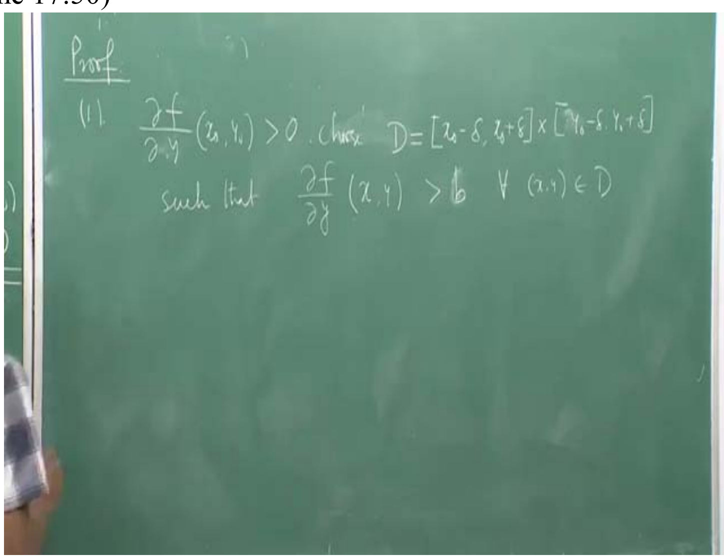


, this is by continuity

And”del f” “del x”, (x,y) is

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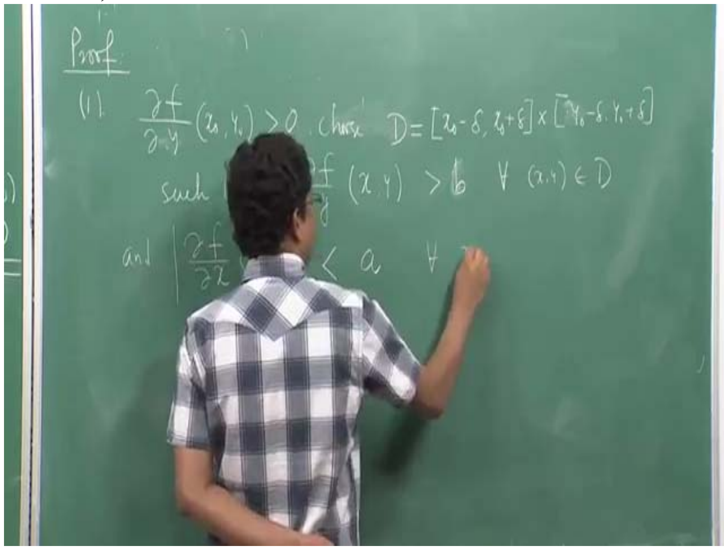

less than some number a for all x, y, right in D

It is a continuous function,"del f" "del x" in a closed interval, so it will have a maxima and a minima,

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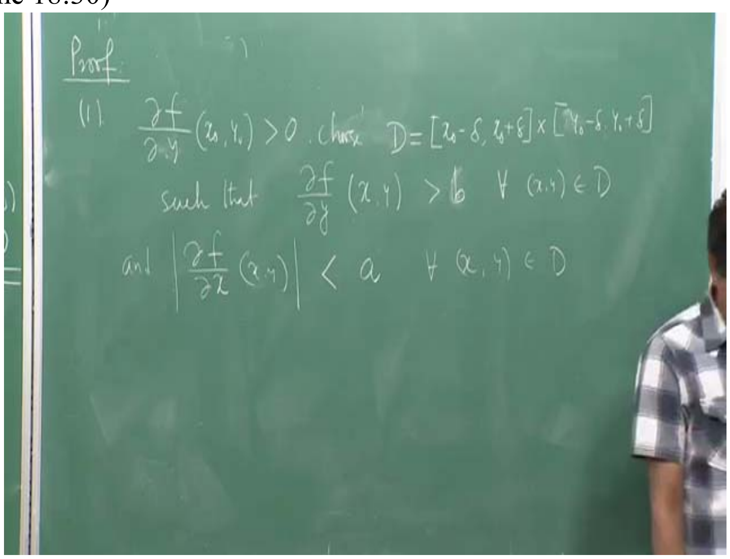


so I check the maxima and there it is the minima

Ok?

Ok, so let us look at f of "x naught", "y naught" minus delta minus f of "x naught" "y naught" and apply MBT, apply MBT for "y naught" , "x naught" is fixed.

So MBT on "y naught" variable

What is it, this is equal to...how much..."del f" "del x" at some point "x naught" y 1 into uh...Ok I can apply MBT to any x Ok.

Ok, not “x naught”

So x minus “x naught” plus dell f “del y” at “x 1” “y 1” into “y naught” minus delta minus “y naught” , that is minus delta, Ok

for x in this interval, “x naught” minus delta, “x naught” plus delta

Now what is f “x naught” “y naught”?

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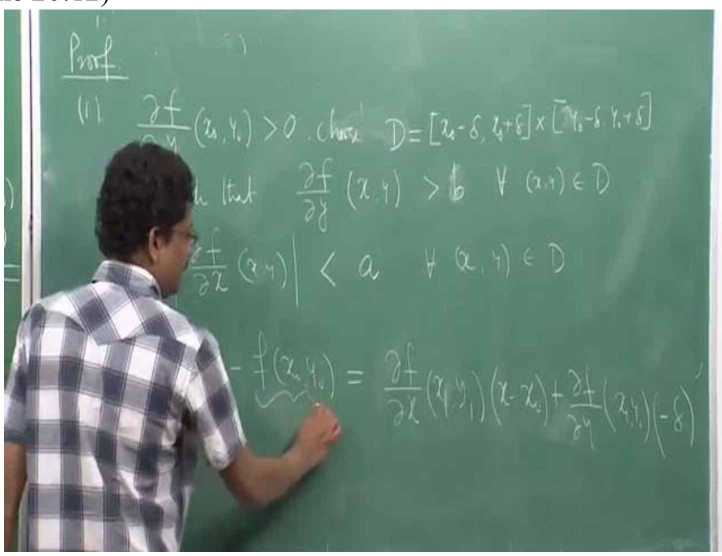


This is zero

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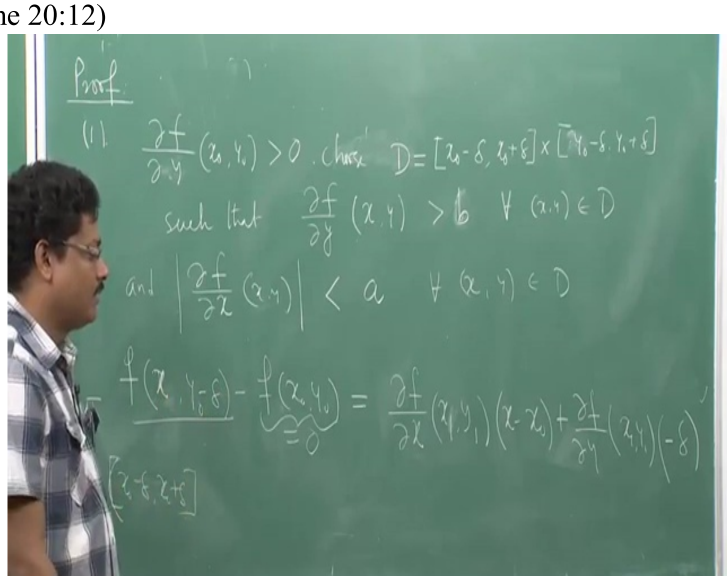


So left hand side is just this fellow, f of “x naught” “y naught” plus delta Ok?

Now what I will do, I will make this delta

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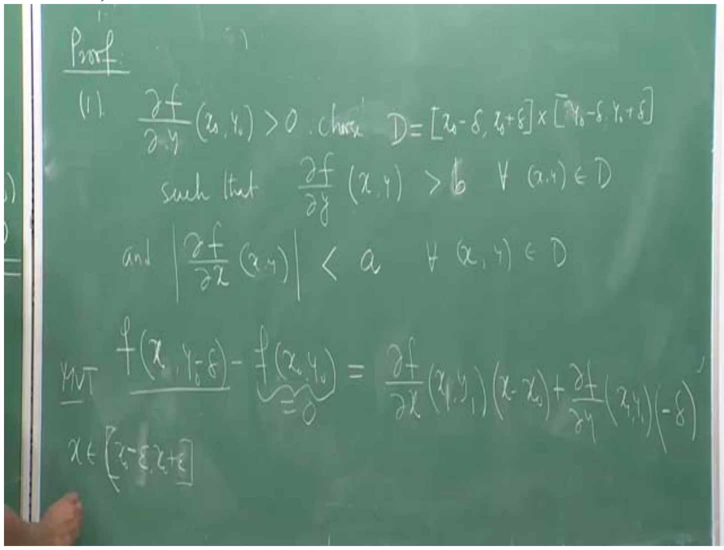


, I will choose a smaller interval

"epsilon" is less than, which "epsilon", "epsilon" is less than delta such that I still have"del f" "del x" is less than a

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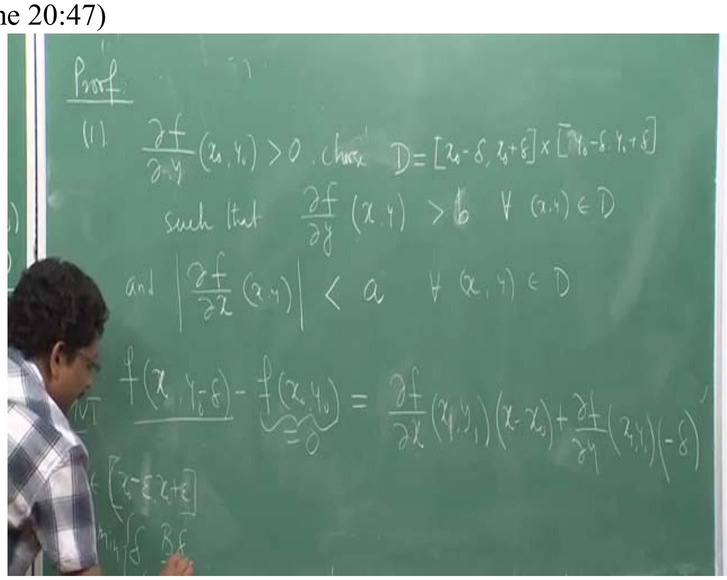

And I will also make "epsilon" minimum of b delta by a

Then if you look at this, this fellow is less than

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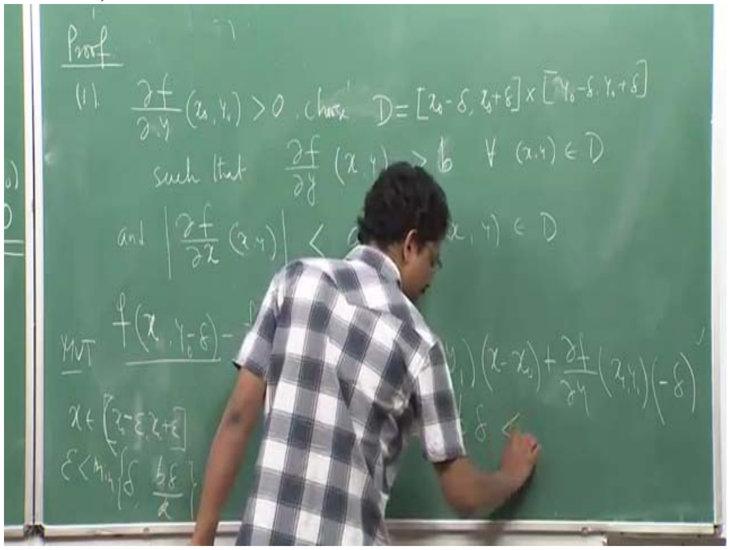

a “epsilon” less than equal to a “epsilon” or strictly less than a “epsilon” minus b delta and this will be strictly less than 0; it will be less than or equal to that...less than, yeah, this is strictly less than

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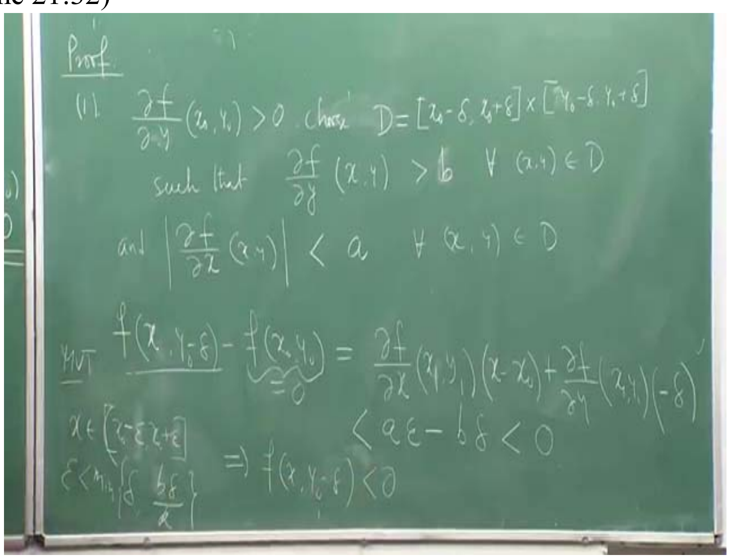


So what is this, this is f of (x,y) naught minus delta is less than 0

So I could have..I can apply MBT for each x in “x naught” minus delta to “x naught” plus delta but I make smaller interval such that I get f x, for x in this smaller interval, “x naught” plus x minus “epsilon”, “x naught” plus “epsilon”, “epsilon” is less than delta and less than this fellow such that this is less than 0

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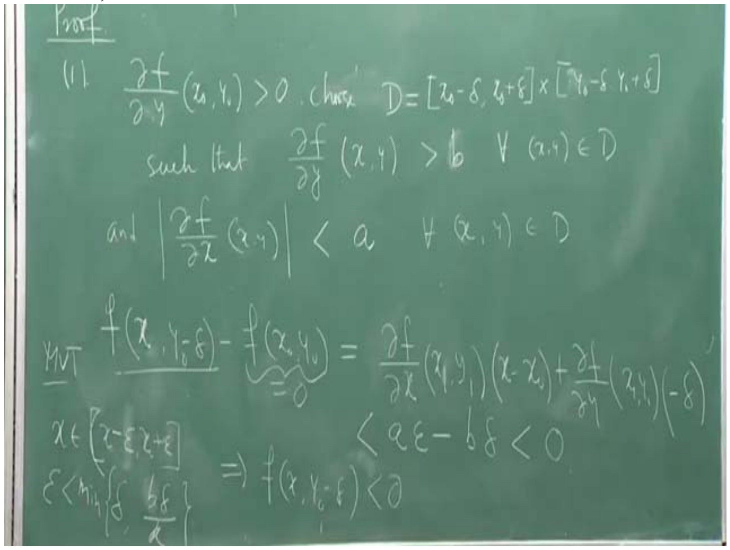


Now a similar calculation, so this is “x 1” “y 1” you know, how I get....”x 1” “y 1” is in the

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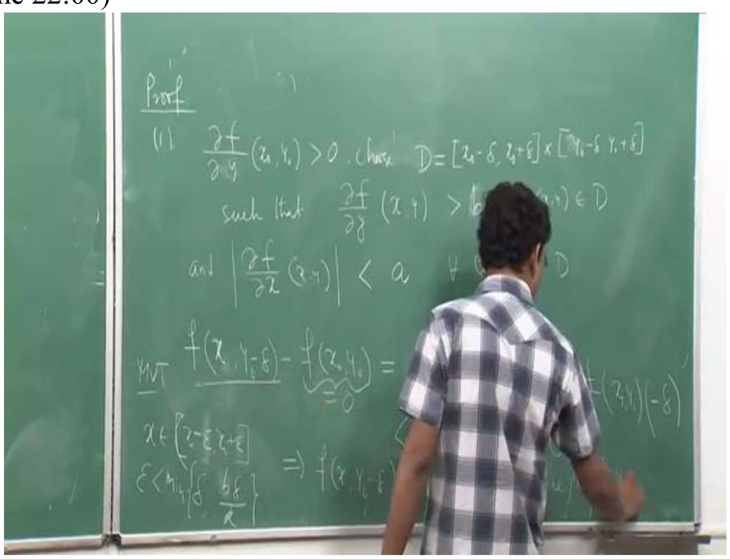


line joining (x,y) naught minus delta and “x naught” “y naught”

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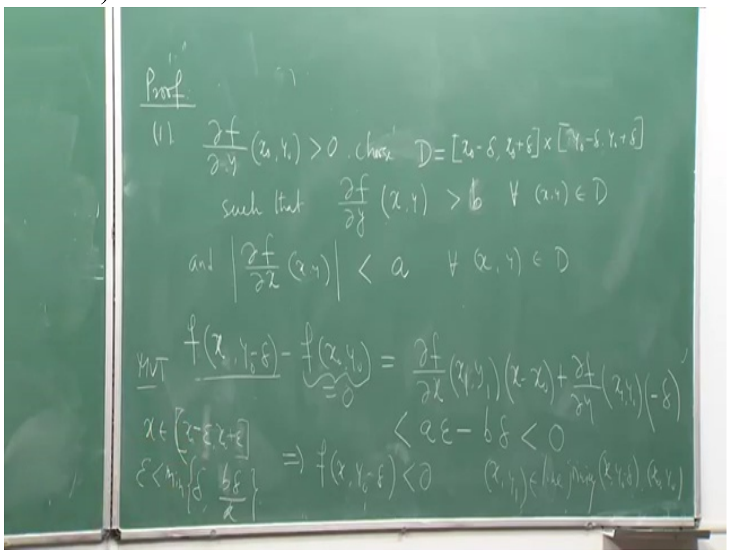


So I have to play a little strict, I have to shrink the interval there

Now what I do, so similar calculation will show

it will follow that f of (x,y) naught plus delta, this will be bigger than 0 for x in “x naught” minus “epsilon”, “x naught” plus “epsilon”

So plus delta bigger than 0, “x naught” minus delta bigger than 0, so therefore

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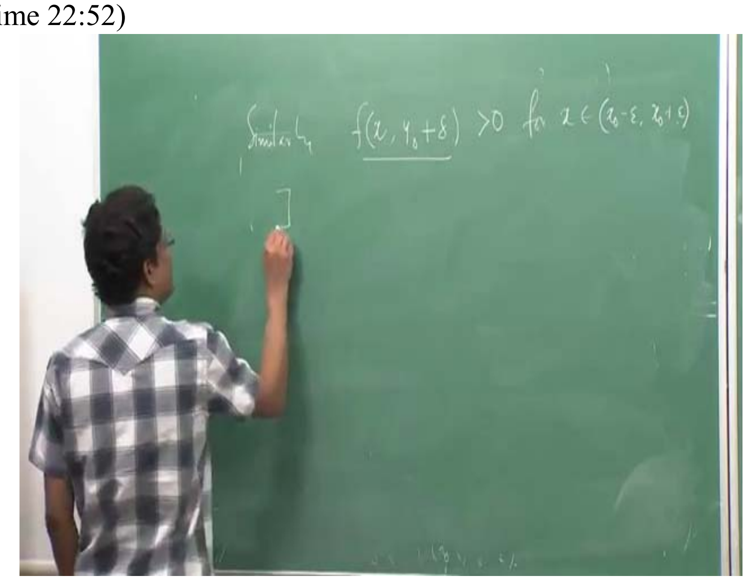


there exists unique, why unique because for fixed x, “x naught” minus “epsilon”, “x naught” plus “epsilon”

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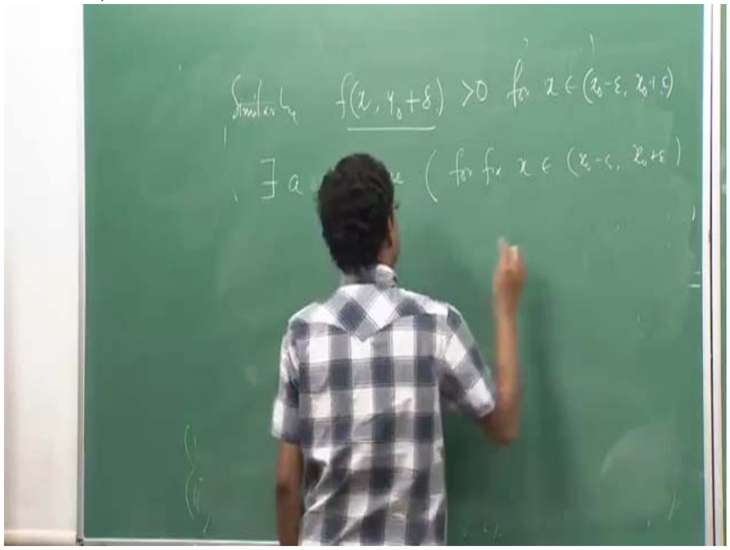

if f x of x, y is strictly increasing

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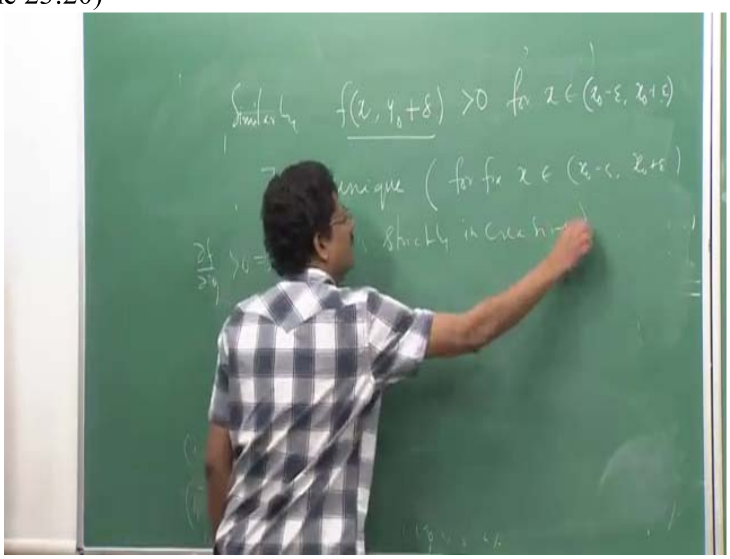

this is implied by"del f" "del x","del f" "del y" bigger than 0

There exists a unique y in the open interval "y naught" plus minus delta, "y naught" plus delta

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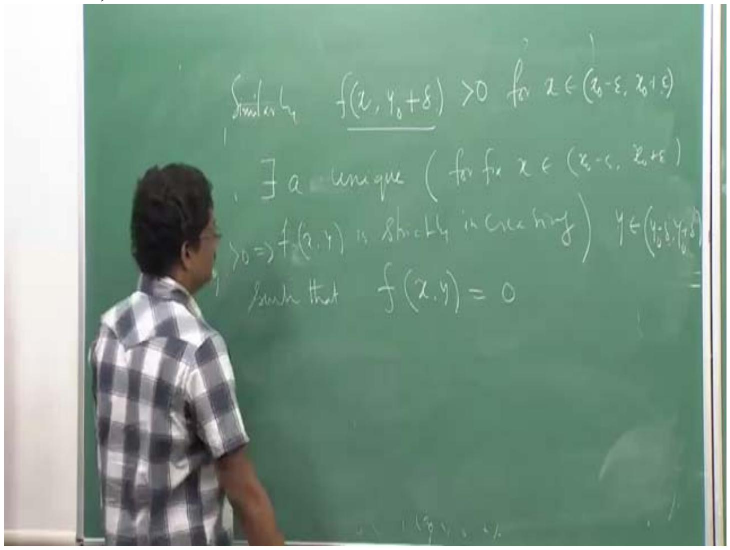


such that f of (x,y) equal to 0

So I define g of x equal to this y, x in this interval

So I got this interval

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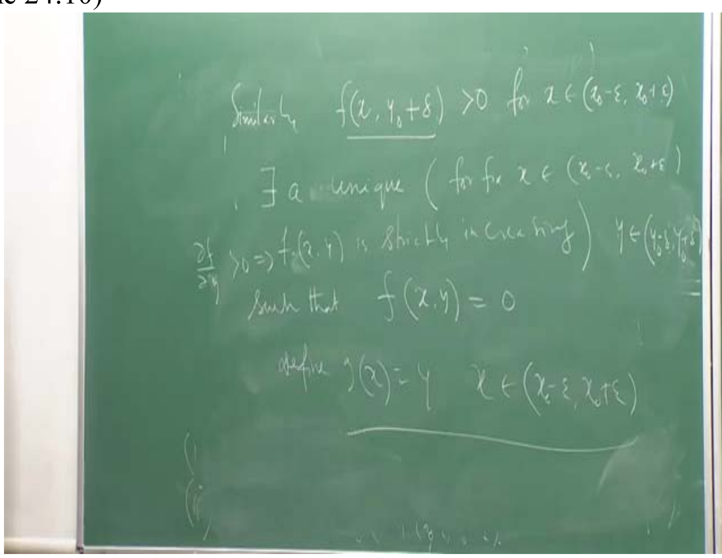


So I got definition of g

Only thing remains to prove is, differentiability of g.

But for differentiability, we will show continuity and then differentiability, I mean both will come. Ok?

So I got the first part, right?

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Second part, differentiability of g

Well, let's check "x 1" and "x 2" in this interval and let us put "y 1" equal to g of "x 1" and "y 2" equal to g of "x 2"

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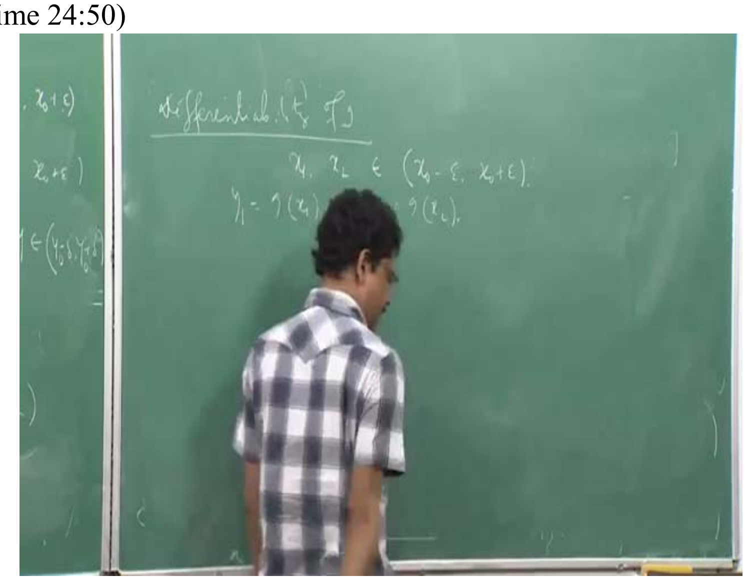

Ok?

Now look at this fellow

f of "x 1" "y 1" minus f of "x 2" "y 2"; what is that?

Both of them are on the level surface, so this is 0, this is 0,

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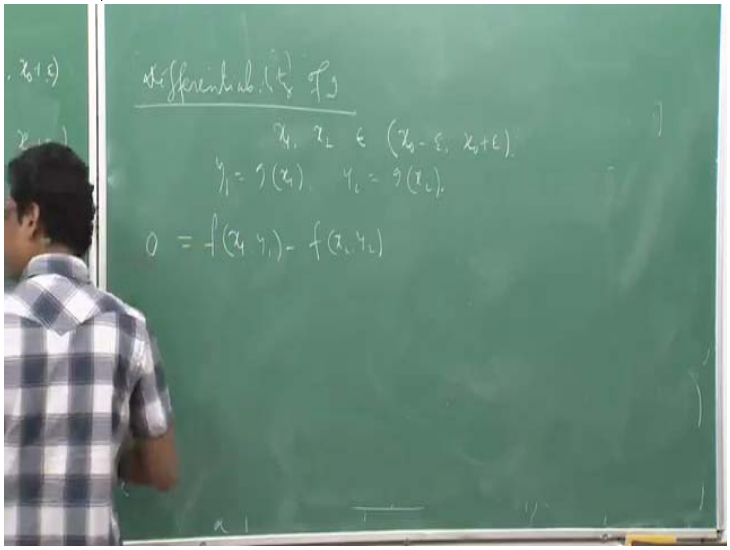

this is 0

Equal to....I apply MBT again

“,”del f” “del x” at some “x 3” “y 3” at “x 1” minus “x 2”, Ok,

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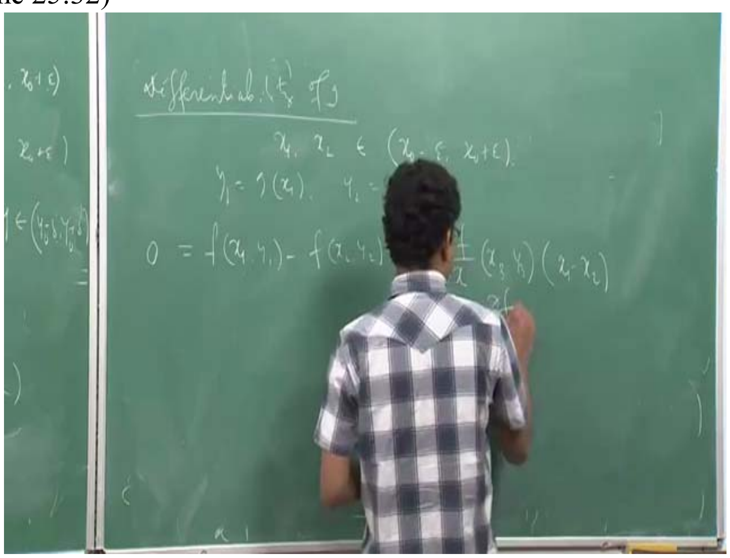

plus”del f” “del y” at some “x 3” “y 3” at “y 1” minus “y 2”

Again, same thing I have to write,

“x 3” “y 3” in the line joining

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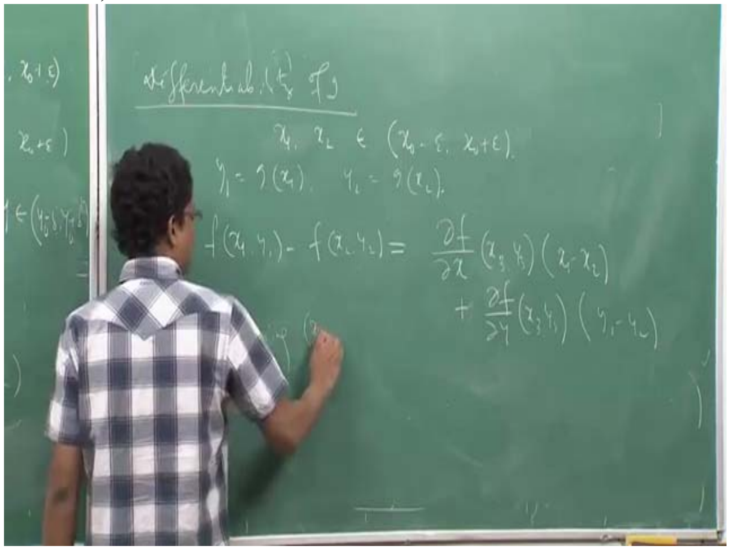

“x 1” “y 1” and “x 2” “y 2”

What does this say, this side is 0

This fellow is less than a, so this will say g of “x 1” minus g of “x 2” which is equal to “y 1” minus “y 2”

Bring everything to this side

is equal to minus”del f” “del x” “x 3” “y 3” divided by ”del f” “del y” “x 3” “y 3” “x 1” “x 2”

Take mod, take mod, take mod here

Ok

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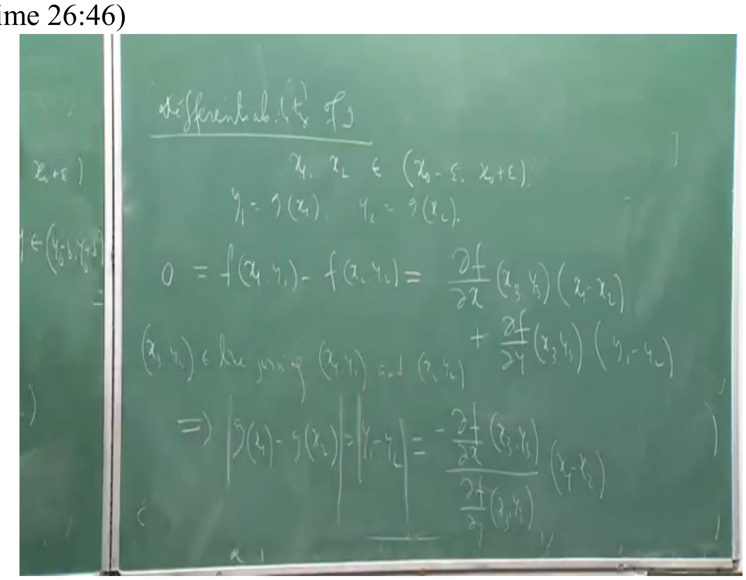

Keep it

This fellow is less than a

This is greater than b

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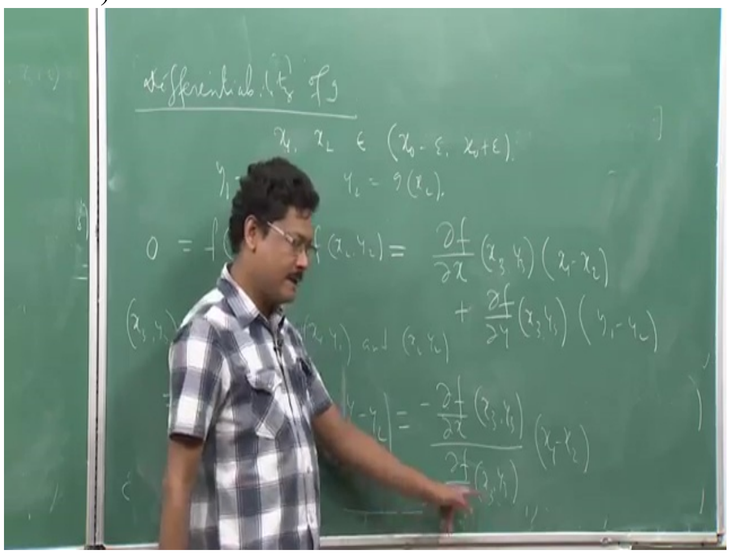

So 1 upon this is less than b

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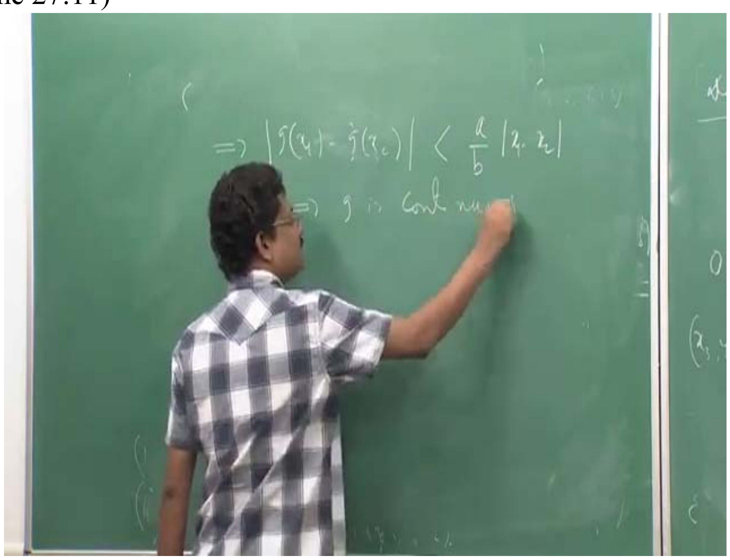

This of course implies g is continuous, right?

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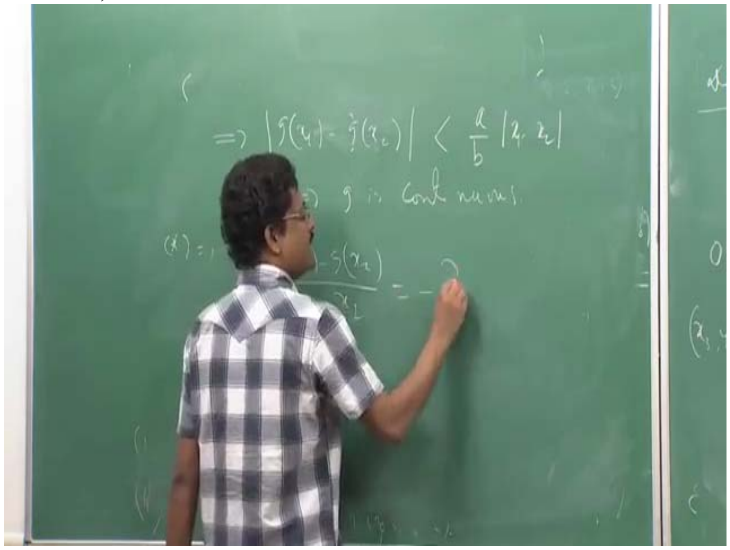

And just, for differentiability, it also implies, g of “x 1” minus g of “x 2” divided by “x 1” minus “x 2” equal to minus ”del f” “del x” at “x 3” “y 3”

“,”del f” “del y” “x 3” “y 3”

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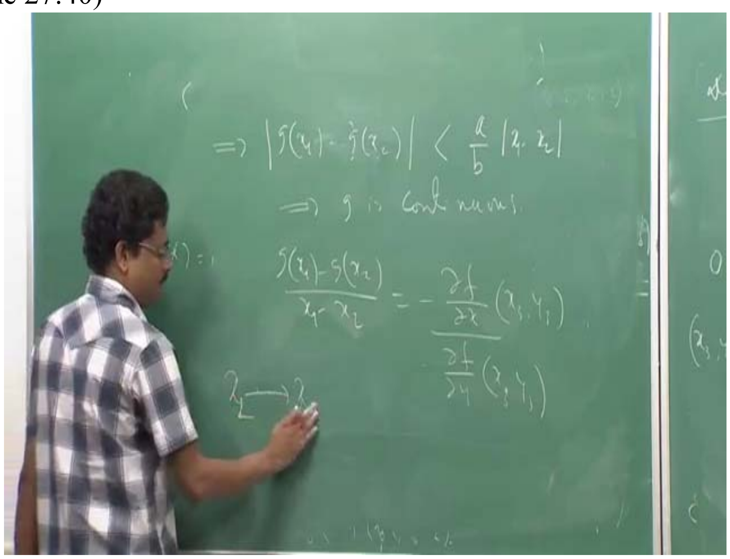

Ok?

So suppose “x 1” goes to “x 2” now.

I take limit, “x 1” goes to “x 2” or “x 2” goes to “x 1” let’s say

What will happen?

Limit “x 2” goes to “x 1” left-hand side, which is actually derivative of g of “x 1”, this is equal to ...

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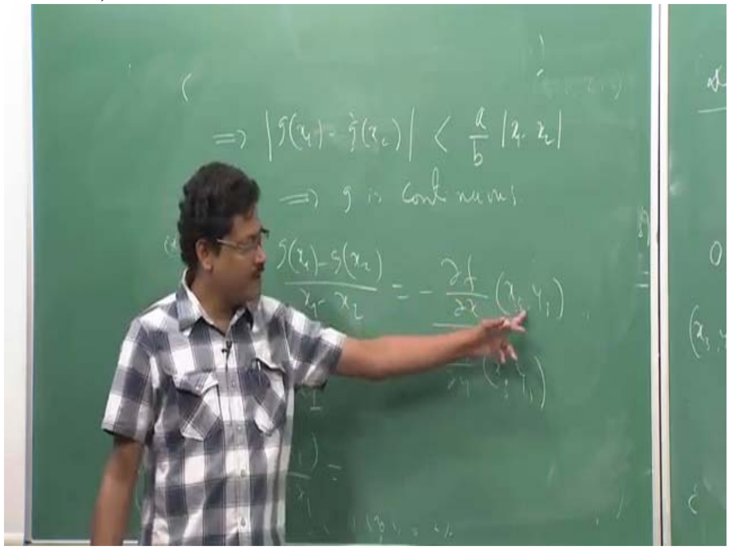

“x 1” goes to “x 2”, “x 3” “y 3” is in the line joining “x 1” “x 2” so “x 3” “y 3” will also collapse to “x 1” “x 2”

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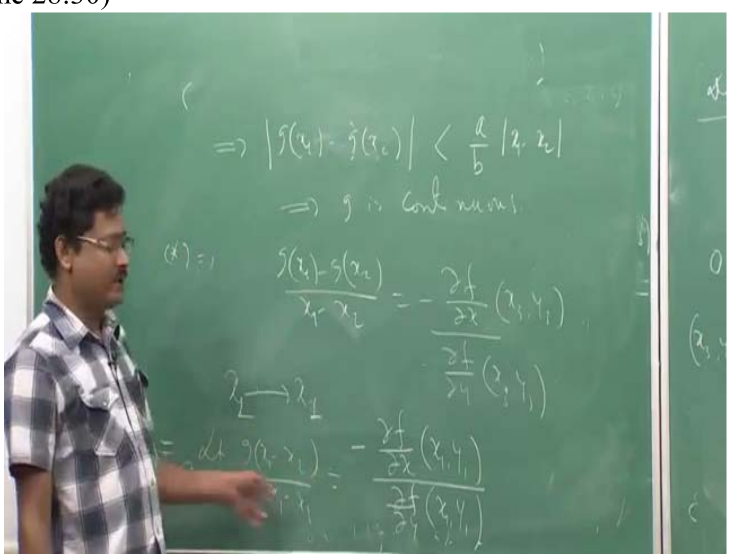

So this is minus ”del f” “del x” “x 1” “y 1” by ”del f” “del y” “x 1” “y 1”

This is non-zero,

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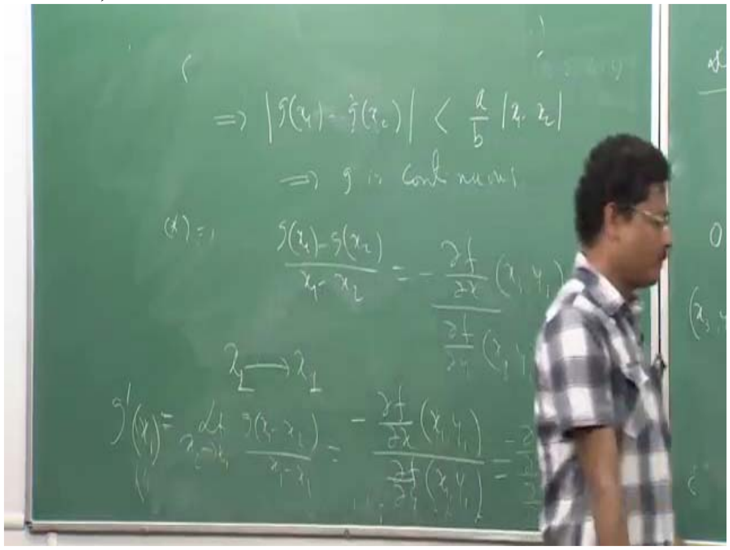

so this exists and g is differentiable

Not only differentiable, we actually get...what it is...it is minus ”del f” “del x” “x 1” g “x 1” divided by” del f” “del x” “x 1”. That’s it. That is the proof of implicit function theorem.

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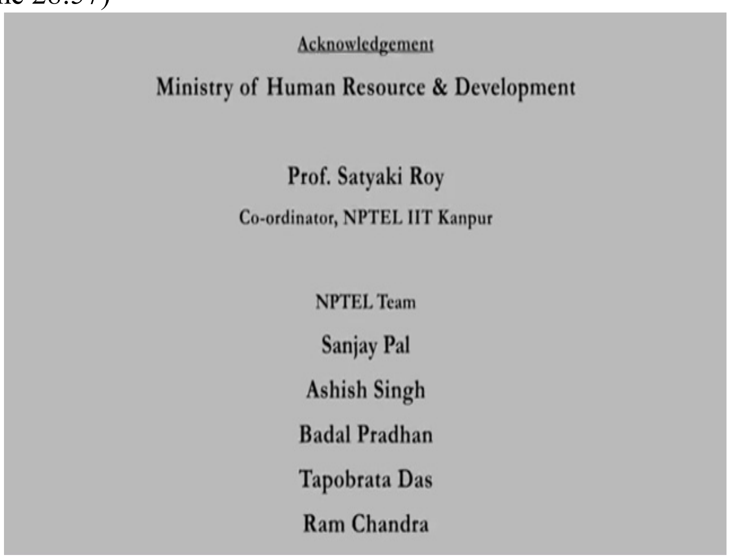

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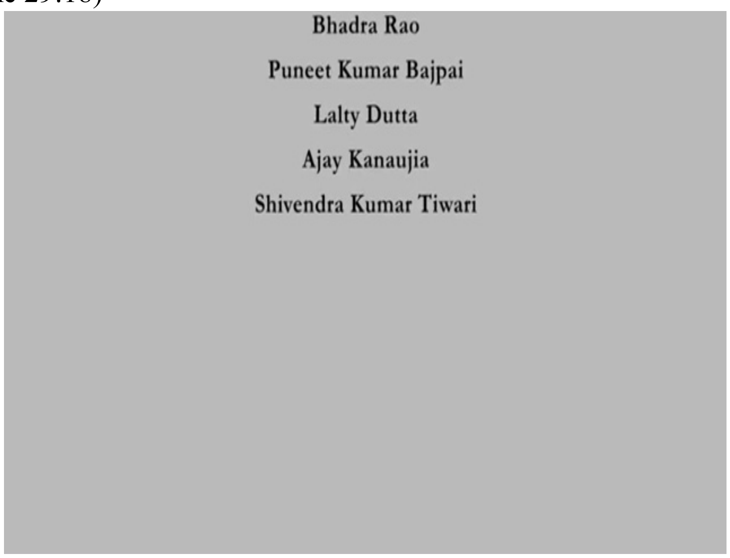

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