Differential Calculus of Several Variables
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Lecture Number 15
Specialisation to Functions of Two Variables

Okay. So let us continue, from where we, umm, left the proof yesterday. So we wrote this theorem for the maxima, minima, saddle point test in terms of the Hessian matrix. Instead of calculating the i-n values, we just look at the principal minors of the, umm, Hessian matrix, and then decide, we'll be able to decide the (some) some critical point is a maxima, minima or a saddle point. So what was the theorem? So I have the same setup, f is from an open set to R, P is a saddle point in U.

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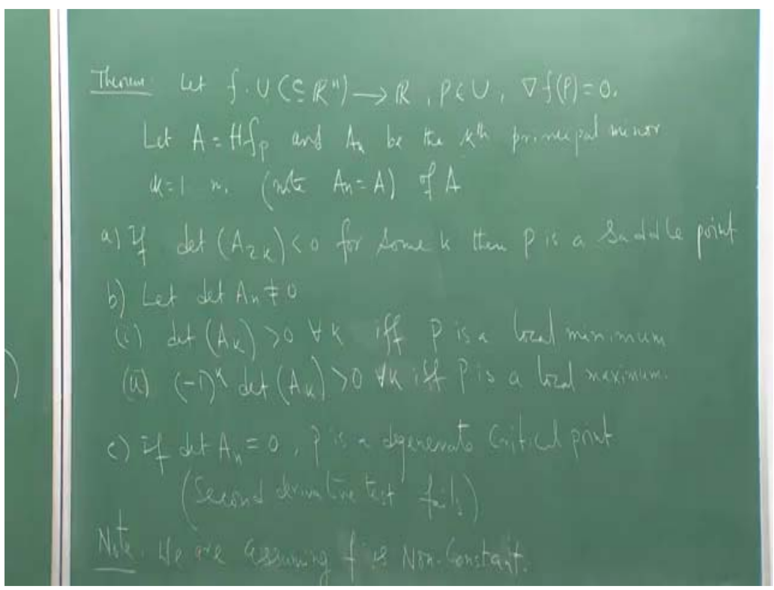

Instead of writing Hfp repeatedly, I'll just denote it for this theorem A equal to Hfp, because once P is fixed, f is fixed, I can just call it A, and Ak be the Kth principal minor of A. So you've defined Kth principal minor, that is a first K row and first K column you take and leave rest of the part. And of course you can see that An is equal to A, nth principal minor is the entire matrix A. So what is the test?

Well, if, for some two case of even number, the principal minor A2k, the determinant is 0, then P is a saddle point. Now, if we have determinant of A not equal to 0, that is to say, P is a non degenerate critical point, that is An is invertible matrix, the Hessian is an invertible matrix at P, then P is a local minimum, if and only if every principal minor of Ak has positive determinant. Similarly, P is a local maximum, if minus 1 power k determinant Ak for every K is greater than 0.

And of course, if determinant of An is equal to 0 then P is a non, P is a degenerate critical point, and in that case we cannot conclude anything about 'second derivative test'. Anything from 'second derivative test'. 'Second derivative test' fails, we have to do something else there. We have to go for higher derivatives or maybe we have to look, do something by observation. Umm, I forgot to note something yesterday, that actually we're dealing with a non constant functions, because constant function, grad f is 0 for (every) every P, so there is no test for that.

And so actually I can add here, that P is a strict local minima, P is a strict local maxima, that is, there's no point nearby in the neighborhood where it is equal to fp. Okay, so what about the proof? In the proof, let's in 'a' part. So let us have determinant of A2k less than 0 for some 2k between 2 and n. I've denoted P with these co ordinates, P1 to P2k, 2k plus 1 Pn, and Q is P1 to P2k, so I've chopped off this '2k plus 1 to Pn' part.

Then we'll define this open set in R2k, which is x1, x2, x2k, (())(03:30) such that x1, x2, x2k, (())(03:34) with P2k plus 1 to Pn. See that P is fixed, so these co ordinates are fixed, that belongs to U. And we considered this function g from V to R, which is g(x1, x2, x2k) is equal to f of (x1, x2, x2k), this is a free variable, and this 2k (plus) 1 to n are fixed. Then it is very easy to note that since these are fixed, that Hessian matrix of g at Q is del square f del xi del xj, k cross k matrix, which is precisely the 2k principal minor.

Now under (assu) assumption, that okay, we have assumed that all, I think yesterday we wrote down that all (de), all derivatives, all second order partial derivatives are continuous, these are symmetric matrix again, so A2k is a (symme) real symmetric matrix. It has 2k many i-n values, may be with repetition, beta 1, beta 2, beta 2k, some of them may be equal. But I can write, counting multiplicity I can write, 2k many i-n values, and all of you know, the determinant of 2k is a product of the i-n values.

And this is given to be less than 0. Now here comes the catch; that I have even number of real numbers, here, i-n values, whose product is less than 0. So that says, that, there must be one which is positive, and there must be one, because if all of them are negative, since there are even number of terms, the product will be positive. And if all are positive, product is anyway positive. So, I can conclude from here, that there exist i and j, such that, beta i is negative, and, beta j is positive.

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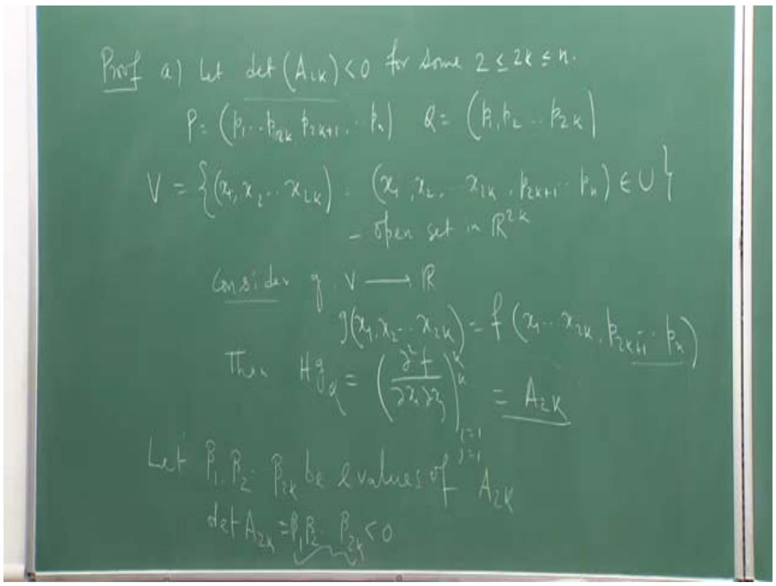

So one i-n value negative, one i-n value positive, okay? Given i-n value, I can of course talk of, I have an corresponding i-n vector. So let, u be a i-n vector for beta i and v be i-n vector for beta j. That is, HgQu equal to beta iu, and (H) here, HgQv is beta jv. What is u and v? u is a vector in R2k, v is a vector, in R2k, right? Okay. Now just consider this vector. Y1 u1, u2, u2k, and 0, upto nth co ordinate. 2k plus 1 to n. And Y2 equal to v1, v2, v2k, and rest of the part 0.

So these two vectors are in Rn. So 2k plus 1 to n co ordinates who are missing there, so I put them 0. Okay. Now look at what happens. Y1 prime AY1, Y1 has this co ordinate 0. So this will be, how much? You just calculate, this will be uH, uA2k, which is, H(Q) u, which is equal to, so this is A, I should have written A here, right? A2k here. Which is equal to beta i u prime u. Now, it doesn't matter if you take u, norm 1 or not, u prime u is always positive, or all the time you can take norm of i-n vector is always norm 1.

Usually we take i-n vectors norm 1. In that case, u one u is 1, which is beta i, this is less than 0. And what about Y2A, Y2 prime Y2, for the same reason this is v prime A2kv, which is beta j v prime v, which is beta j bigger than 0. So what we conclude? Okay now, something is less than

0, something is bigger than 0. I can make, u was norm 1, so Y, Y1, and Y2, both are same norm as u, because I have just added 0. That implies, minimum of Y norm 1, YA, Y prime AY is less than 0, less than maximum v equal to 1, sorry, Y equal to 1, Y prime AY.

Because there is something positive, one vector for which is positive, so maximum is bigger than 0. One vector which is negative, so minimum is less than 0. And you remember, this is precisely the condition for P is a saddle point. Okay? That's the proof for the 'a' part. 'b' and 'b' part one and two is very easy. That's direct from linear algebra. You can check it in your linear algebra notes, that this is simple linear algebra.

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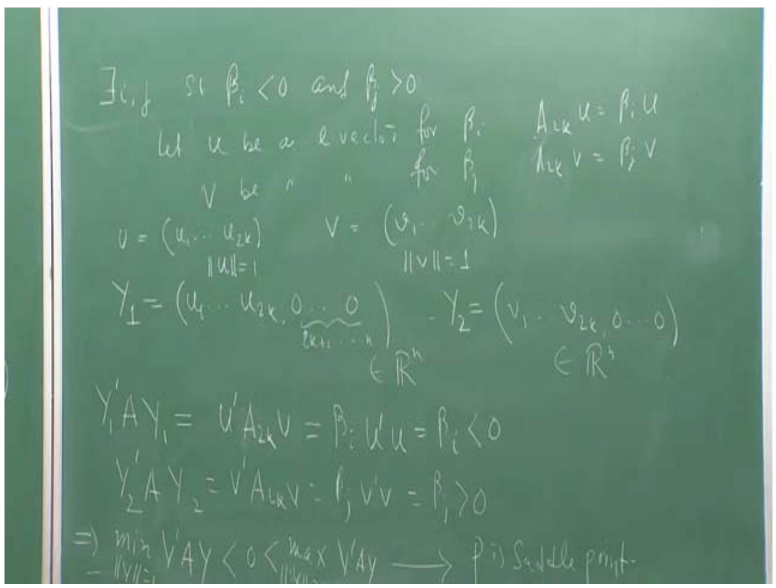

Determinant of Ak greater than 0 for all k implies all i-n values of A are positive. That is to say, if and only if, if and only if, minimum, that the minimum of the i-n values is positive. This is if and only if P is a local minima. Okay? And once we have one part, from local maximum, consider minus f, and we have already observed, Hf, H of minus f at P equal to minus Hf of P. So, A2k for minus A2k corresponding to, the principal minor, the principal minors of, sorry, principal minors of order K corresponding to f, is negative of principal minor corresponding to minus f, and you know determinant of this is simply minus 1 power K, determinant of Ak.

So apply, one. You will get P is local maxima, because a local maximum for f is a local minimum for minus f. Okay? That's the proof. Very good. So this gives you a (ve) uh, one way to check the nature of this critical point, saddle, minimum or maximum. For the rest of (the) today's lecture, and that's the end of this third module, we'll specialize this theorem, with examples of course, to n equal to 2.

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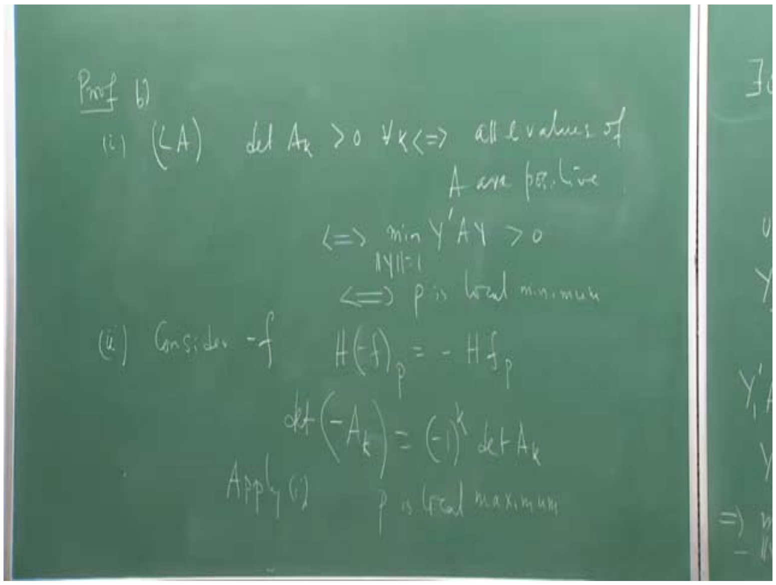

I will directly apply this theorem and see what happens. So, specialized to R2, that is, now I have a function f from an open set U in R2. (())(13:15) are only two variable to R. In R2 we'll see most of the books use some special notation. So if I have some (x0, y0) in U, this grad f at (x0, y0), which is actually del f del x at (x0, y0), del f del y at (x0, y0), people use some special notation for this. This one is written as, this way. So this is notation only.

f(x) (x0, y0) is a partial derivative del f del x, instead of writing this one writes f(x), for R2. So this is, so I'm following some book notations, standard book notation here. Okay? Similarly del square f del x square, this is denoted by notation, again, fxx and del square f del y square equal to fyy, and del square f del xy, del x del y equal to f, first you doing y, so yx. But we are actually assuming that they are continuous in our, so this is fxy, first x then y. So these are the notations, special notations for, usually used in books, they're same thing, right?

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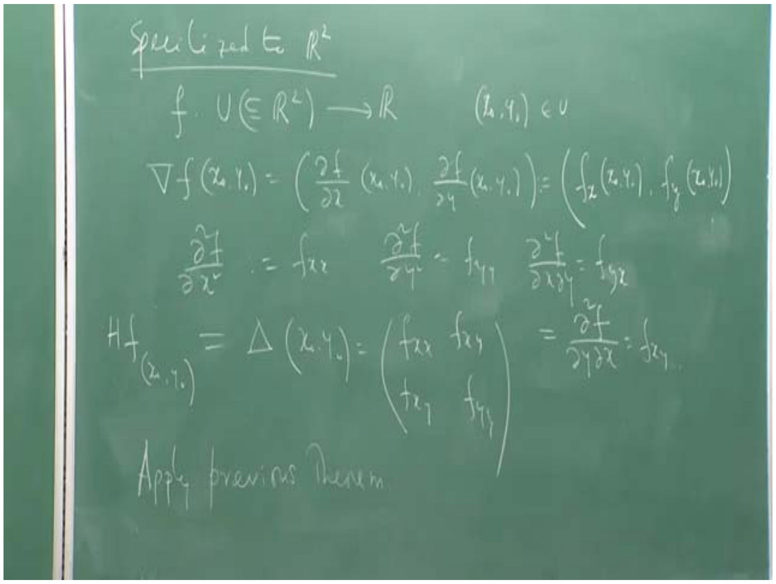

And, this Hessian matrix at (x0, y0) is usually again denoted by this delta (x0, y0), which will be equal to now, in this notation, assuming, second order partial derivatives are continuous, so that the matrix is symmetric. Okay? So what other test says? Apply the theorem. Okay. What will it say, if I apply the theorem? That, you see, A2k even, but there is only one (princ), there are only two principal minor here, delta 1, which is fxx, and delta 2, which is delta itself.

And I should have some delta 2k less than 0 for saddle point, so test says A if delta 2k, but only one possibility, that is a delta itself, is less than 0, then (x0, y0) is a saddle. Okay? Next, delta not equal to 0. All the principal minors are positive, that is, delta greater than 0, and fxx greater than 0, that is if and only if (x0, y0). So this is the first part. So this was the assumption b, so one, if and only if (x0, y0) is local minimum.

Similarly, delta, what will happen? All the principal minors are; and what was the condition there? Minus one power K, remember, so maximum one power K, delta Ak. So now here I have delta 1, K equal to 1, so I should have fxx less than 0, and, 2, so minus 1 power 2, so this was the condition, is 1, so delta must be (gre) and that is, second (principal) minus delta itself, it is greater than 0, if and only if, (x0, y0) local maximum. So you just go back to the theorem, apply it, you get this result.

And of course, third condition is there. c, delta equal to 0. I should have written (deter) determinant everywhere. Determinant, determinant, determinant. Delta equal to 0, then. Okay.

So this is so easy, to remember. So, (max) delta determinant greater than less than 0, saddle. Both the cases, determinant is, determinant is positive. And, so it is determined by the sign of fxx. fxx greater than 0, local minima, fxx less than 0, local maxima.

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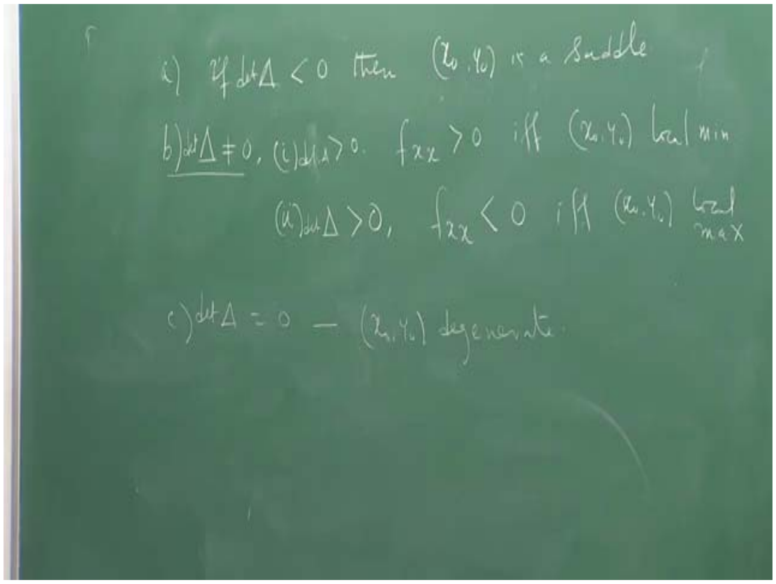

Now, I will do two quick example. Instead of doing, just writing one function, I do it with some problem, (which) where we can apply these things. So this is a problem I found it in some one of the books. Find minimum distance from the point (0, b), b greater than 0, that is, on the y axis, to the parabola x square minus 4y equal to 0. So what is the problem? Here is the point (0, b), this is b, okay? And here is a parabola y, umm, y equal to x square by 4, okay.

I want to find (so) so you can draw different lines from this to the parabola, different points. You have to find the distance, that is, minimum distance, that is, the length which (poi) which (le) length has the minimum distance. So for that, I have to find out a point on the parabola, for which this distance between these two points are minimum. That's what we have to do. So what's, how do I start? So let (x0, y0) be such a point.

Be the point on the parabola, where, minimum distance is attained. Okay? Now what is the distance? I have to minimize this distance. Distance from (0, b) to (x0, y0), which is equal to x0 x square, any point on the parabola, x square plus y minus b square, where, y square equal to 1 over 4x square, because it has to be a point on the parabola. Now, if you see, there's a, distance is a positive function, so instead of minimizing this distance, I could as well minimize distance square, that will minimize distance as well, so I can get rid of this root over.

So that I have, some, (so) derivative calculation is easy. So this information will give us, this is x square plus x square by 4 minus b square, which is x square plus x power 4 by 16, minus xx square b by 2, plus b square. So this is a function of x only. So you now see, we've reduced the problem of by minimizing a function of two variables x, y, to a function, minimizing a function of x only. I want you to complete this yourself, because now we have to check, you have to find a point x0, such that f prime x0 equal to 0, and f double prime x0 is minimum, so greater than 0.

Once you do it, you will find there's some fun in this problem. In a sense that, this position of x0 will depend on the position of b. So complete it. And I request all of you to complete it. That will help you in future, in whatever way you understand it. Okay? I have reduced the problem of finding, minimizing by two variable by two, finding the minimum of one variable function. Second example is also interesting, in the sense that, let's see what happens.

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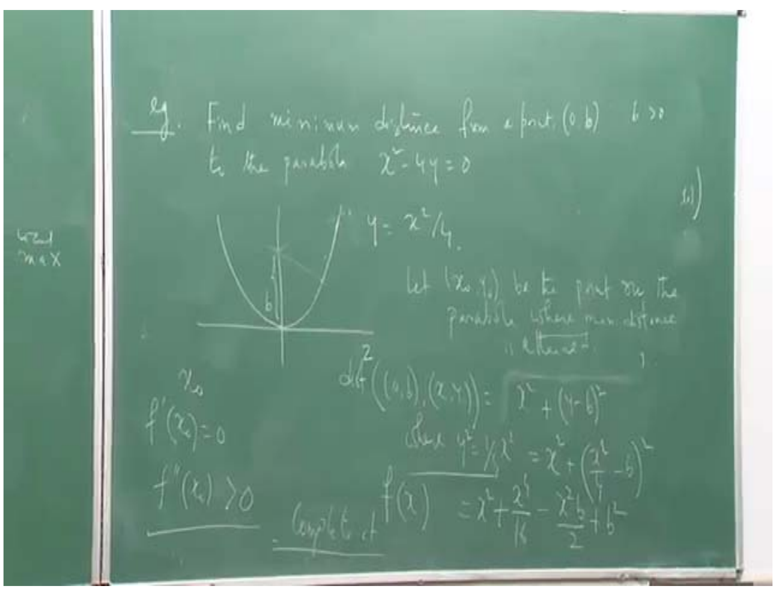

Let's say f(x, y) defined on entire R2. Two variable function. Let's check for minimum, maximum on the entire R2. U equal to R2. Straightforward you can say there's no maximum. Why? Because if you keep on increasing x and y, this function goes to infinity. You take x and y as big as as you f(x, y) you can make very close to infinity, I mean it (())(24:40) as big as you want. So there is no maximum. So I have to check for saddle or minimum. Okay let's see a critical point.

What is grad f(x, y)? This is how much? 4x cube plus 2xy, and, grad f grad y is x square plus 2y. So, for critical point, 4x cube plus 2xy equal to 0, and x square plus 2y equal to 0, you see, that only critical point is (0, 0). That satisfy these two equation. Okay. This is my fx, this is my fy. Let's see what is fxx. I have to go, go for the test. This is equal to 12x square plus 2y, so fxx at (0,0) is 0. fxy is how much?

First, okay fyx I'll have to calculate, first x then y. So fx is this fellow, and then y, that is 2x. So fyx (0, 0) is 0, and I don't need fyy, still I do, it is 2. So delta equal to, delta at (0, 0) is 0, 0, 0, 2. Determinant of delta is 0. So (0, 0) is a degenerate critical point. So here we cannot apply the test, but we can make a little observation at, as I was saying, that if the second derivative test fails, one tries to make observation and see if we can still include some, conclude something.

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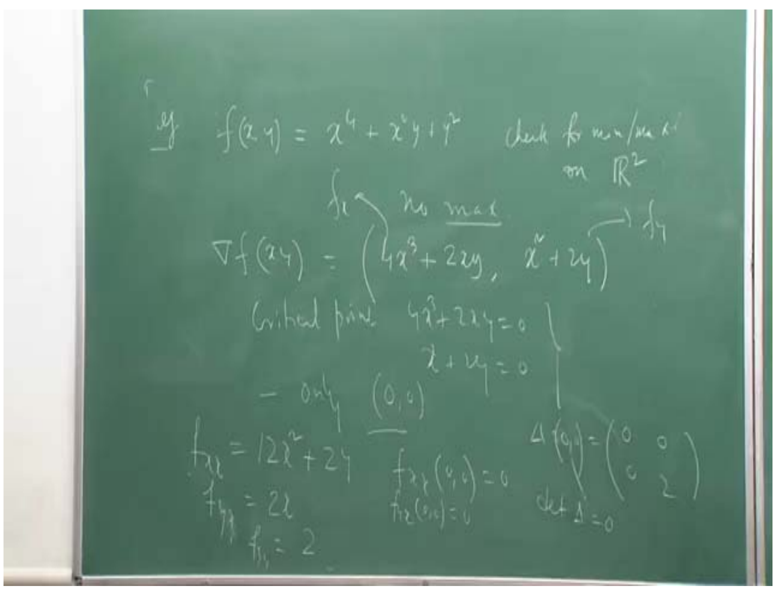

You look at this function. My claim is, (0, 0) is local minima, in fact it is a global minimum, because f of (0, 0) is 0, and you see, x square y, modulus of this, this is of course the positive square root of x4 y square, correct? So this is the geometric mean of x4 and y square. So this is less than equal to, less than equal to x square plus y square by x power 4 by y square by 2, arithmetic mean, which is strictly less than, both are positive, x4 plus y square. That says, x4 plus x square y plus y square is always greater than 0, for whatever your x and y are.

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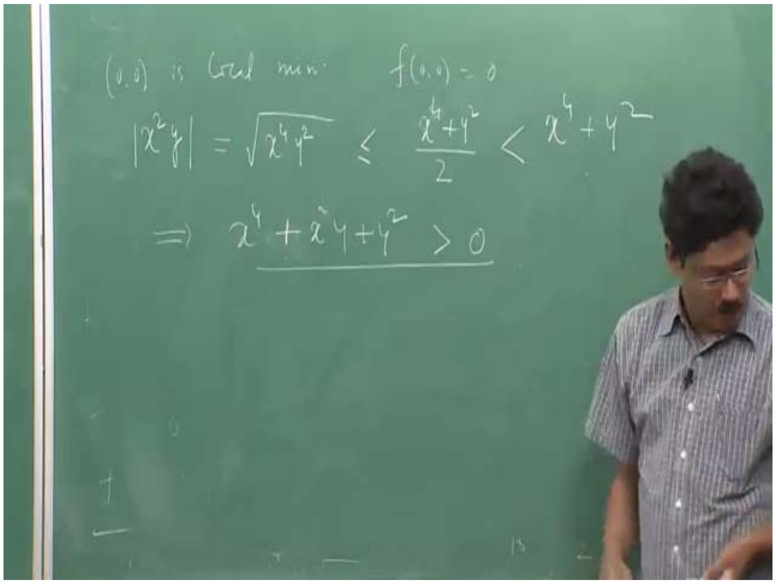

But f(0, 0) is 0, it’s a local minima. So in case of degenerate it is not still that hopeless. You can do something. Thank you. That ends up third module.