Differential Calculus of Several Variables
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Lecture Number 12
Maximum and Minimum

Okay! So, today we'll discuss about maximum or minimum for real valued function. As I said that I have to consider elder because maxima minima always involves comparison. So what is the setup? Again, we have f, so here I emphasize again. Always we take, but here I emphasize. U is open set in Rn, f is a function from U to R. Problem is to find, to find a local maximum, or, local minimum.

You've done this problem for function from intervals too. Here we are going to do it for function of several variable. What is this local part, local maxima or minima? This means f is maximum locally, that is, x naught in U is a local maximum or minimum if there exist a delta greater than zero, such that, for all y in B x naught delta, of course we are inside U, so this ball should be inside U, I should not go outside. For all y in B x naught delta, maximum means, f of x naught should be greater than equal to f(y).

So around that x naught, x naught is maximum. Value of f x naught is maximum. The f is local. It, outside this ball, anything may happen, but, around this ball, there exist such a ball such that this thing happens, and for minimum, or for minimum part, f of x naught is less than equal to f(y). This is the definition. You know, if I have a function of one variable, so recall, if f is a function of one variable, and x naught is a local maximum or minimum, that, then you immediately have something that, if, f is differentiable, at x naught. So when you find maxima minima of a one by whole functional, you first try to find it's zeroes of this f prime, right?

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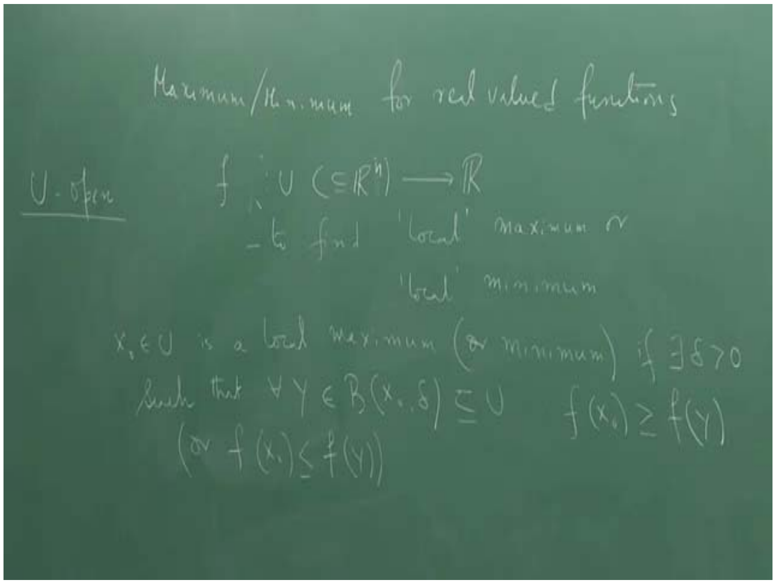

That’s what the way you do it. That you given a function f, you try to find it’s zeroes, or so called stationary point or critical point, and as you know maxima minima it will be f prime x naught it will be zero. This is the first necessary condition. You know the proof also I (prese), but, we’ll prove it for in invariable, just, proof is just one line. Which works for n equal to one as well. But in invariable, actually you don’t even need f to be differentiable. There’s a little bit of difference here. So what happens that, same thing will happen here.

If f from this U in Rn to R, and x naught is local maxima or minima, then, grad f at x naught, this is zero. Zero vector, right? What is the difference? I have not written f is differentiable here. You remember grad f is what. So, directional derivative in a canonical directions. So, as you’ve seen in a very early week, that existence of the, (direc), directional derivative doesn’t make f differentiable. That is grad, this, all this del f del f del xi may exist, f may not be differentiable, but in this case actually, we have that f is even it is not differentiable, then, it is, grad f will be zero.

What happens here if grad f this exist is one variable, if this exist, f is automatically differentiable. It doesn’t happen in invariable, okay? So therefore, this comes automatically for n equal to one. How do you prove it? If I prove this one, then I of course I have proved this one. Okay let’s prove it for, let’s fix one deduction. Fix any reaction ei. Look at the reaction on derivative. What I know?

f of x naught plus t ei minus f of x naught equal to grad f xi, at x naught into t plus mod t into Ei(h), where, Ei of h, Ei of t, t, t; h is, t Ei. Ei of t Ei goes to zero as t goes to zero. Now what happens? Suppose f x naught is local maxima. Okay? Now what happens? This is, and, so there exist delta, such that all y in B x naught delta f(y) is less than equal to f x naught. So if I take in particular t is less than delta, naught t less than delta, then f of x naught plus t Ei, this is in this ball. So this fellow, is, less than this.

So sign of this is always negative. Always negative, for mod t less than delta. Sign of left hand side is always negative. Come to the right hand side. This fellow goes to zero. So t is very small. Sign of this expression depends on sign of this expression; because whatever it be this is, this I can make it very very small, so sign will be depending on this expression.

Now del f, del xi, x naught, this is positive or negative I do not know. Suppose positive. But t I can approach from negative and (beco), positive both side, because I have just, I just need to ensure mod t is less than delta. So I can take mod t at t, negative. So in that case, this will be negative, as well as I can take t positive. So this will be positive. So right hand side will have both the signs. According to t, if this is positive, I can take t positive to make it positive, I can take take negative to make it negative.

So it will have both the sign, and this part will not matter, because this will be anyway, very very small, as I take t to be very small. But this side always remain negative. It cannot happen right? So, the only possibility is, so right hand side changes sign according to t. So this, but this side always negative. The only possible way is, del f del xi at x naught is zero. But this is true for arbitrary i, so grad f x naught is zero.

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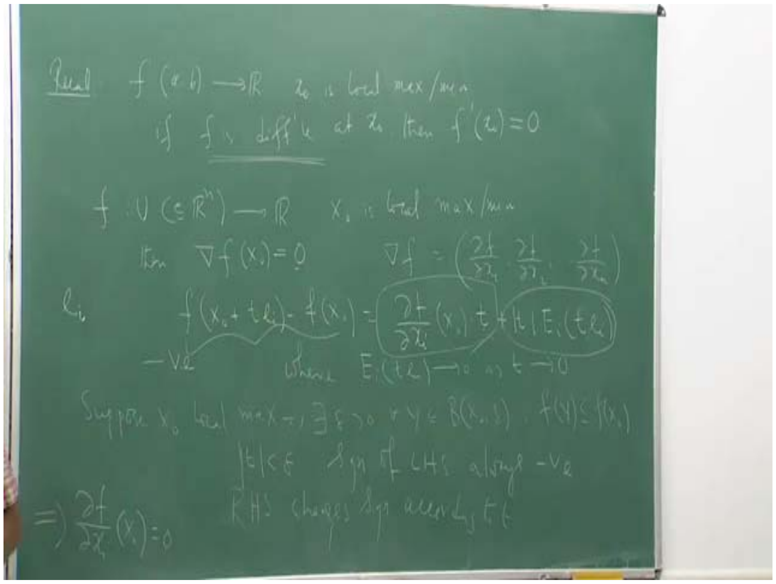

So this is a necessary condition for a local maxima or minima, where grad f x naught has to be zero, and, this condition is not sufficient, and that is the one example that is, one very classical example for one variable. You do it. Let me recall it again, that it cannot, it is not a sufficient condition, that grad f may be zero, but it's neither a local maxima or minima. So you know some classical example of, for f from R to R, this famous function, f(x) equal to x cube. Sorry. This looks like; so f zero is zero.

Should be look nice enough. f prime at zero is also zero. Two x, three x square at zero is zero. But if you take any neighbourhood around zero, this side, if x is positive, this is positive, this is negative. So there is no neighbourhood where f prime zero is bigger than or less than any point in neighbourhood of zero. Right? No neighbourhood, so here I can conclude zero is not max, local maxima or local minima. Similar example I can do for function of several variable.

You can make examples in any n, I'll say we make it for two. f of (x,y) equal to let's say, y minus two x square y minus x square. What is the domain? Domain is entire R two. Now if you look at f of (x,y), is y minus two x square, y minus x square. y minus x square is a parabola. This is y minus x square. y minus two x square is a parabola again, but this is little steeper. This is y minus two x square. And look at zero.

f zero, zero is zero, and you calculate grad f at zero, zero, this is also zero. But what about zero? Okay let's take any neighbourhood around zero. Any delta, doesn't matter which delta. You see, if x and y comes from here, this is above this parabola y x square, so automatically above this parabola, so y is bigger than two x square, y is bigger than x square, so f is positive. Here, it is below this, y minus x square, so it’s automatically below y minus two x square, so this is negative, this is negative, so f is positive.

But in this region, in between region, f is bigger than y (minu), this fellow, y is bigger than x square, but y is less than two x square. So here, f of (x,y) will be negative. See in this shaded region, f of (x,y) is negative. So around zero, zero, you will have points, where f of (x,y) is bigger than zero. What is the value of f at zero, zero? As well as points on here, which is less than zero. So, that makes zero, zero, is neither maxima, neither local maxima, nor minimum. And such a point is called (())(14:43) ‘saddle points’.

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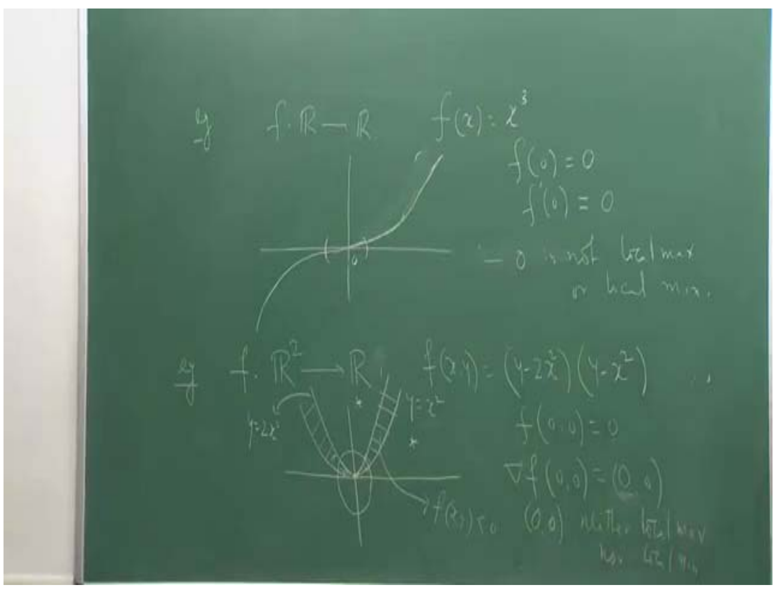

So let me make a definition here. So if, grad of f(x) at zero, then x naught is a possible candidate for local maxima, so x naught is called a ‘stationary point’, or stationary point or ‘critical point’, both the terminology is used. Critical point. If, for every delta greater than zero, there exist y and z in B x naught delta, so, so x naught is critical point, this is the assumption, and if, for every delta greater than y and z, there, every delta greater than zer, zero; so if you look at any neighbourhood around x naught, there exist y and z, such that f of y is bigger than x f x naught, and f of z is less than f x naught.

So condition like this that both the cases are possible that it's neither loc maxima local min, neither maxima nor local minima, x naught is called a 'saddle point'. Saddle means once you climbing the mountain or some on the hills, so there are places where it's plain, so this side you go, this is, you go down, this side you go, you go up, and in between there is plain. So it's called saddle. So for that reason it is called 'saddle point'. Okay.

So this terminology actually borrowed from function of one variable. Now, you know for function of one variable, to actually check maxima minima, you go for so called, second derivative test. So let me recall what you do there. So what you do there? First you check f of x naught equal to zero. Then, you look at f double prime x naught. If it is greater than zero, you call it, then you say it's a local minima, right?

I'm recalling what is (happ), what you have done already, in, one variable. You call it local maxima. Well you have seen the proof or not, it doesn't matter. But this is the what you do, and you see there's a connection. If you look at f of x naught plus h, minus f of x naught, h is the neighbourhood of x, so, h less than delta. Then, if f x naught is a local minima, then this sign will be always positive. f x naught is bigger, less than anything around it.

So sign of f x naught plus h minus f x naught is actually a sign of f double prime x naught. Similarly, if it is local maxima, this will be always negative, that is a sign of f double prime x naught. So, and, this is local maxima minima depends on in the neighbourhood, this is always positive or negative, and that sign depends on the sign of f prime x naught, f double prime at x naught. Why such a thing is happening? Let us try to see one justification for such a thing in case of function with function several variable, with technical value in R, and, that will of course, say, what happens there, the special case of n equal to one.

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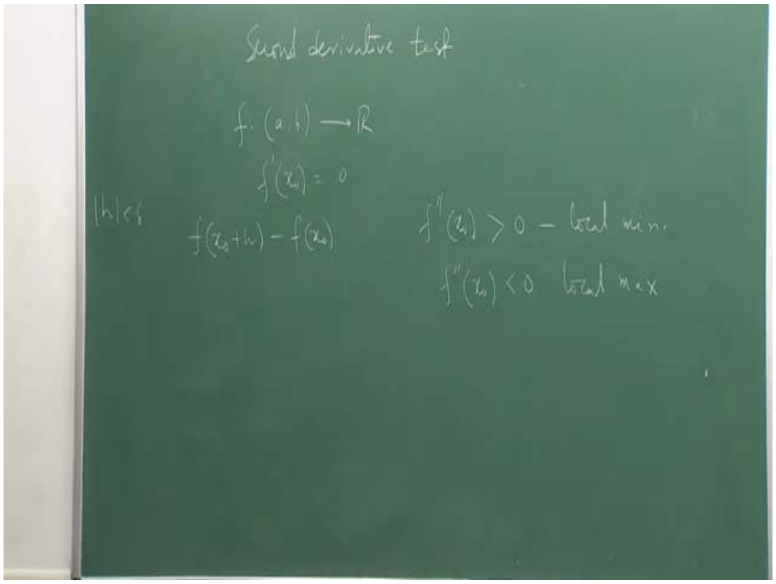

Well, we look at in Taylor's formula, for n equal to two. So how do you write? Instead of y, I write f of x naught plus h minus f of x naught. What is that? If you recall, summation i, so that is actually grad f at x naught h plus half f double prime (z, H). Right? Where, z is a point in the line joining x and x naught plus h, or other way round. Correct? Well, okay.

Now, x naught, if local minima or maxima, local maxima or minima, then we have already wrote grad f x naught is zero. So f x naught plus h minus f of x naught, this is actually half of f double prime (z, H). Now write it, I write it in this formula. Summation i equal to one to n, j equal to one to n, del square f, del xi, del xj at z hj, hi. Okay? So sign of this fellow will depends on sign of this quantity. Here.

Now, suppose, suppose, del square f, del xi del xj are continuous, for all i, j. Then what I do, I write it as, half of. Let me write it in this form. I just subtract what I have added. Okay? So, this fellow is here, half of f double prime at (x, H). Oh, hj, hi was missing. (x, H) plus half of H square E(H). Now look at this.

This is less than half of xi, xj. x naught. I made a mistake here. This is x naught. This was the correct expression. Please correct your notes or if you're taking, if you're looking back, just look back at the expression there. This would have been correct. I just missed it. I hope you followed. Now see what is happening. If I assume this is (())(25:25) continuous, j (())(25:28) in between this interval.

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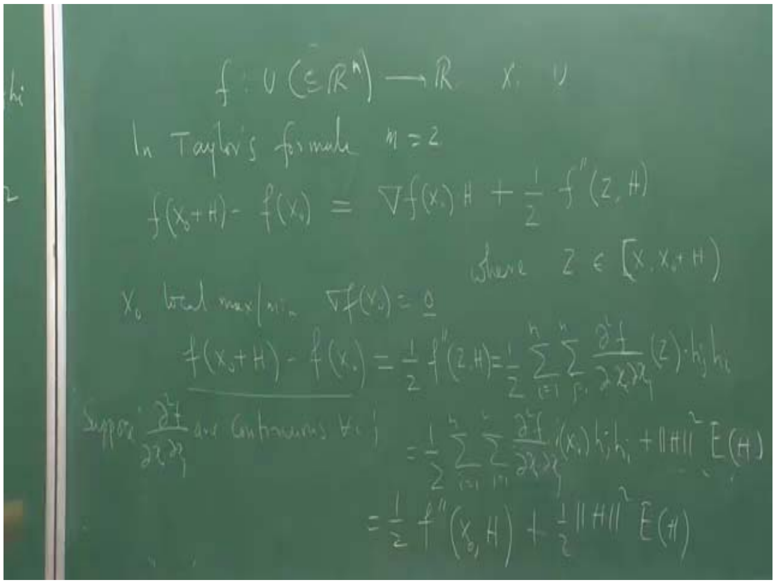

So I can choose H very small, so I choose delta such that if this is small, then, sign of f of x naught plus h minus f of x naught is determined by sign of, okay? So that's it. The, informal explanation of why this maxima minima, the sign is determined by the second derivative. Instead of second derivative, here I have f double prime x naught, H. So next day, next lecture, we'll make this argent, because this is just informally I've written down.

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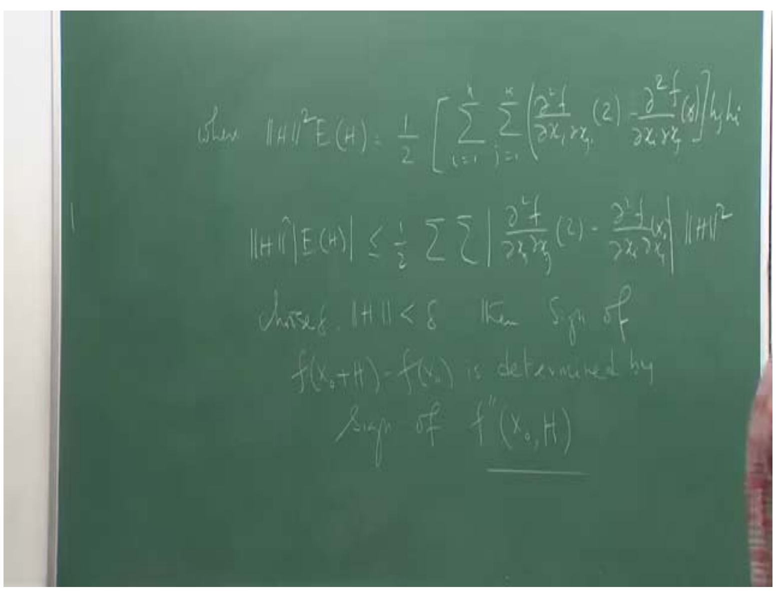

We'll make this argent formal, write it in a better notation and try to derive the test for, actual second derivative test for checking maxima minima. So that is, next lecture. Thank you.