Differential Calculus of Several Variables
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Module 02
Lecture No 10
High Order Derivatives.

Okay, I will come to the last lecture of this module. Here we will discuss about, well in several variables calculus you don't stop at taking first derivative, right. You then take second derivative and third derivative and finally arrive, finally you talk about function which has all derivatives and then you talk about Taylor's theorem and Taylor's formula. We will do that next week for several variables but for time being this lecture we concentrate on higher order derivatives.

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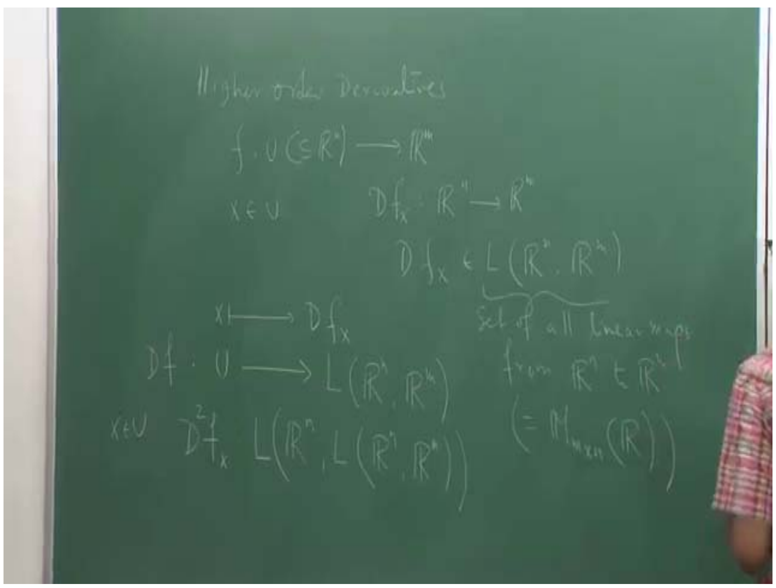

So the same setup, F is from U open connected set, maybe connectedness is not essential for this lecture RM, and we know we can define for some X in U, we can define this derivative, DFX which is a linear map from RN to RM, let me write it in this way. DFX belongs to L RN to RM, so this is the set of all linear maps from RN to RM and all in the, if I write in matrix form these are all, if I fix bases from RN and RM then this will give us all M cross matrixes, real matrixes.

So if you talk about second derivative you must consider this X going to DFX, but now you will see this is the map from U to so DF if I call this map DF, this is the map from U to DFX belongs to L of RN to RM which is itself a vector's space. If you now want to talk about the second derivative that is this square F, so X in U, this square F at X that is a derivative of this map, this will be a linear map from U is in RN to L of RN to RM.

Well, you see it is already matrix value, this fellow is already matrix. Now you are defining maps from RN to matrixes so it will be, again you can say it is a matrix, huge matrix where each entry is a matrix of order (())(3:33). So total matrix will be M cross N into N. So this is very difficult to write down and people still do analysis but let us not bother too much about this thing because I think in matrix form it doesn't help much.

If you want to do some analysis you have to think of some other ways, maybe we encounter or may not in this course, but there is one thing if we take real valued function this becomes very easier.

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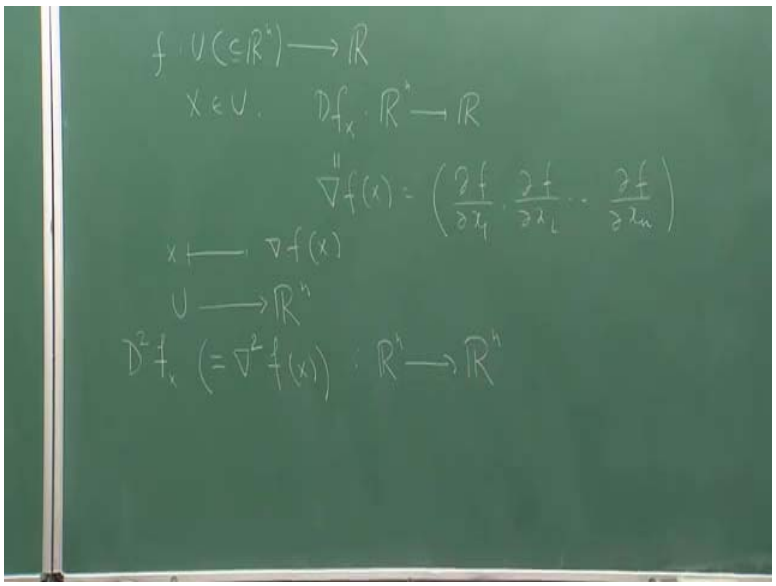

In the sense that suppose of F from U in RN to R, real value and you know actually to talk about differentiable function in F you can write F as F1 F2 FM and you understand the properties or differentiability of each component FI and then you can talk about derivative of F in terms of those. So enough to from all practical purpose if you want to talk about higher derivative you consider F from RN to R and for RN to RM you start from here and then go up with the formula we have it.

But now this DFX for X in U, DFX is a linear map from RN to R, so this is a row vector which we have actually written as, actually it is we have fixed notation Grad F at X which is you know del F dell X1, del F del X2, so on del X del XN, this is the reserved rotation for Grad. So you see now X going to Grad F at X, this now a map from U which is in RN to RN again, so D2 of FX maybe we will write it as a special notation I don't want to use this notation because this is reserved for something else.

Okay, let me write for this course, I am not going to talk about this. This will be in a linear map from RN to RM and this I can write it as M cross N matrix, how, well, Grad FX is from U to RN, Grad F a vector in RN, now I will consider this is a function, so what I will do…

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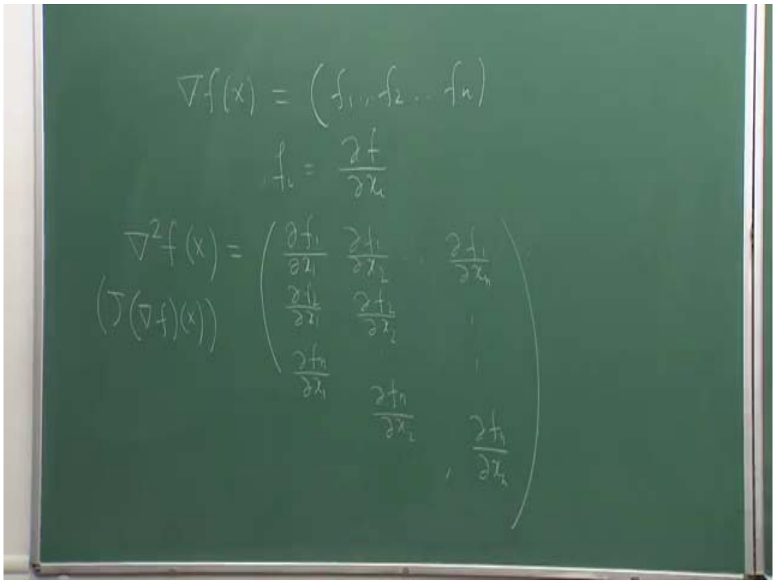

I will write Grad F at X as, it is a RN to RN, del F is map from RN to RN so I can write it as N component, we have done it before. Well each FI is Grad F del XI. Now I know what is the second derivative, derivative of this map, this is we have already known so that will be del FI, so this will be del F1 del X1, del F2 del X1, so on del FN del X1, del F1 del X2, del F2 del X2, del X2 so on del FN del X2, del F1 del XN so one del FN del XN. We know this from before because I have written.Actually this is Jacobian of this map, del F at X., correct.

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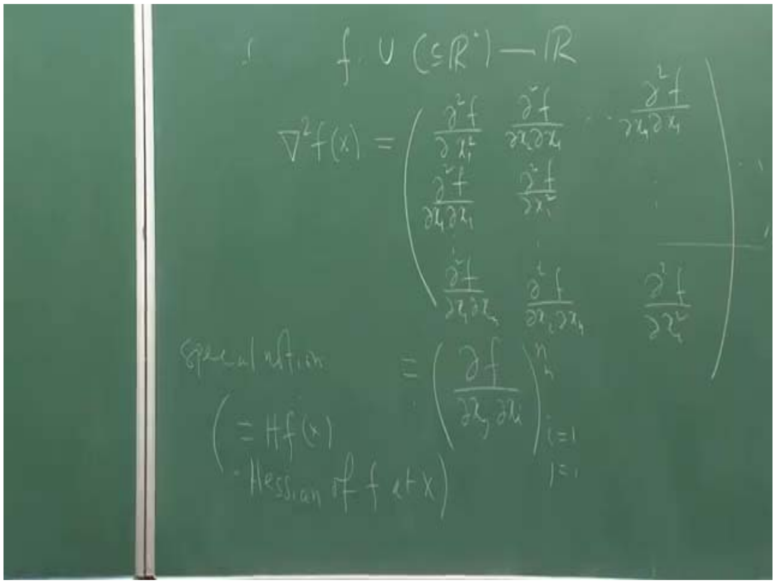

But now I know what is FI, so this matrix will be, I can write it as, I don't have place to write here, maybe I go back to that board. So let me keep the setup, so Grad square F at X is, now I write what is definition of FI, F1 is Grad F del X1, so del square F, del X1 del X1, del X1 square, del F2 is, del F del X2, so that is del X1 del X2 so on, del X del X1 del XN, next del F del X1 del X2, del square, del square F del X2 square, so one del square F del X2 del XN, del square F del X1 del XN so one del square F del XN square.

Which is written as del FI, sorry, del F, del X JXI, I equal to 1 to N, J equal to 1 to N. Some books use a special notation for this, special notation for this we will use it in our next calculation next time, it is called H F at X and written as Hession. Okay, so if you don't remember which way to follow del XJ or del Xi, always go back to this way, that you write del FX is equal to F1 F2 FN, FI equal to this thing and then apply the derivative formula, so there is nothing much to remember here. In one minute you can just write it down if you know, how to write a derivative of a map from RN to RM by component wise. Now this matrix has this property. You see 1 2 (())(11:25) del X2 del X1, and 21 (())(11:29) del X1 del X2 so order is changed.

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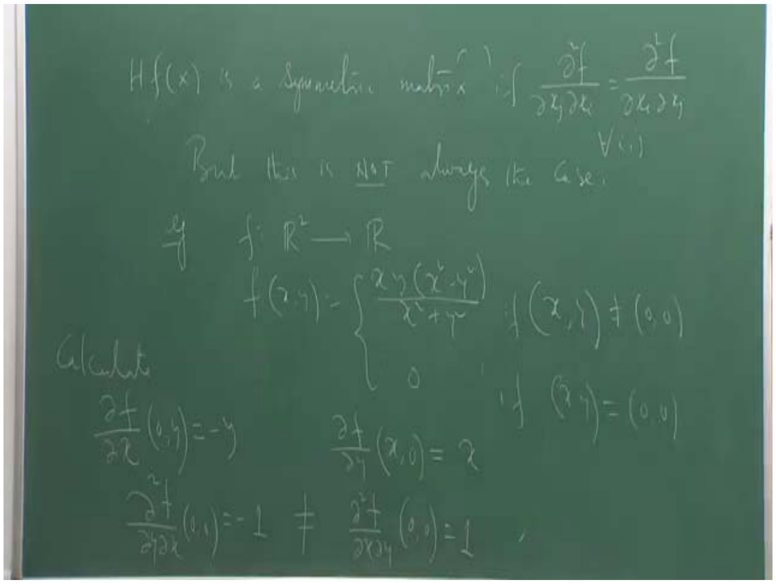

So what I can see from here is that this matrix H is a symmetric matrix if del square F del XJ XI equal to del square F del XI del XJ for all IJ. So if the mixed partial derivative, they are called mixed partial derivative. They are two different components involved, if changing the order are equal then the matrix becomes symmetric.

But this is not always the case. For example, I give you very elementary example, let us say F from entire R2 to R, F of XY equal to X into Y, X square minus Y square divided by X square plus Y square when X and Y both is not equal to 0, is it no origin and 0 at the origin. What you can calculate here, so do this calculation yourself that del F del X, this is equal to, you just calculate it, del F del X equal to, well let us calculate it doesn't matter.

Let us calculate it at 0 Y, this will be equal to minus Y. So del square F del y del X at 0 0 so you verify this calculation is equal to minus 1, whereas del F del Y at X 0, this will be X, so del square F del X del Y this will be equal to 1, clearly this two are not equal so del square F del Y del X and del square F del X del Y at 0 0 they are not equal, one is minus 1 and as in Y. You do this calculation very easy (())(14:31) to do the derivative.

But this is not a very likely nice situation for us, to go ahead with calculation to go ahead with calculus, we want the mix partial derivative be equal so that I get this matrix symmetric. It becomes very handy when you, next time we talk about Taylor's theorem and also when you talk about maximum and minimum real valued function. So come up with certain criteria of check maximum and minimum this symmetric is essential otherwise it gets total mess.

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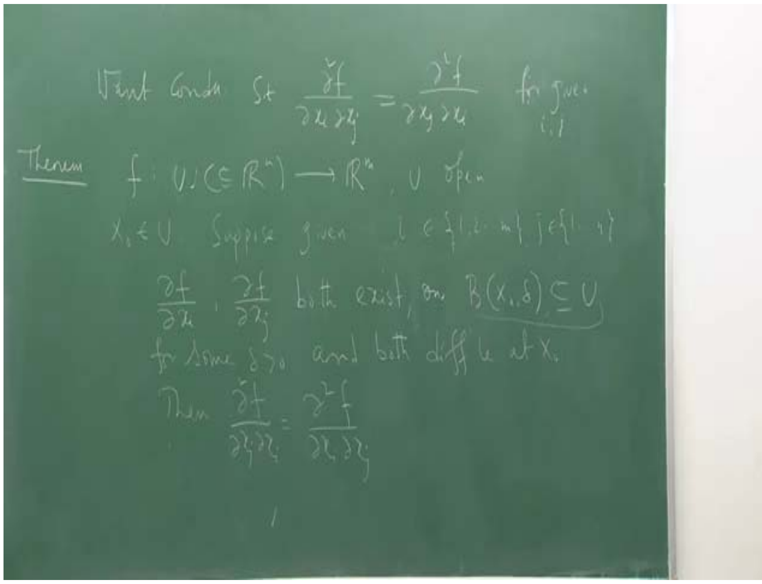

So we actually want condition for equality of mix partial derivative. So we want condition such there del F del XI, del F del XJ and del square F del XJ del XI for given I J. Let me state it as a theorem. I state it for F from U, I state it more general, RN into RM, U is open connected state but open is enough here. Condition says, so let's fix a point, I want to always check at a point, suppose given I and J, given I from I in 1 to M and J in 1 to N, del F del XI, del F del XJ both exists on a ball around delta. That is for all points in the ball around delta for some delta.

See when X not is in U and U is open there will be always a ball but I want a ball such that this del F XI and del XJ both exists on the entire ball, not only at X not, on the entire ball for some delta greater than 0 and both differentiable at X not, that is del F and del XJ differentiable at X not. Then del square F del XJ XI equal to del square F del XI XJ, mix partial derivative at equal.

So what is the condition they must be of course, this must two must exist so they must be differentiable at X not, but the condition is there both exists. I missed most important, and continuous. Why I did not write because I am saying that there is a differentiable but differentiable X not means it is already continuous at X not but I want it continuous on the entire B X not delta, but, if anyways, you can have this or may not have this, continuity of X not is enough.

But it has to exists on the entire ball that is important, so you add N continuous or not it does not make any different, that is what I mean. Okay, the proof is little long and the proof is not going to, we are not going to use the entire idea of the proof any way. But still you need a proof once you write a statement.

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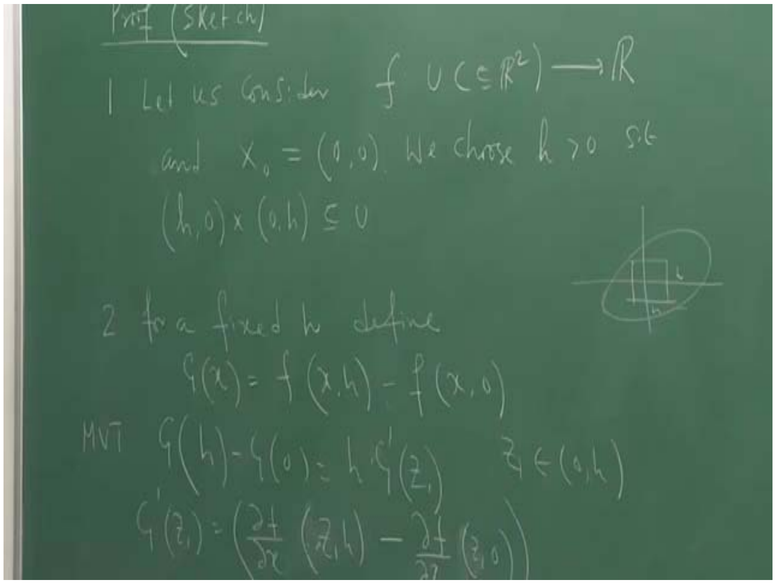

So what I write here is proof, but I will write a sketchy proof. I will give you the idea of the proof you complete the lines in between. Okay. Do the sketching and you go like this. First may your reduction that I am only concerned about 2, XI given XI and XJ, so let us only consider because nothing to be changed for R1 and RM, F from You in R2 because two variables are involved at R. For RM what you do you apply component wise, two variables are involved do R2 is enough. If we have RN you do component wise.

Okay and without loss of (())(20:09) your X not is 0 0, this is just to make the calculation easy, because U is any set in R2, here is the origin but you can always shift origin such a way that 0 0 belongs here. There is a transformation of origin and that does not change any property differentiability, continuity anything.

So we choose, we want to go into (())(20:41) hypothesis so some H greater than 0 such that H 0 cross 0 H, this set is in U, so here is my U from You around origin I choose H such that this square of length H and height H, this is completely named. With that what you do, for a fixed H define G of X equal to F of X H minus F of XJ.

Then we apply Mean Value theorem to get, this is equal to H into G prime at some Z1, right. Z1 is 0 to H, but immediately you can write G prime Z1 equal to del F del X1 at some Z1H minus del F del X1, F at Z1 0 and apply the definition of what is del X1? That was my second step. See verify all those steps, I am not writing the entire proof.

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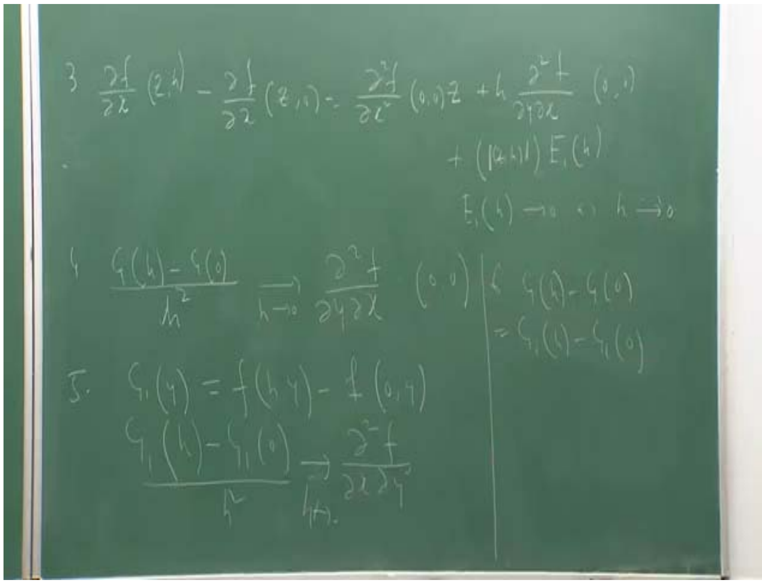

Third step is you again write del F del X as applying MVT del square del X square at 0 0 at Z, not applying, applying the condition of derivative, sorry, applying this is differentiable, del F del x at X 0 as del square F del X square 0 0 at Z plus H into del square F del Y del X 0 0 plus mod Z H into (())(23:58) U and H, U and H will go to 0 as H goes to 0 because as H goes to 0 Z also goes to 0, Z is between 0 and H.

Okay, from that what we do, we calculate GH minus G 0 it will come out to be del square F del, okay, let me skip this step because this need some technicality, what I actually want to show is that this divided by H square this goes to del square F by del Y del X at 0 0. And what you do in the next step you repeat the entire thing by defining G1 Y equal to F of HY minus F of ZY, so here I take XH XO with X and I did G1 and G2.

And you again show by the same step G1H minus G10 divided by H square, it will be just ulta, just other way round, sorry, this goes to 0, this thing as H goes to 0, this as H goes to 0 and final observation is GH minus G0 equal to G1H minus G10, so this two Limits will be equal because numerator are equal denominators are H square, so there is the idea of the proof. You fill up the details.

Thank you!