Differential Calculus of Several Variables
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Module 1
Lecture No 1
Introduction of Functions of Several Variables and Notion in Distance in RN.

Welcome you all to the first lecture of this course, which is Several Variable Calculus, a Differential Calculus of Several Variables. So I presume all of you have done a course of One Variable Calculus where you deal with functions…

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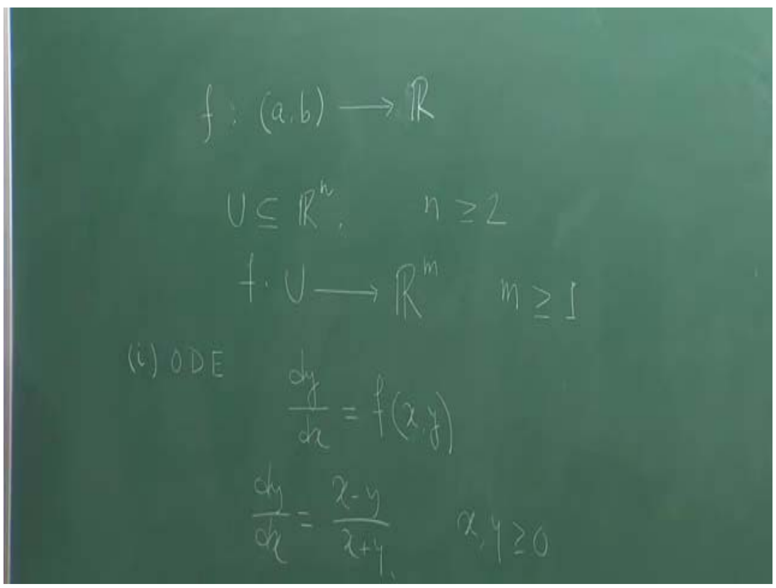

…if define on some intervals A B, maybe closed, when one side maybe both side. A and B are allowed to have go to infinity to R. So called real valuable functions of one variable and you talk about their continuity, differentiability.

Properties of continuous functions like mean value property, intermediate value property and so on. It is okay but very often we need to deal with functions of several variable that is functions which has arguments x, y, z or more. That is function define on art, sub sets of R2 or R3 or RN for (())(1:35). So our set up is that you have a sub set U in RN and typically our N is bigger than and equal to 2.

Function is defined from U to some RM and M is typically bigger than equal to 1, if it is 1 then is a real value function, but arguments are several variable. Okay. Let me just sight few examples that you might have encounter such a thing already. For instance when you solve ODEs or in a differential equation even for first order, typically your equation is given by this way, that is DYDX, Y is function of X that is not a problem, but DYDX the rate of changing of functions which respect to X is function of two variable XY.

Where F is defined on certain domain in R2? For example, one differential equation like that which you can solve very easily and you have done it I am sure. Well let's say X and Y greater than or equal to 0, so your FXY is this function X minus Y upon X plus Y and the domain is upper half plane that is called first quadrant of the R2 that is X and Y both greater than 0. I mean I am just giving an example.

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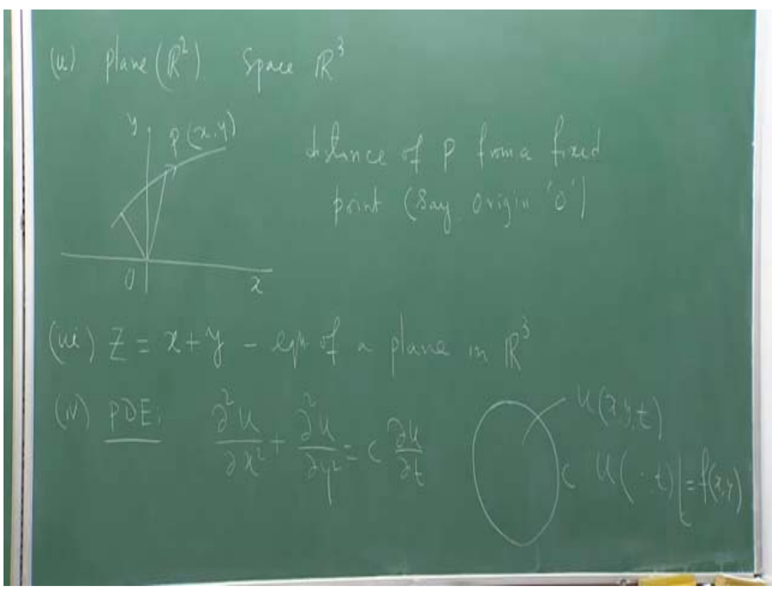

Similarly suppose in space so or plane, so for R's plane is R2, and space is R3. So just say in plane the point is moving. And you want to find so a point P this is moving along a (())(4:12) and you want to find the distance of P from a fixed point, say origin. So you see at this point the distance is this, while it moves to here the distance is this, so P is typically presented by two coordinates X and Y and while the point is moving both X and Y are changing.

I will write down and you can also easily write down the function, the distance function, but this function is function of two variables again. Also you might have seen (())(5:14) coordinate geometry, equation like Z equal to X plus Y. this is actually saying that another function of X and Y whose value is X plus Y and suppose I want to plot it on R3 then all of you know this is the equation of a plane in space that is R3.

One an sight many more examples and particularly when you talk about PDE, I write it typical PDE – Partial Differential Equation which is dealt in any course in mathematics, some constant which can be assumed to be 1 or any depending on the situation. This is a famous equation which actually let us say, I have a circle on plane. And it a (met), think of it is a circular metal late, this is the boundary and this UXT, XYT this metal plate is heated and the boundary is kept so as listed the boundary is C, boundary is kept at a fixed temperature at a fixed level which is a function of X and Y and USD denotes the heat distribution of any point XY at time T.

And this is the famous heat equation which is the origin of many theories in mathematics. And here also you have to deal with variable of 3 coordinates X, Y and T. I am sure you can think of many examples from other disciplines like not only in mathematics but in physics, chemistry, biology and economics where you have actually have to deal with functions like this where functions are define on RN to RM.

Now in calculus we are mostly concerned with functions which have nice property. I mean we have to deal with discontinuous functions sometimes, but calculus cannot be done, or functions which are discontinued cannot be done (())(8:24) through technique of calculus, they need some more advanced technique. So as far as calculus is concerned we are mostly interested in continuous functions and functions which have better property, I mean differentiable.

So with that view, let us try to see what does continuity mean for a function of several variables? I will start humbly with the continuity, recalling continuity of function of single variable and let us see we can generalize that idea or how to generalize that idea to function of two or more variables.

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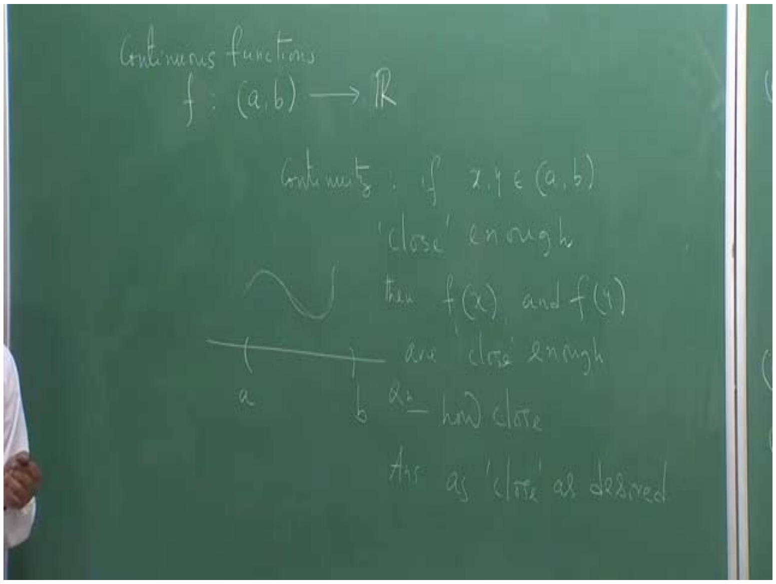

So for our lecture today we deal with continuous functions. All of you can recall what is a function of, what does it mean by continuity of a function of one variable? What does it say, in Lena's language it is simple and I tell you once in for all. Continuity means, let us say function is defined from an interval AB to R, continuity means simply (listen to me carefully), in Lena's language you start from the left of A, this is your interval AB.

You start from left from A, if it is close you can start at A. Pick up the point F, where the F takes values and go all the way to, sorry, start from the right of A, pick up where F is and go all the way to left of B and you, continuity means you will be able to draw the function without lifting your pencil if you doing, using an exercise book or as I am doing it in a board without lifting my chalk.

This is very Lena's description because all of you know that every function cannot be plotted. So what you mean here, mathematically we mean that continuity is if X and Y in AB two points, they are 'close' enough then FX and FY are 'close' enough. But you see this term 'close' enough is not very mathematical very precise, so immediately question is – How close? Answer is – As 'close' as possible as desired.

This looks better, this description looks better than my first description that you can draw the function without lifting your pencil. Now to actually formulating mathematically, actually all of you know it but still I am emphasizing.

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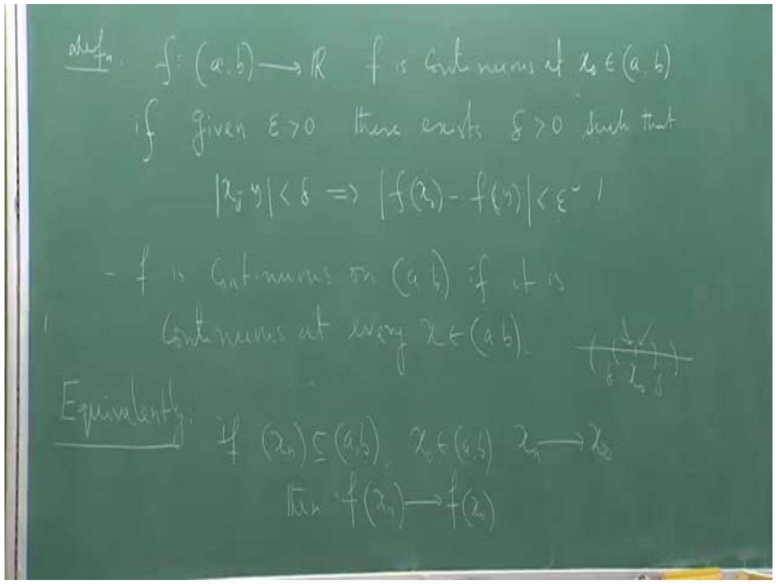

Actually formulated mathematically you take help of mathematical language and the formulation of continuity of one variable in mathematics is, this is the definition – F from AB to R, F is continuous at some point X not in AB, if (all of you have seen this definition) given epsilon greater than 0, epsilon is not specified you give me any epsilon.

There exists delta greater than 0, delta of course depend on epsilon and the point is not, such that X not minus Y less than delta implies FX minus FY, FX not F of Y, X less than epsilon. This is precisely the formulation of this what I have written on this board. X and Y 'close' enough and FX and FY are 'close' enough and F is continuous on AB, F is continuous on AB if it is continuous at every X in it. This is the formal definition of continuity but all of you also learn that there is equivalent formulation.

Equivalently there is a sequential formulation if XN is a sequence in AB, this notation is a XN is a sequent of points in AB, X not in AB, XN converges to X not that means XN is getting 'close' enough to X not, that is precisely this term. That here is X not, here is your AB, you have a delta interval here, and XNs are eventually coming to this interval, then F of XN converges to F of X not. So the value of F at XN is converges to F of X not.

And I am sure all of you have solved this, this problem that these two formulations are equivalent, very easy. Now you see to lift this from a function of sum sub set of RN to RM, I need the (dist) concept of distance here. So 'close' is a concept, 'close' is a adjective associated to distance. So I need a concept of distance in RL. Fortunately all of you or all of us know there is a world define notion of distance which is called Eucledian distance.

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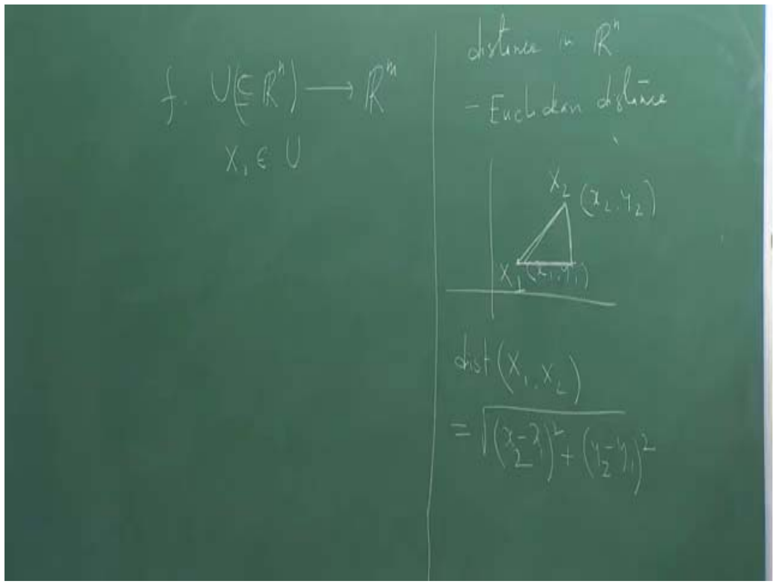

So let us get to U which is a sub set in RN to RM a function. And I have a point say X not in U. I want to define continuity of F at X not so I need a notion of distance. Distance in RN for us, for time being who will be concerned with Eucledian distance. What is that? Geometrically if I have a point here, if I have a point here, Eucleadian distance is join is by straight line, the length of this straight line. And how do you write it?

Let us say this point is X1Y1, this point is X2Y2, this is X point X1, this is point X2, then distance between X1X2 is X1 minus, X2 minus X1 square plus Y2 minus Y1 square. Actually we were measuring this length using Pythagorean Theorem. This length is X2 minus Y1, this length is Y2 minus Y1 and this is the hypotenuse. So this is given by this formula.

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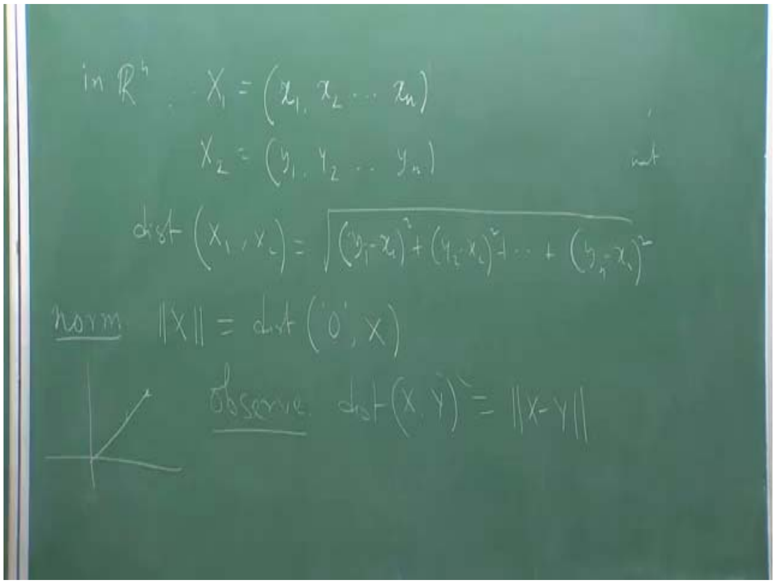

So in general if I have in RN (Where do I write, I will write her), so in RN I have two points X1 which is of course N coordinates and X2 which also has N coordinates, distance between X1 and X2 is square root of Y1 minus X1 square, Y2 minus X2 square, YN minus X square. So you have a notion of distance in any RN and, okay maybe later on we have to deal with some more distance ideas, but for time being, it is fine.

With distance there comes another notion called norm, which is the norm of X is simply distance from origin to X. That if I have a point here the norm is this line, length of this line. There is a bit of catch here, because there are spaces other than R2, R3 and RN we don’t have to bother about them for this course. That there may be distance but no norm, but that is not our concern. And observe with this distance between X and Y, any two points is actually norm of X minus Y. Why?

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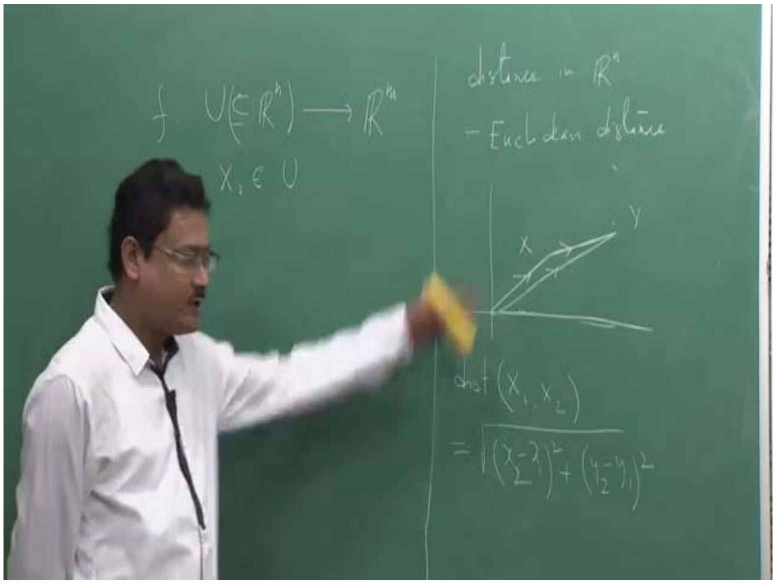

Look here, let me draw it little bigger. Here is your X, let us say here is your Y, I want this distance and now you know there are formula vector edition. This vector plus this vector gives rise to this vector, so this minus this will be length of this line, which is the distance between X and Y. Now we have notion of distance so I can try to lift the exact definition of continuity from one variable.

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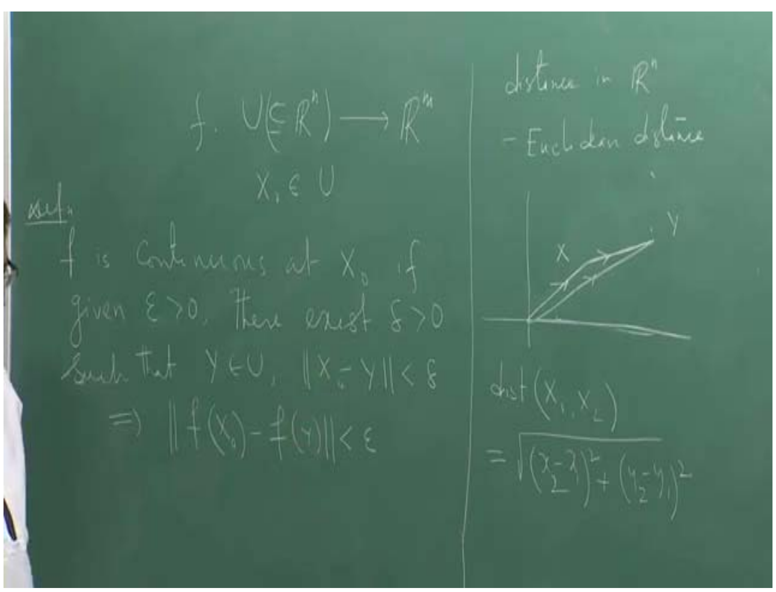

So definition, same definition, F is continuous at X not if given epsilon greater than 0, there exists delta greater than 0 such that Y in U, X not minus Y less than delta implies FX not minus FY less than epsilon.

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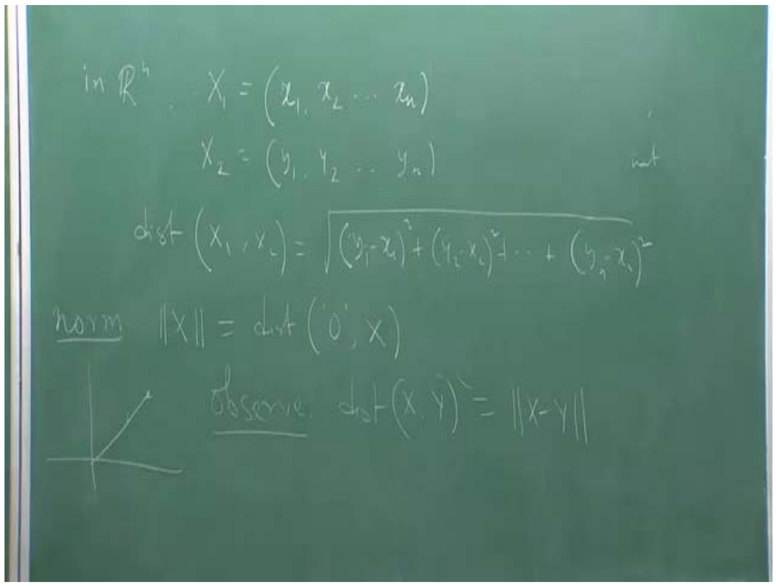

This norm notation I am following from the other board. Distance is norm of X minus Y. Correct?

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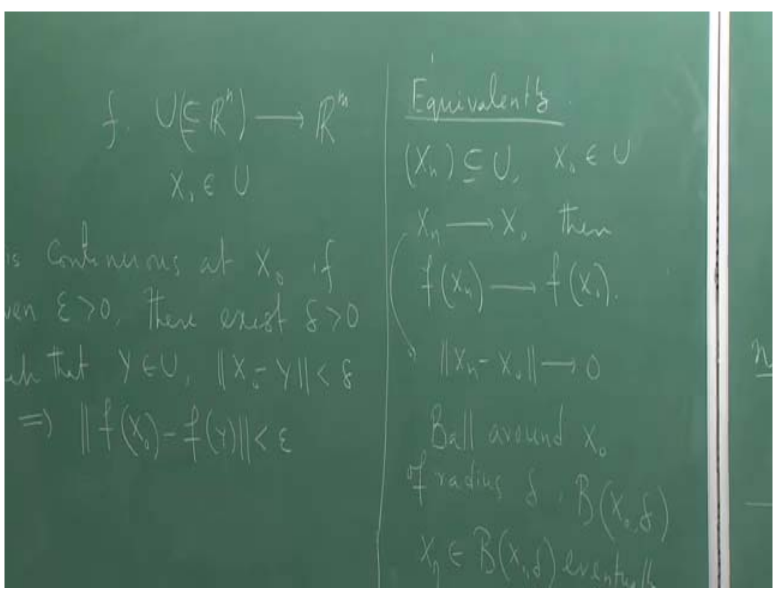

So let us make again an equivalent formulation like that XN is a sequence in U, X not in U, XN converges to X not then F of XN converges to F of X not. Now just give a little bit of reflection here whilst you say real number converges to X not it means it coming to an interval around X not. You take any interval around X not, so a real number converges to X not, means you take any interval around X not eventually the sequence is here. Here XN converges to X not, you are in space or you are in plane or in N dimension then actually you mean this distance XN minus X not.

This is going to 0 so this implies XN minus X not going to 0 so it means it is not an interval now what is means that you take a ball around X not of radius delta which is denoted by B X not delta and XN are in B X not delta eventually. Eventually after some in all of them are there, so instead of interval we have to replace it by ball. Here the picture is this and there the picture…

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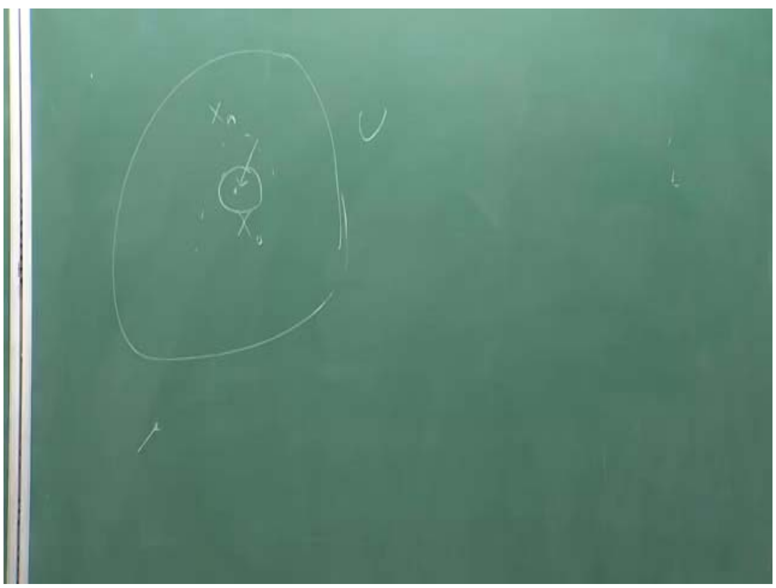

Let us say in the plane, I have a (play) this is my U, here is my X not, XN convergence XN is a sequence here and you draw any ball around X not of radius, arbitrary radius, XN will eventually enter this ball that is what means this is a convergence main scene, RN. Okay, this is the end of first lecture. Next lecture we will look properties of continuous functions and example of continuous and discontinuous functions.

Thank you!