Basic Calculus for Engineers, Scientists and Economists
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Lecture – 06
Limits and Continuity-2

We have already spoken about what is the meaning of function approaching and the limit when it goes to a, about what is the meaning of continuity. But, it will be interesting to know what sort of properties continuous functions have, have they something interesting to tell about themselves.

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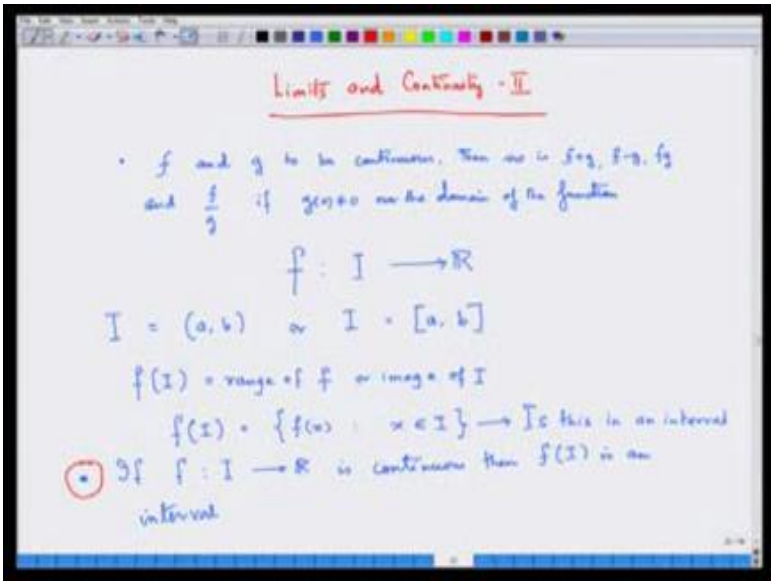

Some properties are very simple that we would just mention in the very beginning that if you take f and g, two functions to be continuous, then so is f plus g, so is f minus g, so is f g and f by g. If g x is not equal to 0, for what is the domain of the function. So, now we will try to give some descriptions about important properties of functions.

So, we will now consider a function f defined from an interval I to R. I would say that you can take I in your mind either as the open interval or you can take I as a closed interval. So, the range of f is called image of I; f I, which is the range of f or image of I. So, f of I is the set of all f of x in r. So, that x belongs to the interval. Either closed or open, does not matter. A very important question is this range; an interval is this. An interval. It is not so obvious. You would like it to be, but it is not so obvious. So, that is why you need proofs in Mathematics that you need to really vigorously justify what statements you are making.

The important result is the following; that if f is continuous, if f from I to R is continuous that is continuous at each and every point. Remember if it is a closed interval, then when you are talking about continuity at the point a, then you are talking about continuity from the left side; because you only take out, can take the limit from the right side. You can take the limit from the right side. You can approach from the right side.

And, if you are talking about point b, you can approach only from the left side. You cannot approach from the other side because the function has no definition outside a, b. So, if this is continuous, then f of I is an interval. This is a very fundamental result. Truly important. I would rather say I will put a red sign around it to tell you that this result is very important. What is the conclusion of this result? What? Can you give me something more? OK, this is very interesting. That image is also an interval. So, how nice the function might be. Of course, but can you tell me something more? This leads to a very famous property about continuous functions.

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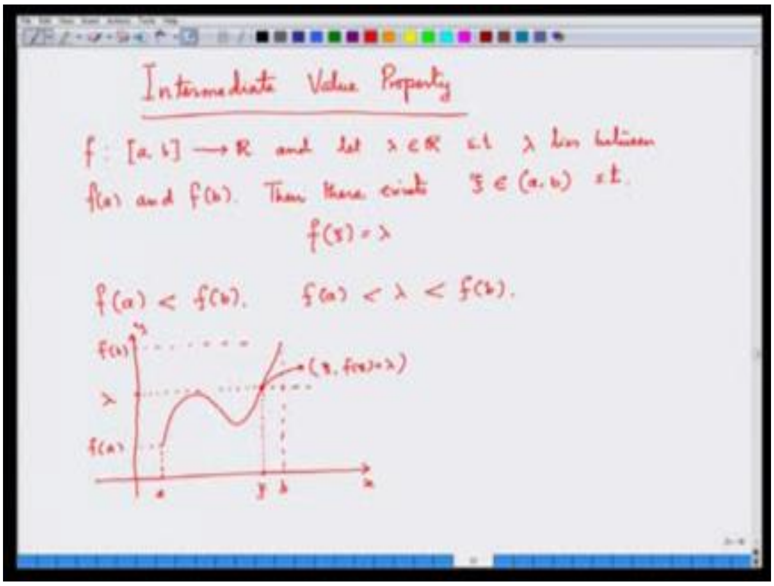

And, those who know some Mathematics could immediately realize. It is called the intermediate value property. Let us see what is it.

This is just a consequence of the fact that an interval is mapped into an interval by a continuous function. Now, what does this say? What is the meaning of intermediate value properties? So, I will write in red. Let me instead of going for open or closed interval, let me just take this. I just want to make one point here. You can take I as this or that. But, when you want to have this thing, you have to remember that I have to consider I as a closed interval. So, when you are talking about a function, you can either take this interval or take this interval.

But, if you want to talk about these important properties, then it is better to just concentrate on closed interval. So, that is why I want to state here that I will just consider on this interval; f from a, b to R. Now, from now onwards study would be concentrated on this type of functions. This is actually a larger property. Those who knows some more Mathematics would realise what I am trying to say is that the continuous image of a compact set is a compact, but forget about all these things.

So if I is of this form that I is continuous, then f I is an interval, if I is given like this. But, if I say a to b, then I cannot talk about this. I cannot conclude this result. So, for me the interval, from now on should be just this. This, taking this interval I a b is much more easier, when you are talking about derivatives. There are reasons for it, which will not go. But for me, when I am talking about continuous function, I am only talking about this interval. So, I now write this one. So, this result which I said is very important, I just forgot. Sorry for that. That this result holds only if I is given in the form of a closed interval.

Then, this fact is a very clear fact that if this is an interval, then f I is an interval. Of course, it is a closed interval, alright. It is a closed interval. I want to make it more clear. So it is a closed interval, if I is this. This is the fundamental importance.

Once that is done, so you come to this. It says OK; let us have a function like this. And, let lambda the element of R, such that lambda lies between f and b because I do not know whether f a is big or f b is big or whatever it is. Lambda should lie between f a and f b, whichever is bigger or small, that is not a big issue. Lambda lies between f a and f b. Then, there exists psi belonging to the open interval a, b; because you know at if because I want lambda to lie strictly between f a and f b. There exists a psi between a and b, such that f of psi is equal to lambda.

Suppose f a is strictly less than f b and I am expecting a lambda to be of this form; lambda is this. Of course, there is no harm if you take equality. Then, of course if you take equality and if lambda is f a and you already have a, then lambda is; so, you can say okay, see remember this psi if lambda is strictly between this, this psi has to lie strictly between this. The reason is, suppose psi is equal a and f of a is that same, f of psi is equal to lambda what psi is equal to a, then this will not be a function. It will break the definition of the function.

So, if psi is strictly in a, b and lambda is strictly in this, if lambda is strictly between f a and f b, then psi must be strictly between a and b. This is very important thing about intermediate value theorem. This, one has to keep in mind because if you say no, I will take psi in a closed interval a b, then an f psi equal to lambda; which means you have to keeping the opportunity, fact that psi could be a. But, then it would be; f a would be equal to lambda. So, f a would have two values; one is f a and one is the lambda because I have said that lambda is different from f a. So, this is something very important.

If both have equality, then you can have a closed interval. So, but see, we need not bother about f a and f b because we know the points better. Now, this is a very important result.

It is; if you draw a picture, it looks like this. So, there is some continuous function. This is my b and this is my a. So, this is my f of b and this is f of a. Now, take a lambda which is here. So, how do I know that there will be a psi lying between a and b? How do I find from this graph that psi? We just draw a line parallel to the x axis through lambda and see where it cuts the graph of the function. It cuts it here. And then, from there you drop that it hits somewhere between a and b.

And, this is your psi. Plus f of psi is lambda because this point is nothing but psi lambda. Co-ordinate of this point is psi, f psi and f psi is equal to lambda. That is the geometric interpretation of the intermediate value theorem. And that is going to happen every time. So, this property leads to very important property of functions from a closed interval to R. That property is called the property of f been bounded; boundedness property.

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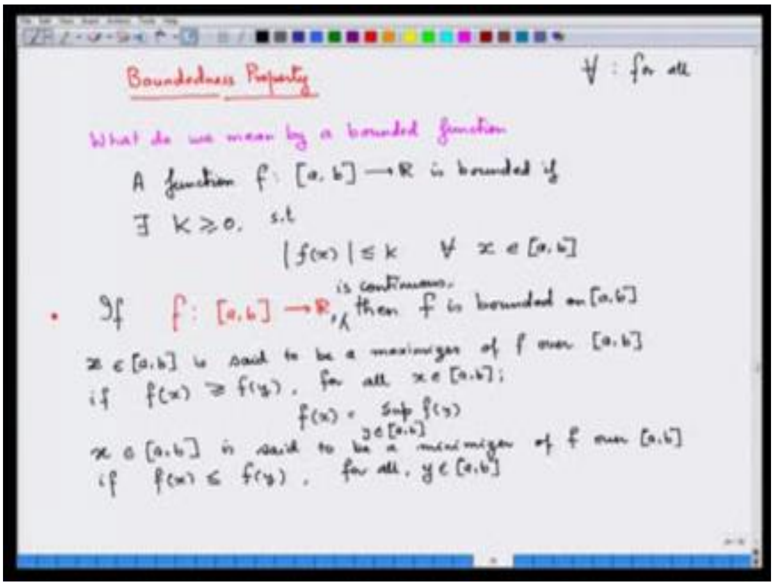

Boundedness property means, so what do we mean by a bounded function? A function if from a, b to R is bounded, if there exists K greater than or equal to 0, such that mod of f x is less than equal to K, for all, this is the sign for "for all". I am giving this sign. And, I am possibly not making it amply clear. This actually means for all x, which is lying in the interval a b. That is the meaning of main function is bounded.

A very important conclusion which comes out of this intermediate value property is that and the fact that intervals are mapped intervals. And, it has to be a red marked property. If f is the function from a, b to R, then f is bounded on a b. That simply means exactly this, the fact that we have just wrote. But, this has a more deeper consequence.

So, we will not now come to tell you some stories about maxima and minima. Tomorrow's, hope in the next week's first class on limits and continuity would be rather about examples. But, today we will talk about very important.

So, x element of a b is said to be a maximize; maximizer of f over a b. If f of x is less than equal to f of y, for all x element of a b. If you want to look smarter, then you can write f of x supremum of f x f y y element of a b. Similarly, you can write, talk about the minimizer. The function value, sorry here minimizer, I said maximizer. So, means this function value f x is bigger than all possible function values at all other points. So, x is the maximizer over a b, if the function value at f x, that is, f x is bigger than the function value at any other point y in a b.

Now, x element of a b is said to be a minimizer of f over a b, if f of x is less than equal to f of y. This is greater than and less than, for all x in a b. That is, the function value at x is smaller than the function value at any other point in a, b.

So, what is great about the maximizer or minimizer? Maximizer or minimizer may not exist. Now, you see. You might wonder that if the statement that I have written that if f is from a to b to R, then f is bounded.

Of course, remember we are only talking about continuous function. So, this f of a to b to R is continuous. So, maybe I am not very clear on this. Let me write it down. If f from a, b to R is continuous, then f is bounded. That is so. But remember we are only talking about continuous functions, and not talking about any other functions. If I miss also 'continuity' in this statement because as I go on along with the flow, forgive me for that. But, take f to be continuous; all statements here pertain to continuous f. It does not pertain to discontinuous f at all.

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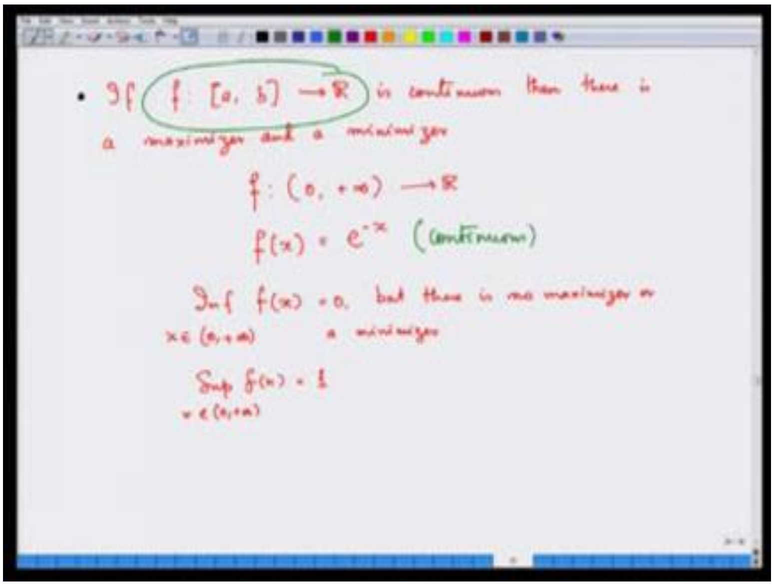

So, what is our conclusion? Why I need to talk about a maximizer and minimizer? Basically, then you know that at what point the maximum value is there and at what point the minimum value is there. So, it does it happen when you have a continuous function from a, b to R.

If f from a, b to R is continuous; so if a, b to R is continuous, then there is a maximizer and the minimizer. Maximizer and minimizer, in the sense we have just defined it. There is a maximizer and a minimizer. But, this is not true if I, for example, want to talk about a function from a non-closed interval to R. That is, take a function f from say 0 to plus infinity; this interval. The whole real line, other than 0; all the positive numbers to R, and f x, so of course it is not a closed and bounded interval. f x is equal to e to the power minus x. Then, the infimum of f when x is running over 0 to plus infinity, where, of course 0 and plus infinity is of course not a number.

So, on this interval, open interval, and then the infimum value is 0. But, there is no maximizer or a minimizer. See, infimum value is 0. And, in fact you can check out the supremum value of f x as x goes to; is in this interval is e to the power 0, which is 1. But in this interval, 0 to plus infinity, where 0 is excluded from the interval, there is neither x, no x for which f x value is one or no x for f x value is 0. So, this closed and boundedness is a very important concept. So, you have to understand that we are only talking about functions of this form, continuous functions of this form.

Note, in this interval 0 to plus infinity e to the power minus x is actually a continuous function. So, with this little examples and it is very important; we have learnt some very important properties of continuous functions, we are going to close our discussion. And, in the next week we are going to start giving examples of what we have just learnt and then move on to derivatives.

Thank you.