Introduction to Probability Theory and its Applications
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Lecture - 13
Independent R. V. and Their Sums

I will start by talking about Independence of Random Variables, when they are jointly distributed.

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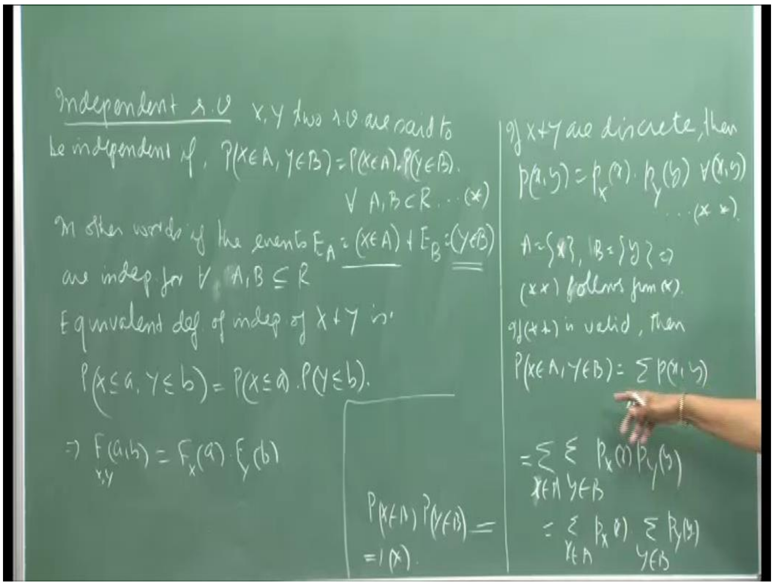

So, just again an extension of the same concept that we talked about for single random variables, now here one can write it in two, three ways, x belongs to A and y belongs to B, so their two subsets a b of the real line. And then a probability this should be equal to probability x belonging to A into to probability y belonging to B, and this should hold for all possible subsets A, B of R.

Now, in other words if the events, so another way of saying is that this event and this event are independent for all A, B subsets of R another way of saying it. Then an equivalent definition of independence of x, y would be that you now allow you take real numbers a, b and then you are saying that x is less than or equal to a, y is less than or equal to b is equal to probability x less than or equal to a in to, this should hold for all and b and this is if and only if.

This or you, then by using the 3 axioms you can show that this and this are the same definitions that you can, then can take to real numbers and you can describe your events in that way. So, this will immediately employ that you are distribution function, cumulative distribution function for the joint cumulative distribution function can be written as a product of the individual cumulative distribution functions. Now, if x and y are discrete, then we say that we can equivalent way of writing it is or this is how we can p x, y is p x (x) into p y (y) that means, individual probabilities for all x, y.

And surely this definition and this definition are the same, because this is the general one which covers both continuous and discrete. Now, if you just can take A as single term x and B as the single term y, then this follow from here, because x equal to A, so A, B that means x, y, if A is simply a single term, B is also single term y, then this becomes x, y and this is probability x into probability y of course, with respect to x and respect to y.

So, I mean two follows from one, when you choose the sets A, B to be single terms and the other way it is valid, because if star is valid for all x, y; then you can you can write this definition as the first part of star as summation p x y, x belonging to a y belonging to p that is our definition. But, since this is x and y are said to be independent, then this can be written as p x (x) into p y (y), where x summing over all x's in A and summing overall y in B, then you can separate out the summations you can write out this way and therefore, this is probability x equal to I did not… So, probability x belonging to A into probability y belonging to B, which is the same as this, so if this is valid star follows and if star is valid, then double star follows, so this is whole idea.

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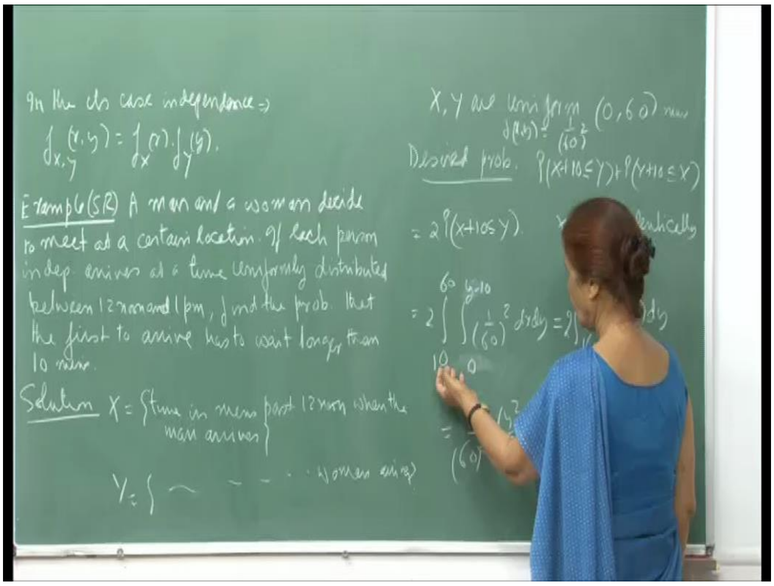

Now, and of course, in equivalent condition for this, for the continuous case would be, that the joint PDF, can be written as the product of the individual PDF, so now you have so many equivalent ways of expressing the same concept. I will take this example function Sheldon's law, a interesting to again show the computations and the how we make use of independence of random variables.

So, here a man and a women decide to meet at a certain location, if each person in dependently arrives at a time which is uniformly distributed between 12 noon and 1 pm. So, they arrival time of both of them is a random, I mean for the man and the women both are random variables and this is the time is between 12 noon and 1 pm, now you have to find the probability that the first who arrive has to wait longer than 10 minutes, so therefore that is why I converted the PDF as 1 by 60 for each x and...

So, here let me define the random variable x as time and minutes past 12 noon when the man arrives and similarly, time in minutes past 12 noon when the women arrives at the fix part, so both are random variables. And they uniformly distributed between 0 and 60, so that means the PDF will be 1 by 60 for each, so when you take the joint PDF that will be 1 by 60 square; and will say that x and y bit vary between 0 and 60. So, the probability that you are asking for is x is the probability for the man to arrive, so if the man arrives first. then x plus 10 should be less than or equal to y that means, this is the time when the women arrives.

So, the man has to wait because he is the first to arrive he has to wait for longer than 10, that is probability x plus 10 less than y, but since x and y are continuous random variables computation of the required probabilities alright. And similarly, if the women arrives first then y plus 10 should be less than or equal to x, but since x and y are identically distributed independent assumes. So, therefore, the two events are the same, so that means, the probability will be exactly the same from symmetry.

So, twice probability x plus 10 less than or equal to y, this what we have to compute and so you write down the limits you see for x, for x the limits are 0 to y minus 10, x is less than or equal to y minus 10 and x varies from because the arrival, the man can arrive at 12 noon itself, so x will be 0. So, x actually varies from 0 to 60, so here it is 0 to y minus 10 and the women can arrive has to, because the man has to wait 10 minutes or more, then it should be 10 to 60; so the arrival time of the women would from 10 to 60, so this is what is important.

So, once you fix the limits and the joint PDF, then computing the probabilities is not a problem, so one has to spend time in just thinking as to how to event is described very clearly. So, here the limits for y for the women arrival time would be have to from 10 to 60 and this is from 0 to y minus 10 here, so therefore you integrate respect to x and they simply x, so y minus 10 by this thing. And then when you integrate respect to y it will be y square by 2 minus 10 y, from 10 to 60, so this will be 2 by 60 square and this would be 1 by 2 60 square minus 10 square and then minus 10 into 60.

((Refer Time: 08:27)) So, we will continue from here, this was y square by 2 minus 10 y from 10 to 60, so then this becomes 60 square minus 10 square, this one by 2 and minus 10 into 60 plus 10 into, this 10 into 10, so when you compute this form 10 to 60. So, this is minus 10 into 60 plus 10 into 10 and so when you simplify the numbers this here, this would become 2500 upon 60 square and this is 25 by 36. So, desired probability is this much that means, who ever arrives first at the meeting place will have to wait for more than 10 minutes, the probability of that is 20 upon 36.

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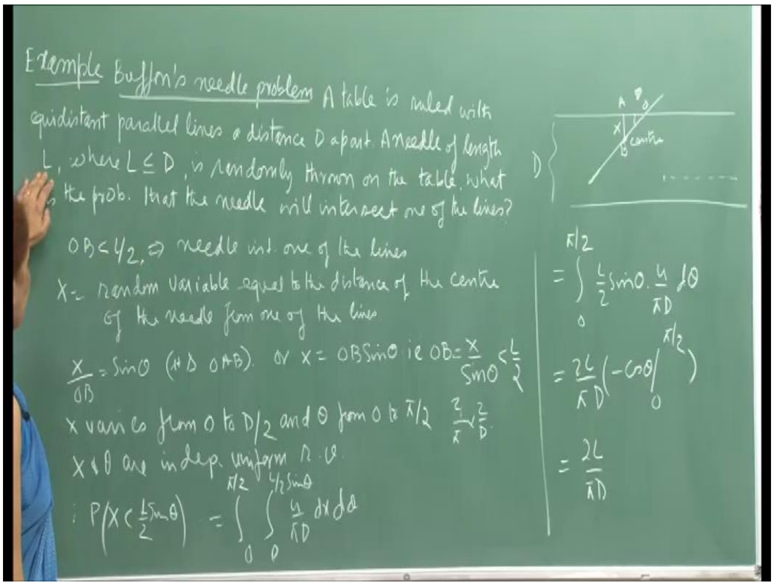

So, let me continue with the with examples of jointly distributed independent random variables, now is the interesting examples, it is called the Buffon's needle problem, Buffon was a French naturalist and so he formulated this problem. It is say that table has parallel lines drawn on it and the distance between two consecutive parallel lines is D, now you drop a needle on the table and of course, one possibility is that the needle lies like this, so it does not intersect one of the lines, but suppose it does intersect one of the lines, so that is what we have to find out.

So, a needle of length capital L, where L is less than or equal to D, the distance between two consecutive parallel lines is randomly thrown on the table, what is the probability that the needle will intersect one of the lines, so we have to compute this probability. And now here I would drawn this diagram, so this is the needle at B is the middle point of the needle and you drop a perpendicular form here to the nearest line, so that is x, so that will be a random variable, because you do not know the position.

So, that mean the two random variables that describe the position of the needle is the distance of the centre of the needle from the closest dearest line and the angle it makes with the nearest line, so that angle I am calling as theta. So, this theta and this is a distance which is x, so these are the two this thing and we are saying that you see, this thing is if this is the centre point, then this is L by 2.

A length of the needle is capital L, so this length is L by 2 and if you are O B is less than L by 2, then the needle will intersect the one of the lines, so that way geometrically this is describing. So, this is making use of geometry to also explain things and probability theory, so here x is a random variable equal to the distance of the centre of the needle from one of the lines. And so now, you can see that x upon in the right angle triangle O A B x upon O B will be sin theta. So, sin theta in the right angle triangle O A B x upon O B sin theta or x is O B sin theta that is O B is x upon sin theta.

And this has to be less than L by 2, so this is our condition for the needle to intersect one of parallel lines drawn on the table. So, and x varies from 0 to D by 2, because it can come up to here, because after that it is a same thing, then the position of the line will become like this. So, it this can up to D by 2 and theta can vary from 0 to that means, either needle almost lies on the parallel line all it makes a, it stands like this, these are the two, so theta varies for 0 to pi by 2, so these are the ranges for the two...

And of course, since any position is equally likely, we will say that both the random variables x and theta are uniformly distributed, so x is uniformly distributed in the interval 0 to D by 2 and theta is uniformly distributed from 0 to pi by 2. So, you see the one of the density PDF for a uniform random variable is the 1 upon length of the interval in which this define, so here it is pi by 2, so 2 by pi and here it is 2 by D, so this is the joint PDF that is 4 upon pi D.

So, now you want integrate, you want to find out the probability x is less than L by 2 sin theta, this is our condition, so therefore theta varies from 0 to pi by 2 and x will vary from 0 to L by 2 sin theta, then this is to be 4 upon pi D d x d theta, so this is the integral which will give you the required probability. And so that is simple enough, because respect to x see this is independent of x and theta both, so here you get integral x and so that will be L by 2 sin theta into 4 by pi D.

And then you integrate respect to D theta here, so that will be minus cos theta 0 to pi by 2 which is equal to 1, so the required probability is 2 L upon pi D. So, one can constructs interesting examples and so here, we use the independence of the two random variables to compute their joint PDF and then which I think I probably be now... So, we have already shown this that the joint PDF will be the product of the two marginal PDF' s independent.

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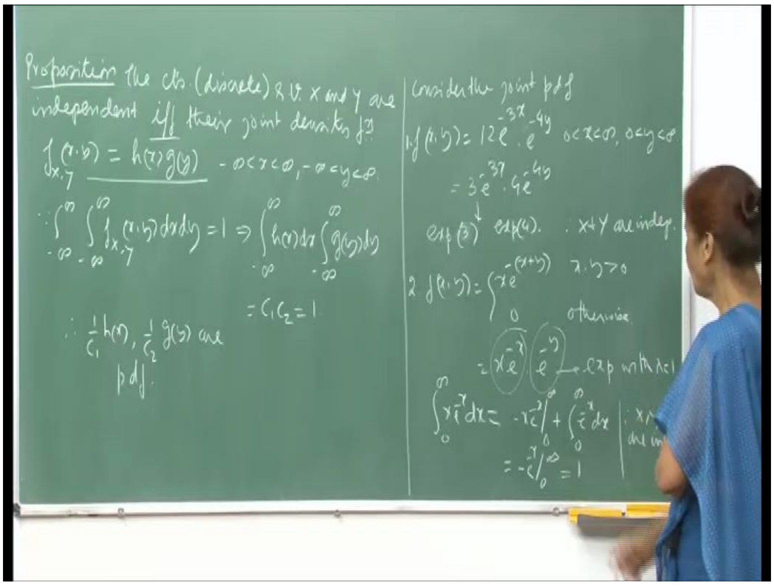

Now, in another preposition and this sometimes makes life easy and you can immediately find out the independence of the random variables, so this if and it holds for discrete and continuous both. So, what it we are saying that random variable x and y are independent, if and only if they join density function f x of x, y can be written as a product of two functions h x and g y. So, this would only a function of x, this is function of y and you see the range the limits are also independent, so x varies from minus infinity to infinity, y varies from minus infinity to infinity.

And since, so what we are saying is that since the integral, double integral here is equal to 1, because this is a PDF, so this implies that when you replace this by h x into g y and the integral separates into two single integrals and so this is this lap to be something like C 1 and this C 2, so this product is 1. So, therefore, see 1 upon C 1 h x will be a PDF and 1 upon C 2 g y will also be a PDF, because by definition this is C 1, so 1 upon c 1 will be 1 and here 1 upon C 2 will be 1.

So, therefore, this is the thing that means, always able to convert these two, since this is the PDF and if it is a product of two such functions, single variable functions, then we will be able to convert both of them into PDF' s this is the idea. So, let us just look at few examples, if your join density function is given as 12 e raise to minus 3 x into e raise to minus 4 x, this is minus 4 x and x between 0 and infinity y between 0 and infinity, when you see I can multiply this by 3 and this by 4.

So, this now is your exponential with parameter 3, immediately recognize it this also is exponential with parameter four and so both are PDF's; and so therefore, you conclude because it says if and only if, see this is the part. So, we have told you already that if they are independent, then the joint PDF will be a product of the individual PDF's and here I am saying that, if f x, y can be written like this, then again x and y are independent. So, therefore, I can immediately conclude that x and y are independent, because your f x, y can be written as the product of two PDF's.

Then if you look at this function here, x e raise to minus x plus y, now here again I can break it up into two functions x into e raise to minus x and this e raise to minus y. So, this I know immediately is exponential with parameter as 1, lambda equal to 1 this 1 and here I can quickly verify that this is also a PDF, which means that integral of x e raise to minus x from 0 to infinity d x and if you now integrate this by parts, then you you get minus e raise to minus x x. Take this as the first function, so 0 to infinity this gives to 0, then you have plus 0 to infinity e raise to minus x d x which integrates to 1. So, therefore, this is the PDF and this also is the PDF, so the proposition again tells us that x and y have to be independent random variables.

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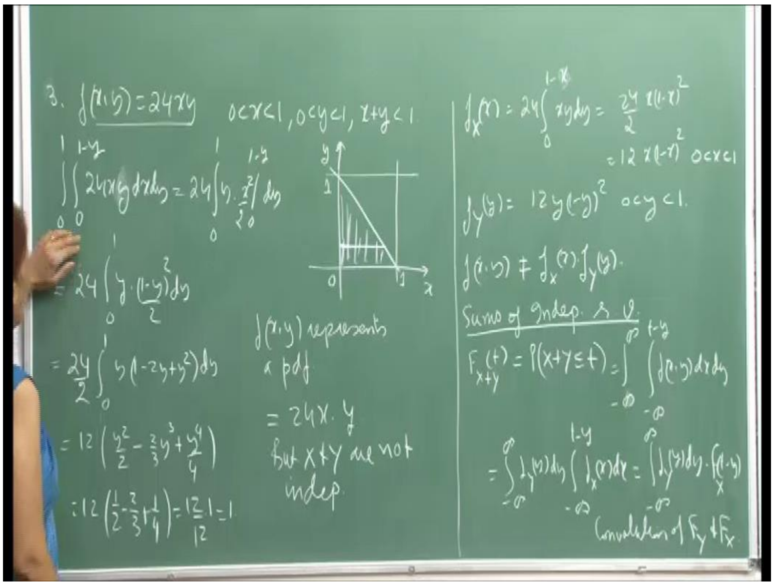

So, let me now show you take another example see f x, y is 24 x y and you see here you can decompose it into a function of x and a function of y. So, let us now, but the thing is that the area of integration, so this is x between 0 and 1 y between 0 and 1, but x plus y less than 1. So, you see here the connection is there and therefore, the suspicion is that the two random variables are not independent and we will see.

So, I have shown here the area and over which the valid region, so this is x plus y less than one, this is the line, so here in the square 1 1, this is the area on which you have to concentrate. So, now, first let me verify that this is the PDF, so therefore see your range for x will be from 0 to; that means, given a value of y, if I am integrating with respect to x then I fix a value of y and then I draw a line, so therefore, I will be my range for x is then from 0 to see when you fix 1 minus y.

So, x varies from fixing a y then y range for x will be 0 to 1 minus y and that is what have written here 0 to 1 minus y 24 x y d x d y and. So, even you integrate respect to x this x square by 2 0 to 1 minus y and so the integral is the value of the integral is 1 minus y whole square by 2. And here just open it up take y inside. And then you integrate you this is what y square by 2 minus 2 y cube by 3 and y of 4 by 4 from this is 0 to 1 right. And, so this will be when you substitute the values at 1 you get this which is equal to 1.

So, we have verified this is the PDF and now you compute the marginal’s and we will show that the product of the marginal's is not equal to the join density function. So, because that condition was if and only if remember, so here when you integrate respect to y. So, gain this will be now 0 to 1 minus x; that means, your fixed x, so once you fix an x then your y will vary like this, so from 0 to fixing x, so y varies from 1 minus form 1 minus x. So, this will be the length of the range for y, so therefore, 0 to 1 minus x, x y d y and again this will be x into 1 minus x whole square the same integral that we did

And, so this will be 12 into x 1 minus x whole square x varies now from 0 to 1, similarly the marginal of y same integral because the symmetric the function is symmetric with respect to x and y both and the limits are also. So, therefore, this will be this and y between 0 and 1, so you see that f x y is not equal to f x into f the marginal's, so therefore, the 2 random variables are not independent. So, when you say that you can break up the joint PDF into individual you know functions of the single variables, then make sure that the range is are also independent.

Otherwise, the random variables will not be independent, so we will continue and then may be will keep coming back to so but independence is a very important concept does simplify lot of things in probability theory. Now, let me start talking about sums of random variables I have already told you earlier that how we can, I know get the distributions for sums of independent random variables. So, here again the catch word is independent we need that, so now for example, if x and y are 2 random variables, which are independent and if f x, y is the joint PDF.

Then, if you want to write the distribution function for the random variable x plus y this will be given by this, which will be, so now, here again the same thing that we used in this case. So, since x plus y less than or equal to t, so if you are integrating with respect to x and y is, ((Refer Time: 23:09)) so this will be minus infinity to t minus y range for x and then the range for y will be minus infinity to infinity. But, since this is x and y are independent, so this can be written as a product of the individual PDF's the marginal PDF's.

So, therefore, I can separate out my integrals minus infinity to infinity f y y d y and minus infinity to t minus y f x x d x. And, so this you know gives you what, this is the probability of capital X less than or equal to t minus y, so therefore, this is your cumulative distribution function for x at t minus y this integral. And this is the integral minus infinity to infinity f y y d y, so this is called convolution of because here this is the distribution function for x at t minus y minus infinity to y, well not exactly you are well I am writing small f y d y. So, anyway this will be called as a convolution because this is at y and this is at t minus y, but I will make things more clear here. See now if you want to if I differentiate this with respect to t then I get the PDF for x plus y.

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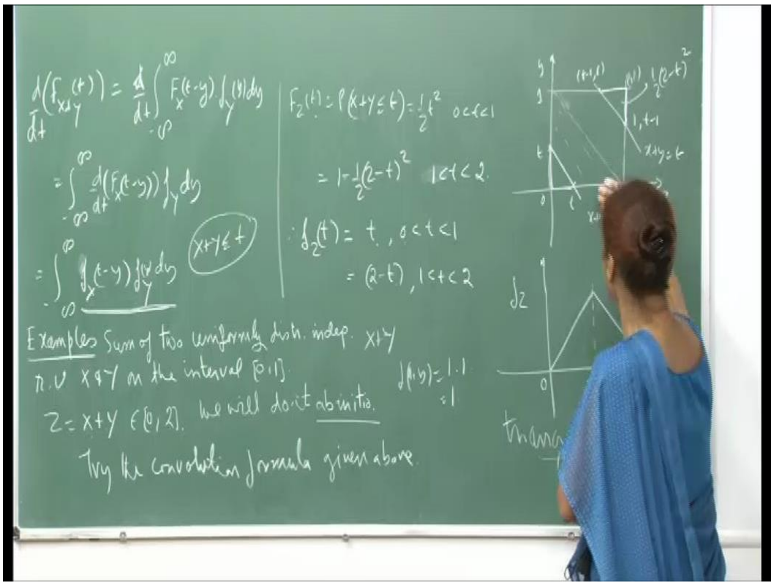

And you say that here this is this and remember we have done it already we can exchange the integral and the derivative sign provided this thing here is differentiable. And, so I take the derivative inside this gives me this and this now f x t minus y d d t will give me the PDF of x at t minus y and this f y y d y minus. So, now, this is also this you can see better as a convolution of the 2 PDF ‘s f x and f y, so f y y then f x is t minus y, where we are looking at the event this less than or equal to t, so this is called the convolution.

And sometimes it comes in handy, but I will try to show you again; that means, my philosophy is that you should try to use geometry and direct methods as much as possible to get answers. The formulae are important and sometimes they are very helpful and I have shown you, that at times it has helped us to use the derive formulae to get the result, but sometimes it also helps to do things directly. So, let us take this example, some of 2 uniformly distributed independent random variables x and y on the intervals 0 1.

So, now, I want to find out the PDF of the random variable x plus y, where both x and y are uniformly distributed on the interval 0 1 and they are independent. So, then the variable z which represents the sum, will now vary from 0 to 2 because this varies from 0 to 1, this varies from 0 to 1, so the sum can be vary from 0 to 2. So, as I am saying we will do it (( )) using the formula directly I will try to compute the cumulative distribution function and see.

So, therefore, the diagram is here 0 to 1, 0 to 1, x axis, y axis, now you see what happens is if your t is less than or equal to 1, see t equal to 1 gives you this line x plus y equal to 1 and this portion is covered by t greater than 1. So, as long as t is less than 1, you see this is uniform, so; that means, uniform mass over this region, so therefore, to get this probability I just need to compute the area of this triangle. So, it would get probability x plus y less than or equal to t, it is simply the area of this.

Because, the joint density function is what, your join density function is 1 into 1, which is 1 because both are uniform on the intervals 0 1. So, the mass spread over is you know is same uniform unit and so the area of the valid region would be or probability, so here as long as t is less than or equal to 1. The area was therefore, if you take this or you take this, then it will always with the area of this triangle.

So, therefore, this is half t square because this side is t and this is side t, so base and height are both t, so half t square for 0 less than t less than 1. Now, when t becomes more than 1, then you see it is no longer because it is this a region which is a triangle, but you need this area. So, therefore, what I will do is from the total are 1, I will subtract the area of this triangle and that will give me the required probability, so this line is x plus y equal to t and therefore, it intersects x equal to 1 at t minus, so therefore, the y point is t minus 1 because t is greater than 1.

Similarly, this is t minus 1 1, so then both these lengths this point is 1 1, so when you do it 1 minus t of minus 1 plus 1, so 2 minus t. So, this length is 2 minus t, this is length also 2 minus t, so the area of that triangle is half 2 minus t whole square and therefore, the required probability is 1 minus 1 by 2 into 2 minus t whole square, when t lies between 1 and 2. So, see looking at the diagram things become really simple and therefore, when you differentiate this respect to t, in this case it becomes t 0 less than t less than 1.

And for this because this is a minus sign minus sign, so 2 2 cancels 2 minus t as t varies between 1 and 2. So, now, if you draw the picture graph of f to t between 0 and 1 it is given by this line and between 1 and 2 it is given by this line, so it is a triangle, so therefore, this is also known as triangular distributions. Now, I leave it to you to try out the convolution formula and then see that you should get the same answer. So, you can sit down and verify for yourself the formula is clear here for different values of t you will have to. So, here of course, your this thing will be from 0 to 2 because your t can vary from, but looking at the diagram things really become simple and you can, so where ever possible use geometry or direct methods.

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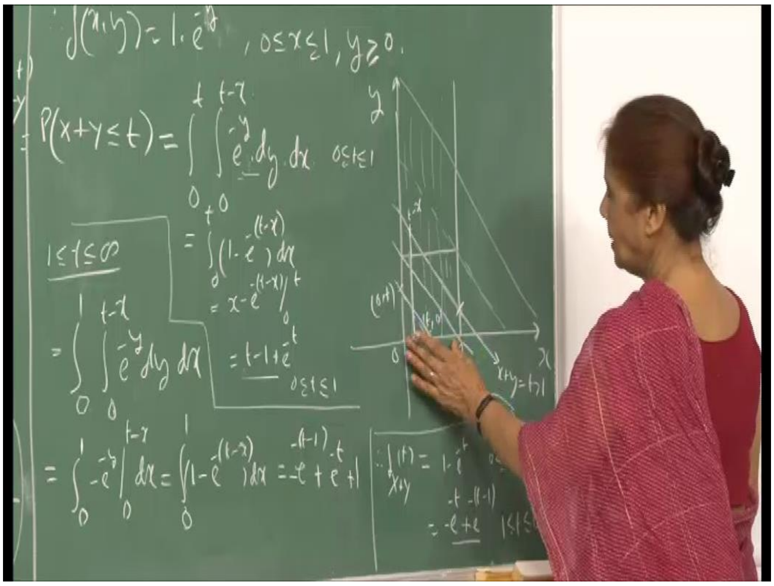

Now, trying to look at this example, where you have x is uniform (0, 1) and y is exponential with parameter 1 and x and y are independent. So, now we want to look at the, so the joint density function of x and y will be just the product of the individual PDF's, so this is for the uniform it is 1 and for the exponential it is e raise to minus should have written e minus y, so this it is ok here I just wrote here e minus y. And so your y varies from 0 to infinity x is between 0 and 1, so now you want to find out the probability that x plus y is less than or equal to t that means, this is your F of x plus y t, this what you found want to find out.

And so you see now here what I have right now written is when 0 is less than t, less than or equal to 1, so you see the region of integration as I had told you is x between 0 and 1 and y non-negative, so going to infinite. So, this is the region for integration and I was trying to tell you that we have to separate out the integration into two parts, because you see for x plus y less than or equal to t as long as t is between 0 and 1, then this is this area which will has to be covered. So, therefore, for example if you take the line x plus y equal to t, if this the line, then the limits for x are from 0 to t and for y will be from 0 to t minus x.

So, therefore, t between 0 and 1 these are the limits and so you can, therefore integrate respect to y first, so that will be 1 minus e raise to minus t minus x, because integral of this will be minus e raise to minus y and so from 0 become 1 minus e of minus of t minus x d x. And then you integrate respect to x the limits are 0 to t and so this will be x and because this is plus x, so this will remain minus e raise to minus t of minus x from 0 to t and therefore, this is your cumulative density function for t between 0 and 1. Now, for t greater than 1 with lines would be like this, express y equal to t which is greater than 1.

So, in that case you are for a given x your y will vary like this form 0 to t minus x, and x will vary from 0 to 1, because when you are talking of t greater than 1 this line, then x will vary from 0 to 1 and t will vary from 0 to t minus x. And therefore, for t between 1 and infinity it will be 0 to 1, 0 to t minus x, e raise to minus y d y d x and again the same integration. This one is exactly the same and then from 0 to 1 with respect to x, so this will be minus e raise to minus t minus 1 plus e raise to minus t plus 1.

So, please just verify that these are valid cumulative density functions of course, here as t goes to 0, this goes to 0, t goes to 0 this is 0 and this is 1, so 1 minus 1 is 0 and as t and from here as t goes to infinity you can see that this will go to 1. Because, this will go to 0, this will go to 0 and you will be left with 1, have I written it correctly here let us just make sure, let us t goes to infinity then this goes to infinity also and so e raise to minus of t minus 1 also goes to 0, so therefore the limiting value is 1. So, this is the valid cumulative density function and therefore, now you want to compute the PDF then just differentiate.

So, this will be 1 minus e raise to minus t as t is between 0 and 1 and this one here would be minus e raise to minus t and then this will be plus e raise to minus of t minus 1, so this is your PDF for... So, therefore, all these things as you work out and get experience, you can get a experience. you can get feeling how to go about big and the diagram in of course, you can do it in two or three dimensions. So, therefore, the diagram helps to tell you that you have to break up the integration into two parts, so between 0 and 1 for t between 0 and 1, the limits for x are different and when t is greater than 1, the limits become difference. So, therefore, this can only you can get the feeling by looking at the figure and then you know deciding accordingly, so we continue with some more of these examples.

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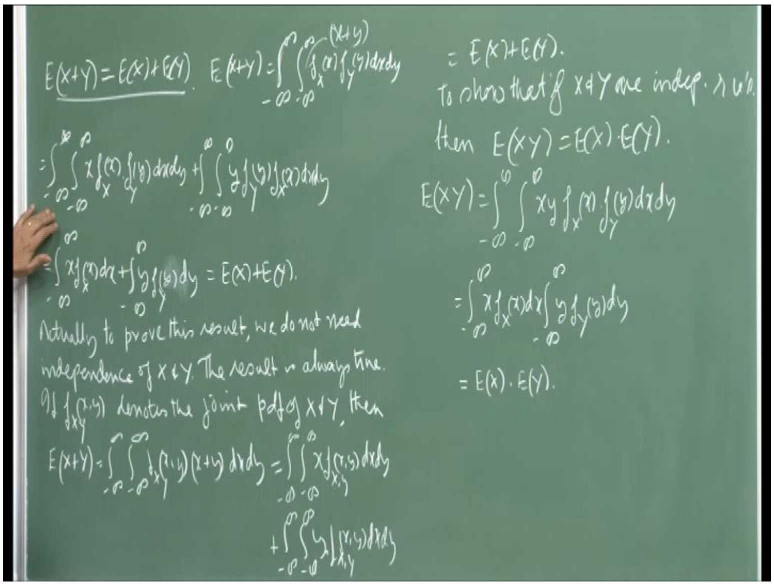

So, expectation of x plus y is equal to expectation x plus expectation of y, now let us just try to first to show this under independence of x and y, so then I can write the joint density function as the product of individual density functions. And so e of x plus y, this should be x plus also see your are integrating x plus y integral minus infinity to infinity minus infinity to infinity x plus y into f x (x) into f y (y) d x d y. So, I have written the joint density function as the product of individual density function and so therefore, one can then separate out the integrals.

So, this would be x f x (x) f y (y) d x d y and here it will be y, now then since if you integrate respect to y, then integral f y of y d y will be 1, because this is a PDF of y; and so you will be left with x f x (x) d x minus infinity to infinity. And similarly, here integration respect to x will result in 1 and so you will get integral minus infinity to infinity y f y by d y and this is E x plus e y. But, I want to point out that it is not necessary to show that this result, you do not need independence of x and y actually it is always true.

And that you can immediately verify, because if x, y is the joint density PDF of x, y, joint PDF of x and y, then e of x plus y we write in this way, integral double integral minus infinity to infinity x plus y f x y, this is this of x, y. Then again I write this as sum of two integrals which is this and this, but we know making the same argument that minus infinity to infinity of f (x, y) x y d y will give you f x of the marginal density of x.

And then so it will be come out to be integral x into marginal of x density function of x plus, similarly y you first integrate respect to x here to get the marginal of y and then you get this.

And therefore, this of E x plus E y, so for showing that E of x plus y is E x plus e y you do not need independence of x and y, but under independence we can show another result, which is that expectation of x y, product of x and y is actually product of the expectations. So, there is a E x into e y and this for this I will need independence, because now I will write E (x, y) as this double integration x, y and the joint can be written as the product of the marginal's, so f x (x) into f y (y) d x d y.

And now here again I can separate out the integrals, so this is integral x of x d x minus infinity to infinity into minus infinity to infinity y f y d y; and now each of this, this is E x and this is E y, hence y gets these are the product. And this result now I will prove in while showing that the under independence variance of x plus y is equal to variance of x plus variance of y, which holds only when x and y are independent. So, therefore, once we have shown this result we can now derive that result.

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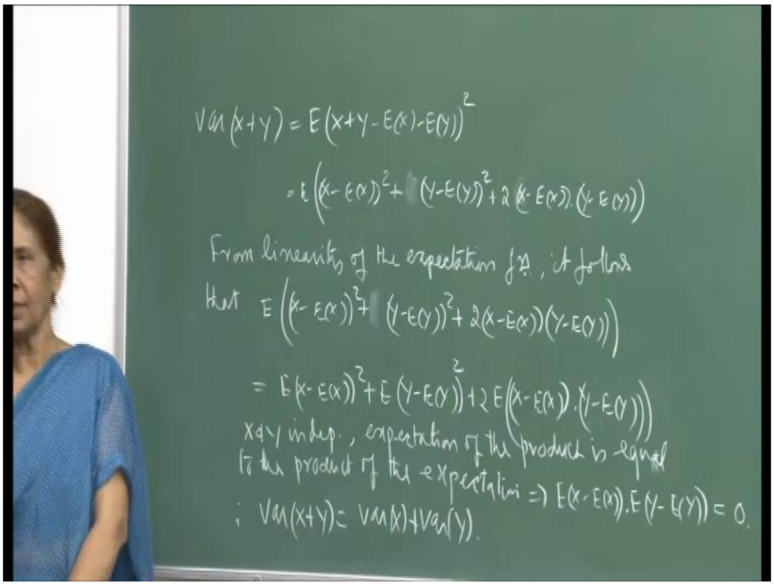

To compute the variance of x plus y, we will require the result that I am giving right now, we will use independence of the variables x and y. So, let me just right down variance x plus y is expected value of x plus y minus E x minus E y whole square. So, I open up the square term, then this will be x minus E x whole square plus, this should not there plus y minus E y whole square plus twice x minus E x into y minus E y. Now, from linearity of the expectation function we have seen it earlier, that it expected function is a linear function.

So, it follows that the expected value of the whole expression I can take expectation inside and therefore, this will be expectation of x minus E x whole square plus expectation of y minus E y whole square plus twice expectation of x minus E x into y minus E y. Now, this is the place where I will use the linearity of x and y, because if x and y are the independence of x and y, if x and y are independent and x minus E x into y minus E y are also independent this being a constant.

So, it can immediately be concluded that, if x and y are independent and then x minus E x and y minus E y are also independent, therefore the expectation of the product can be is equal to the product of the expectations. And therefore, expectation of this product is equal to expectation of x minus E x into expectation of y minus E y and so now, here you see that this would be E x minus E x which is 0 into E y minus E y which is also 0, so with the whole thing is 0. And so this expression reduces to simply expectation of x minus E x whole square plus expectation of y minus E y whole square, which is equal to variance x plus variance y. So, when trying to compute the variance of sum of two variables, two random variables, then I need to be able to write it like this, I need x and y to be independent.

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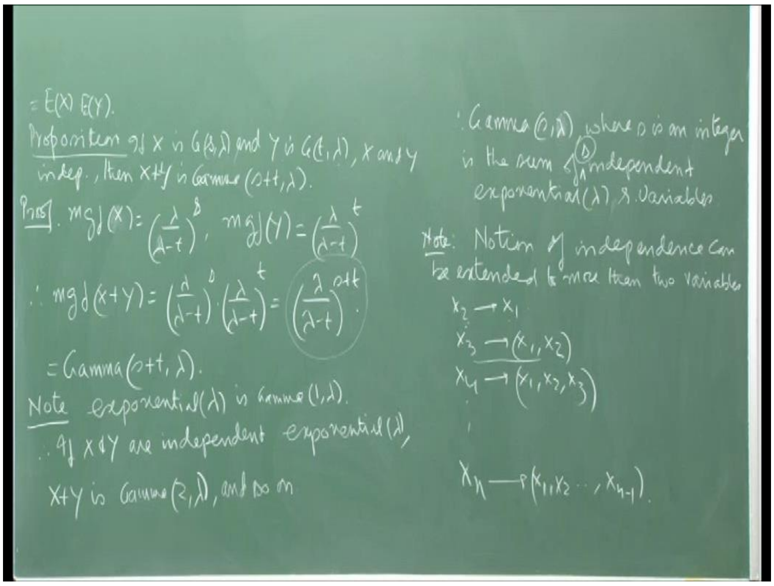

Now, let me show you what independence means to us and of course, interesting relationship between various PDF's that we have discussed or into special kinds of random variables. Now, if x is gamma s lambda that means, parameter s and lambda and y is with gamma with parameters t and lambda, so lambda is the same, but these two numbers differ, then x and y are independent. Then x plus y is gamma s plus t, lambda that means, the first parameters either second parameter is the same, then the first one get added up, when you add up the two random variables, at this I will show you using the m g f, because I missed out on the m g f.

So, m g f is also straight forward because this is expectation of e raise to t x plus y. So, then this way because I will write it as e into t x and into e into t y right this function I can write as this then again I just told you that. So, you might say that y are these independent that can also be shown that if 2 function are independent, then their functions are also independent. Of course, under certain condition, but e raise to t x and e raise to t y are also independent random variables, so when I write expectation of the product that will be the product of the expectations.

See this nothing much to you can actually use the again write the joint this thing, because joint density function will again with the product of individual PDF’s. So, even when you write this function here as an integral you will separate out the integrals and again just as you prove this result exactly you can show by this expectation is equal to the product of the expectations right. So, then; that means, if x and y are independent then there m g f of the sum is the product of the individual movement generating functions.

So, there is a important result and that is what I am going to use here, so I am showing you that we know that when x is gamma s lambda. It is movement generating function is given by lambda upon lambda minus t raises to s and the movement generating function of g t, lambda is lambda upon lambda minus t rise to t. And so the sum would be the product of the m g f ‘s, so the m g f of the sum is the product of the m g f ‘s and so this becomes lambda upon lambda minus t raise to s plus t.

So, now, I am concluding immediately that this; that means, they p d f of x plus y is gamma with parameters s plus t, lambda while defining m g f for you and looking at it is properties I had mentioned that m g f uniquely characterizes the p d f of the random variable, if you know the m g f of a random variable then can tell from what is the p d f of the random variable this can also be proved, but I will simply use the result here.

And so therefore, once I know that the m g f of x plus y of this form then I will conclude that this; that means, the p d f of x plus y must be s plus t, lambda. And now let me this, I had already mentioned this relationship between the exponential and gamma I said that exponential lambda is gamma 1, lambda; that means, the first parameter is 1 and the this is the common thing. So, when you have a exponential random variable it is also gamma 1, lambda.

Now, if x and y are independent exponential lambda random variables both having the same parameter, then by this result x plus y will be gamma 2, lambda. So, I am using this result here that the first parameters get added up if the second parameter is the same, since the exponential x and y both are exponential with, so they are both gamma 1, lambda, so their sum will be gamma 2, lambda and so on right.

So, and now what I am trying to say here is that concept of independence of two variables can be extended to many more in the discrete case I already shown you that characterizing the independence of more than 2 variables becomes tedious, but we can anyway use the end results. So, here see the thing is that you can sequentially use this result you know about m g f's about variance and expectations. So, what we saying is that if x 2 first of all you have 2 variables x 2 and x 1 are independent then once you have this then you can apply that x 3 is independent of x 1, x 2. So, then you can apply the result; that means, you can go on adding.

So, what I am trying to say here is that if you have two exponential random variables both with the same parameter I add them. So, I get a gamma to 2, lambda then add x 3 to it so; that means, I will be talking of x 1 plus x 2 plus x 3, so this will become 3, lambda. So, the concept can be recursively used, so here then you will say that x 4 is independent of these 3 and so on. So, finally, x 1 is independent of x 1 x 2 up to x n minus 1, so this is the whole idea.

So, therefore, now I can say that when you have gamma s, lambda where s is an integer. So, if you look at this result then s is a integer, so gamma s, lambda is the sum of s independent exponential lambda random variables. So, when s is an integer gamma distribution has been built up by adding exponential distributions as of them right and if you remember when I was talking of the gamma distribution and then we were looking at the, so we had said that suppose there is a service counter.

And they are people I had a few who are n minus 1 you are the another person then gamma and lambda you can; that means, the time that you have to spend and we had said that if this time taken to service each customer is exponential with parameter lambda. Then the total time till the another person gets serviced; that means, this is this includes the servicing of n minus 1 people ahead of him and then the n the person is this person. So, when then total n people get serviced that will become gamma and lambda.

So, this was the connection and so we I had use this and I had told you that we will be able to show this prove this result also that the n exponential distributions with the same parameter lambda will add up to gamma distribution with parameters n, lambda. So, I will continue this exploration more in the coming lectures.