Applied Multivariate Analysis

Prof. Amit Mitra
Prof. Shramishtha Mitra
Department of Mathematics and Statistics

Indian Institute of Technology, Kanpur

Lecture No. # 07

Some Problems on Multivariate Distributions - II

In the last lecture we were looking at some by elementary problems on Multivariate Analysis and Some Problems on Multivariate Normal Distribution let us continue with this problems until we solve all these problems.

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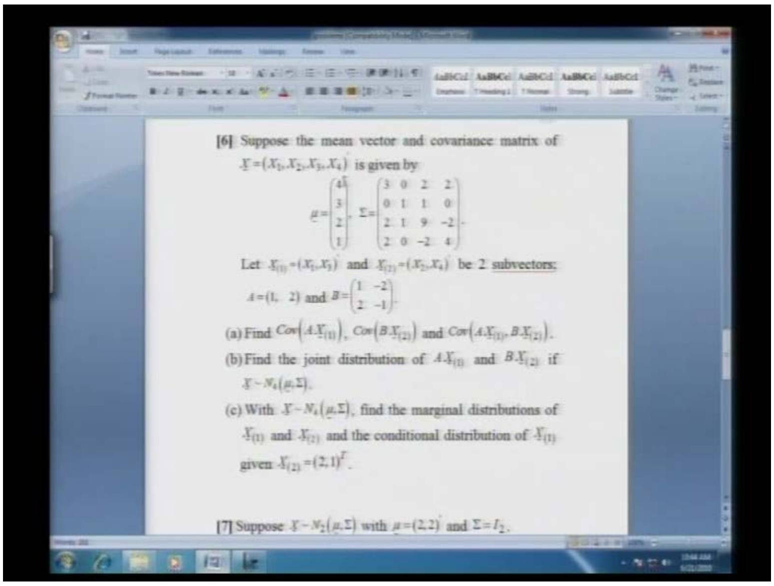

We were discussing this problem number 6 here where we given for the 4 variate distribution the mean vector given by this mu vector and the covariance matrix given by this sigma matrix.

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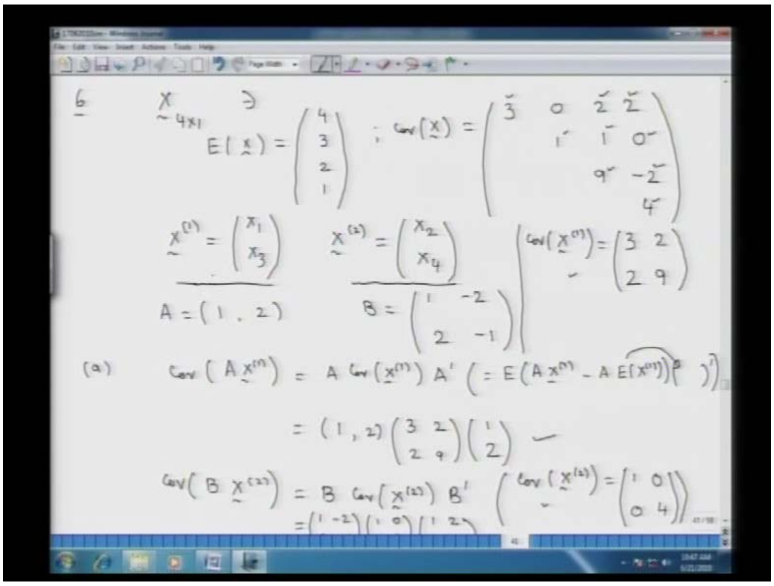

We had let us see we had completed problem number a of this that is we had looked at what is the covariance of A X 1 derived from that multivariate vector and covariance of B X 2 and also further what we had done was to find out what is the covariance between the two components A X 1 and B X 2.

Let us now move on to this problem number b of problem number 6 part b of this problem. Here we are trying to find out what is the joint distribution of A X 1 and B X 2 if we have the joint distribution X following a multivariate normal 4 dimension.

Let us see how to obtain that.

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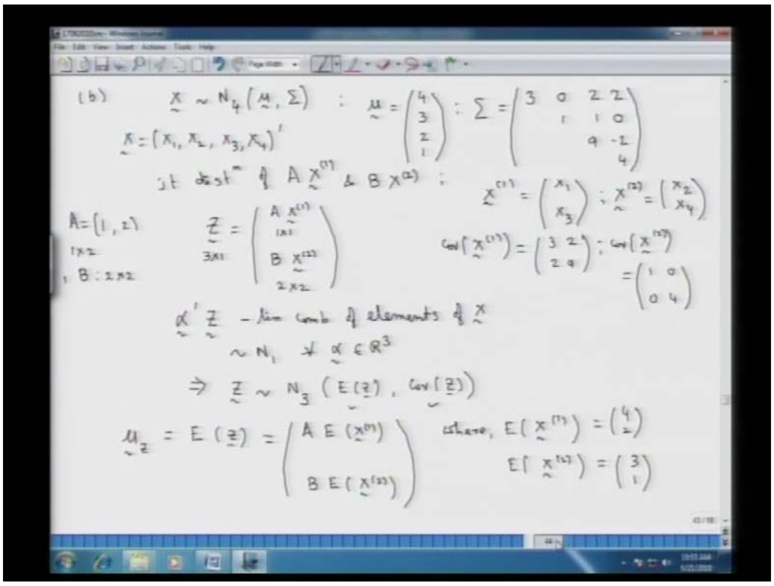

For this b part here the given conditions are that that we have X N 4 variate normal with a mean vector mu and a covariance matrix sigma where this mean vector mu was as given earlier. That is 4 3 2 1.This is 4 3 2 1.We had this sigma matrix the 4 by 4 symmetric positive definite matrix which was given by this particular quantity here 3 0 2 2.

Then the other elements where 1 1 0 then it is 9 minus 2 and 4 . This is the symmetric matrix. No need to write the lower diagonal part here .Now what in this problem we are trying to find out the joint distribution of A X 1 sub vector and B X 2 the second sub vector.

Now, note that these sub vectors where defined to be the following quantities that X 1 was given by this and X 2 was given by this and accordingly what we had seen is that the covariance matrix of X 1 sub vector was given by this 3 2 2 9 matrix and the covariance matrix of X 2 sub vector derived from this covariance matrix of X was given by this 1 0 0 4.

This is what we have now in order to find the joint distribution what we will define is this whose joint distribution is required to be obtained B X 2 .Now here let me make it complete that this X 1 sub vector was let see that was X 1 X 3. This was X 1 X 3 the two components and this X 2 sub vector is this X 2 and X 4 and what we had also obtained in the last lecture was covariance matrix of X 1 to be 3 2 2 9 . The covariance matrix of this X 2 sub vector was the write as 1 0 0 4 .

Now, if we look at this Z vector here which is having this has the first component and this has the second component then if we look at alpha prime Z .Now this is going to be linear combination of the elements of the original X vector is 4 dimensional X 1 X 2 X 3 and X 4 transpose.

It is a 4 by 1 vector which is having a now a multivariate normal distribution. If we look at this particular alpha prime Z now this is going to be linear combination of the elements of X only.

This is nothing, but linear combination of elements of X .Now X is got a multivariate normal distribution. By the definition of multivariate normality that X will have a multivariate normal distribution if and only if every linear combination is going to be a N 1 random variate and hence this alpha prime Z which is nothing, but linear combination of elements X is going to be distributed as N 1.

This is going to be distributed as N 1 this is true for every alpha in the appropriate phase generated by this dimension of this particular vector what is the dimension .We may note that this A was given by let us see A is 1 by 2 and this B is 2 by 2 A was 1 2. This is 1 by 2 this B was a 2 by 2 matrix and hence the order of this particular sub vector is going to be 2 by 2 and what about this one. Now this is going to be A multiplied by that two dimensional vector.

This is 1 by 1.Its scalar random variate. This is not has got the order then it is 3 dimensional and hence this alpha that is what we are taking for checking whether it is a multivariate normal distribution is this alpha belonging to R to the power 3. For every alpha belonging to R to the power 3 this alpha prime Z nothing, but a linear combination of the elements of the X vector the original 4 dimensional vector is going to be N1.

This would imply that this Z vector which is 3 dimension is going to be a multivariate normal 3 dimension with expectation of Z and the covariance matrix of Z . Let us now find out what is this expectation Z and this covariance matrix of Z that will actually complete this particular problem.

Expectation of this Z vector is nothing, but expectation of this particular element out here. That that would be given by A times expectation of X 1 and this is B.A and B are non-stochastic matrices. This is going to be given by this particular element.

Now, expectation of X 1 and expectation of X 2 are simple to be obtained we had expectation of X vector to be given by this. If X 1 is X 1 X 3 then expectation of X 1 would be 4 2 that particular sub vector. Here what we have is the following where expectation of this X 1 sub vector is the corresponding elements from there. That is 4 2 and expectation of X 2 sub vector similarly that is going to be given by 3 1 because the elements are 2 4.

That this is 3 1. This is what is the expectation vector of the Z random vector. Let us denote that by mu Z. This is by plugging in the value of the A vector B matrix expectation of X 1 as given here and expectation of X 2 as given here one can obtain what is explicit form of mu Z .Now the last thing that we need to compute is covariance matrix of this Z vector.

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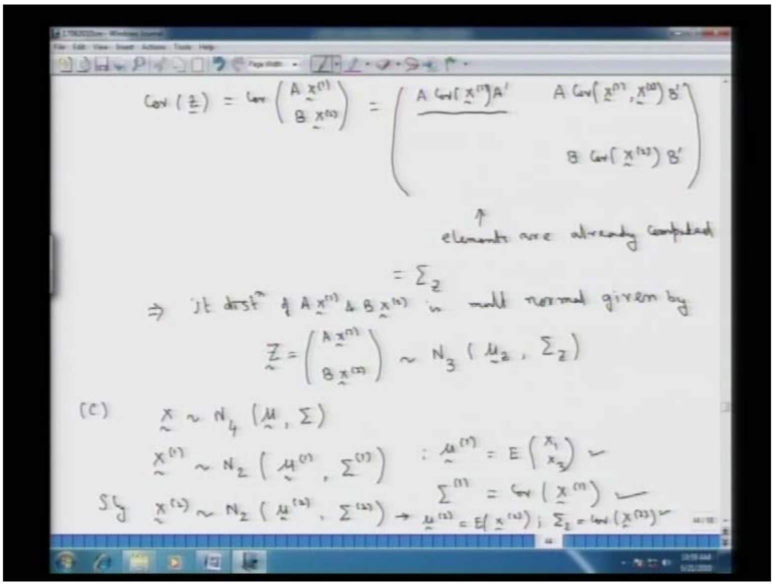

What is that? If we look at this covariance matrix of Z is the covariance matrix of this blocks A X 1 and B X 2. This would be given by we I think we had obtained this term earlier. This the covariance between the 2 A X 1 and B X 2 is going to be A sigma or rather the covariance between X 1 and X 2 which was given by this and this is the B transpose matrix what that is what we had.

Using those what we can write here is that the first block would be the covariance of A X 1. That is going to be A times covariance of this X 1 sub vector times A transpose this is going to be B the covariance matrix of X 2 sub vector multiplied by B transpose and then this part is the covariance between A X 1 and B X 2.

That this is A times covariance of X 1 sub vector with this X 2 sub vector that multiplied by B transpose and this is just the transpose of this particular entry sigma 1 2 out there. Now we have already computed the constituent elements of this A covariance matrix of X 1 into a transpose was given here.

This is covariance of A X 1 which is A covariance matrix of X 1 A transpose which is given by this and we had also obtained covariance matrix of B X 2 which was this particular element here and further more we had this covariance between A X 1 component and B X 2 component given by this and hence we already have all these elements are already computed.

That we can denote this particular term by this sigma Z matrix. That we have the joint distribution this would imply that the joint distribution of A X 1 and B X 2 is multivariate normal given by this Z vector which is A X 1 sub vector B X 2 sub vector that follows a multivariate normal 3 dimension with mu Z as its mean vector and sigma Z as its covariance matrix which is the desired joint distribution as what was required in this particular problem out here.

Now, the c part of this problem is that with the same assumption as what we had made for the b part of the problem that is a 4 dimensional multivariate normal with the given mean vector and the covariance matrix sigma.

Find the marginal distributions of X 1 and X 2 and the conditional distribution of X 1 given X 2 is this particular quantity. This is the c part of this problem .Now we have X following a multivariate normal 4 dimension with mean vector as mu and covariance matrix as sigma. Now X 1 is A sub vector derived from this X here and hence this is going to be a 2 dimensional normal distribution or a bivariate normal distribution with mean as mu 1 sub vector and a covariance matrix as sigma 1.

Now, we have already obtained these elements. This mu 1 is equal to expectation of this X 1 X 3 sub vector which we have already computed and sigma 1 is covariance matrix of

X 1 which also we have computed. This is basically the marginal distribution of X 1. Similarly X 2 the sub vector which is comprising after the second and forth element in this X vector that is also a bivariate normal distribution with mean vector as say mu 2 and covariance matrix as sigma 2 where this mu 2 is expectation of the second sub vector X 2 and sigma 2 is the covariance matrix of the second sub vector sigma 2 is the covariance matrix of X 2 sub vector which once again both these elements we have already computed.

These 2 are the 2 marginal distributions of the 2 sub vector X 1 and X 2 .

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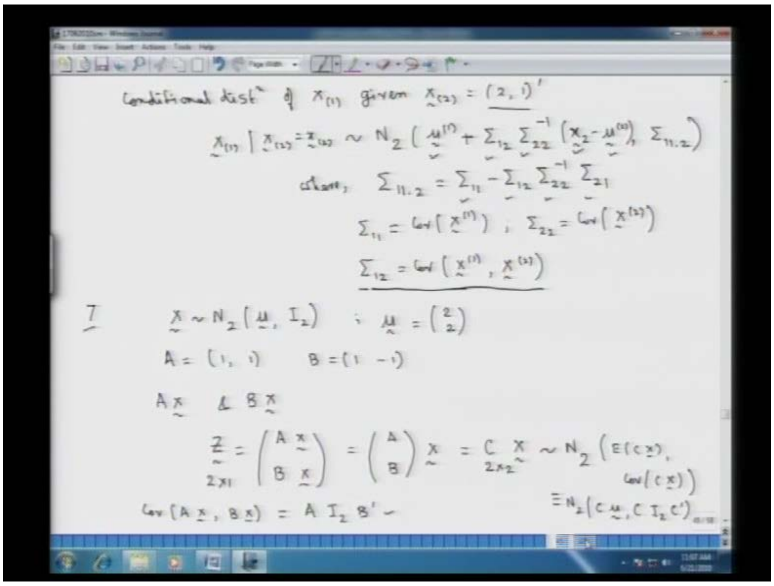

The last part is to look at the conditional distribution. This conditional distribution of X 1 given X 2 sub vector as 2 1. This is what we are trying to find out.

Now, in the theory lectures we have seen that if we have such a partition then this X 1 given X 2 equal to x 2 the given specified vector which is 2 1 for this case is also a multivariate normal with the dimension of the dimension of this X 1 sub vector which is being conditioned by the other sub vector.

This mu 2 would be given by this multivariate normal two dimension will have a mean vector as mu 1 plus sigma 1 2 sigma 2 2 inverse X 2 minus mu 2. This is going to be the mean vector corresponding to this multivariate normal bivariate normal here and the covariance matrix say it is given by sigma 1 1 dot 2 where this sigma 1 1 dot 2 is given by sigma 1 1 minus sigma 1 2 sigma 2 2 inverse sigma 2 1.

Now, these partitions are nothing, but actually is the following where sigma 1 1 is the covariance matrix of X 1 sub vector that is the notation sigma 2 2 is the covariance matrix of the second sub vector which is X 2 and sigma 1 2 is the covariance matrix of X 1 and X 2 and sigma 2 1 is just the transpose of that sigma 1 2 matrix .

If we have these the elements then what we are having was actually this joint distribution of X 1 X 2 rearranged vectors actually rearranged elements forming into the new vector and the conditional distribution X 1 given X 2 equal to x 2 which is this one would be given by this particular multivariate normal distribution with this as its mean vector and this as its covariance matrix.

Now, note that we have already computed mu 1 we know what is sigma 1 2 that is the covariance between X 1 and X 2 sub vector sigma 2 2 inverse is the covariance matrix of X 2 we have also obtained that x 2 is this particular vector 2 1 here mu 2 is the sub vector it is better to use a similar notation. That this mu 1 if that is the sub vector corresponding to the fist element this is the mean sub vector corresponding to the X 2 sub vector. We know all these quantities and sigma 1 1 dot 2 can be computed from here because sigma 1 1 is known to us sigma 1 2 is known to us sigma 2 2 inverse and sigma 2 1 is known to us.

That completes this particular problem of finding out the conditional distribution of X 1 given X 2 equal to this 2 1 .

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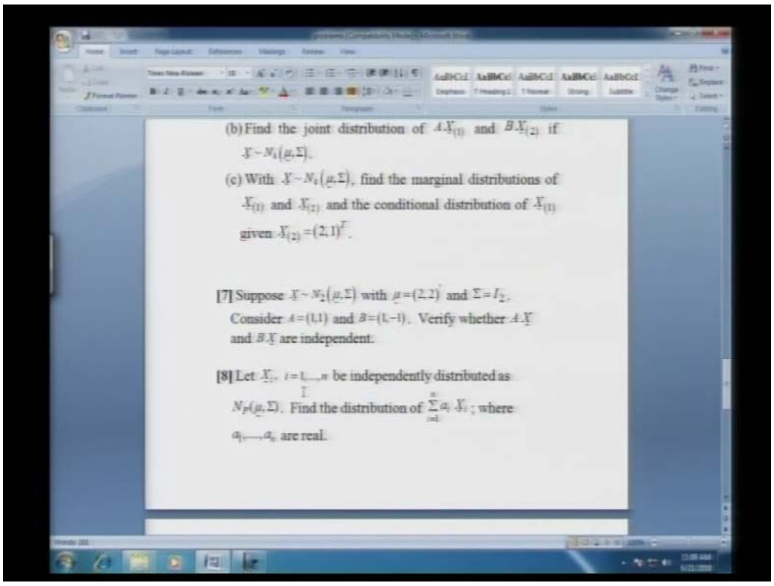

Let us now move on to the next problem we have X A 2 dimensional normal with a mean vector mu and a covariance matrix sigma where mu is given by 2 2 and sigma is an identity matrix of order 2 A is this vector of 1 1 and B is a vector 1 minus 1. Now we the problem is to verify whether A X and B X are independent or not.

Let us see how to solve this particular problem. We have this X vector a 2 dimensional normal with a mean vector mu and a covariance matrix I 2 where this mean vector is having its entries as 2 2 and A is 1 1 and B is 1 minus 1. We have A as 1 1 and B as 1 minus 1.

Now, the 2 quantities in which we are interested in is A X and B X .Now what can be say about A X and B X .Now A X is going to be a normally distributed random variable it is going to be univariate normal because this is A 1 by 2 vector row vector and hence this is going to be distributed as a univariate normal distribution and will be the distribution of B X, but that is not exactly is important in order to verify whether the 2 random variables derived from the random vector X are independent or not the thing that would be of interest is to look at what is the distribution of Z now let us define this Z as A X and B X.

Now, what is going to be the distribution of this particular Z vector. Now we can write this as A B times this X vector. This is some C matrix now what is the order of this C matrix A is 1 by 2 B is 1 by 2 and hence this is 2 by 2 matrix. This is a 2 by 2 matrix C times X.

Now, X has got a multivariate normal distribution in the theory lectures what we had proved is that C X also as a multivariate normal distribution. No need to look at it to fresh and then take linear combinations and argue that every linear combination of that has got univariate normal distribution and hence the joint distribution would be multivariate normal.

This is going to have a multivariate normal distribution with mean vector as expectation of C X and the covariance matrix has the covariance matrix of this C X which we can obtain very easily what is that. This is going to be an N 2 with C times mu as its mean vector C is that matrix and mu is this vector of 2 2 and this is going to be C sigma is this I 2 matrix. C I 2 C prime .This Z vector which is having A X and B X as the 2 constituent elements. This is a 2 dimensional random vector which is having these 2 the 2 quantities in which we are interested in that has got a multivariate normal distribution. Since it has got a multivariate normal distribution in order to see whether they are independently distributed or not well one can actually look at the joint distribution or rather the distribution of Z and then try to find out if the joint distribution of A X and B X is in the form of the product of the 2 respective marginal distributions, but that say combustion way of checking well this is simple in this particular problem.

But better way would be to look at since we have got the joint distribution to the multivariate normal if we just compute what is the covariance of these 2 elements if the covariance is 0 .Then the 2 random variables are going to be independent because we have the joint distribution to be a multivariate normal distribution.

The thing of interest would be to look at what is the covariance of A X and B X. The covariance of A X and B X would be given by A covariance matrix of X that is I 2 and then B transpose . This is the covariance between A X and B X.

Now, let us see what is this equal to.

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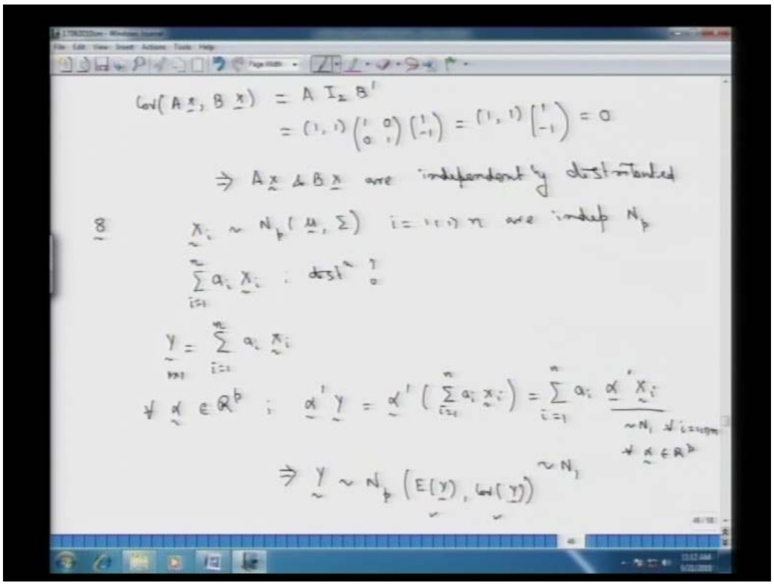

This covariance of A X and B X is given by A sigma now that sigma is what we already have as I 2. A I 2 to B transpose .Now A was given by what 1 1 and B was given by this 1 minus 1 1 1 then this is I 2 and this is 1 minus 1. This is 1 1 1 minus 1. It is 1 minus 1 and that is is equal to 0.

Since we have the covariance between A X and B X equal to 0 and the joint distribution of A X and B X to be a multivariate normal distribution that would imply that A X and B X are independently distributed .That completes this particular problem. This is done.

Let us look at the next problem we have these X i is i equal to 1 to n are independently distributed multivariate normal distribution with mean vector as mu and the covariance matrix as sigma. We are trying to find out the distribution of this particular quantity where these a i is are real constant.

Let me try to look at how to solve this problem this problem number 8 is what we have as X i is following N p with the mean vector mu and the covariance matrix as sigma i equal to 1 2 up to n are independent N p distributions. Now the quantity of interest is a i X i i equal to 1 to n .

We are trying to find out the distribution of this particular quantity summation a i X i. Now let us denote this by a vector Y which is this linear combination of these this actually is the result corresponding to the univariated result where we have we know that linear combination of any univariate normal distribution is also having a univariate normal distribution. This basically is that actually this is the linear combination of n independent multivariate normal distribution. That is also going to be having a multivariate normal distribution.

Now, how do we prove that this is going to be a multivariate normal distribution and what are going to be its parameters. This is this Y is having the same dimension has the dimension of X. This is a p by 1 vector.

For every alpha belonging to R to the power p this alpha prime Y is nothing, but alpha prime of this summation a i X i i equal to 1 to n. This is equal to summation i equal to 1 to n a i alpha prime X i .

Now, each X i is having an univariate normal distribution. This alpha for every alpha belonging to R to the power p as what we have taken this alpha prime X i is going to be an univariate normal distribution. This is going to follow an N 1 for every i equal to 1 to n and for every alpha belonging to R to the power p.

Each of these distributions for every alpha and for every i in this summation are going to have a distribution which is univariate normal and hence the linear combination of those univariate normal distribution is going to be an univariate normal distribution. This follows N 1.

What does that imply for every alpha belonging to R to the power p this alpha prime Y has got an N 1 distribution. This would imply that this vector itself from the definition of multivariate normality that this Y vector will follow a multivariate normal distribution N p with expectation of this Y as its mean vector and the covariance matrix of Y as its covariance matrix.

In order to complete this problem what we would require is find out what is expectation of Y and what is covariance of this Y vector.

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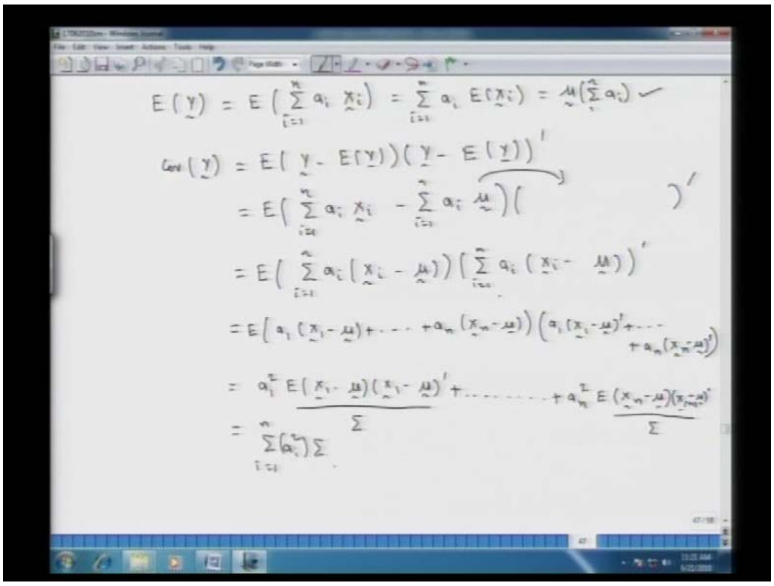

Expectation of this Y vector is nothing, but expectation of this summation a i X i.

Now, each of these a i is are multivariate normal V dimension with mean vector has mu vector. That this is nothing, but we can take expectation term by term. We will have this as expectation of each of these X i vectors and each of them remember as got identical distribution which is that mu vector. This is each one of them are going to be mu vectors. What is this is going to be given by this mu is the constants irrespective of the ith component and then this is summation a i this i is from 1 to n. This completes the first part of it this component expectation of Y computed.

Let us look at what is covariance matrix of Y similarly. Covariance matrix of Y from its definition is expectation of Y minus expectation of Y vector into Y minus expectation of Y transpose .

This is equal to expectation of summation a i X i i equal to 1 to n this minus this particular quantity which we can write as summation a i times this mu vector i equal to 1 to n and then the transpose of this particular entry out here. That this is of the form that it is summation a i i equal to 1 to n X i minus this mean vector mu that multiplied by its transpose say it is a i X i minus mu whole transpose.

Now, if we take expectation of the products that would component form this particular expression there going to be the following. Let us just look at this in a simple way. This is a 1 X 1 minus mu. That is the first term in this n term summation out here and this is the last term a n X n minus this mean and then we will have the n terms corresponding to this entry our here. There are n entries out here.

We will have this as a 1 X 1 minus mu its transpose plus the last term which would be a n X n minus this mu transpose . We are going to be take expectation term by term for the entries of this particular product.

Now, let see what is going to happen if we take expectation term by term if we look at the first entry here now remember that X i is are independently distributed. The first term here when it is multiplied with all these n terms here what happens to the first term is the following that it is a 1 square into expectation of X 1 minus mu into X 1 minus mu its transpose.

Now, the second entry here this element multiplied by all the rest of these n minus 1 entries here will lead us to 0 because X i is are independently distributed and hence the covariance between X 1 and X i for i not equal to 1 would all be equal to 0.

There will not be any contribution when this element is multiplied with the rest of these n minus 1 entries on the second quantity now the same thing is going to happen if we look at any entry from here and then we will have only the corresponding similar entries from the 2 giving us nonzero contribution.

What will be having when we take expectation term by term in this particular product is that the first element when multiplied by all these elements and then expectation being taken only the first entry is nonzero which is given by this and all the rest of the n minus 1 entries will be all zeros and the same thing is going to happen when we look at each of these n entries here the one corresponding the same element corresponding to this when expectation being taken over it is nonzero and similar entries will come up and. The entry corresponding to the nth term where this one multiplied by all these n terms only the product expectation of this particular product is going to remain which is going to given by a n square and then the covariance matrix of X n which is nothing, but the sigma matrix which we started.

This is X n minus mu into X n minus mu its transpose all the rest of the entries will be zeros. This each of these entries now are sigma matrixes because we have X 1 X 2 X n i

.i. d as normal multivariate mu sigma. Each of these entries are sigma and thus this is nothing, but summation a i square i equal to 1 to n times this is a scalar constant that multiplied by this sigma matrix. That is going to be given the covariance matrix of this Y vector which completes the problem where we have obtained that this linear combination of these multivariate normal n i. i. d is having a multivariate normal with expectation of Y given by this and the covariance matrix being given by this.

Let us move on to the next problem which is problem number 9 in this problem set.

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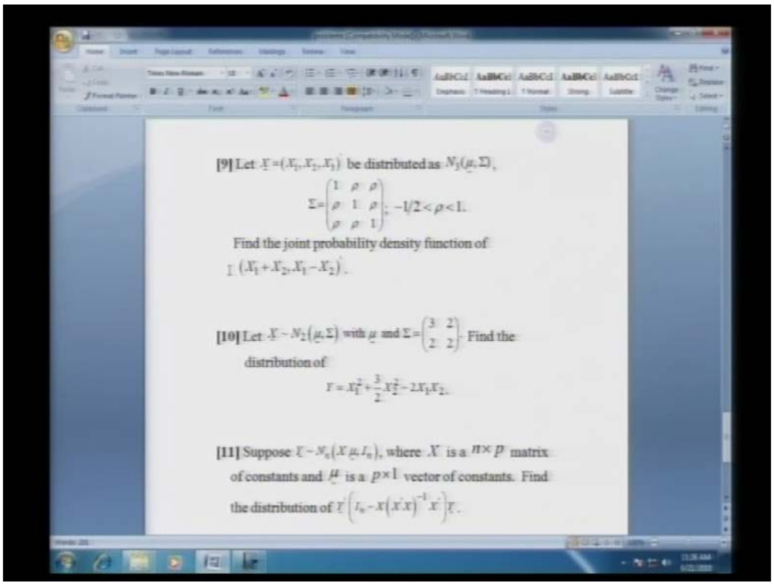

[9] Let $\underline{X}=(X_1,X_2,X_3)'$ be distributed as $N_3(\underline{\mu},\Sigma)$,

$$\Sigma=\begin{pmatrix}1 & \rho & \rho\\ \rho & 1 & \rho\\ \rho & \rho & 1\end{pmatrix};\ -1/2<\rho<1.$$

Find the joint probability density function of $\underline{Y}=(X_1+X_2, X_1-X_2)'$.

[10] Let $\underline{X}\sim N_2(\underline{\mu},\Sigma)$ with $\underline{\mu}$ and $\Sigma=\begin{pmatrix}3 & 2\\ 2 & 2\end{pmatrix}$. Find the distribution of

$$Y=X_1^2+\frac{3}{2}X_2^2-2X_1X_2.$$

[11] Suppose $\underline{Y}\sim N_n(X\underline{\mu},I_n)$, where X is a $n\times p$ matrix of constants and $\underline{\mu}$ is a $p\times 1$ vector of constants. Find the distribution of $\underline{Y}'\left(I_n-X(X'X)^{-1}X'\right)\underline{Y}$.

This is this where we have X a multivariate normal distribution 3 dimension with this has its covariance matrix where row is actually lying between strictly lying between minus half and one note that this particular range of row would be required in order to ensure that this sigma matrix is positive definite.

And we are trying to find out what is the joint distribution of this X 1 plus X 2 first element X 1 minus X 2 the second element .Now this is very straight forward from what we have already solved the problems.

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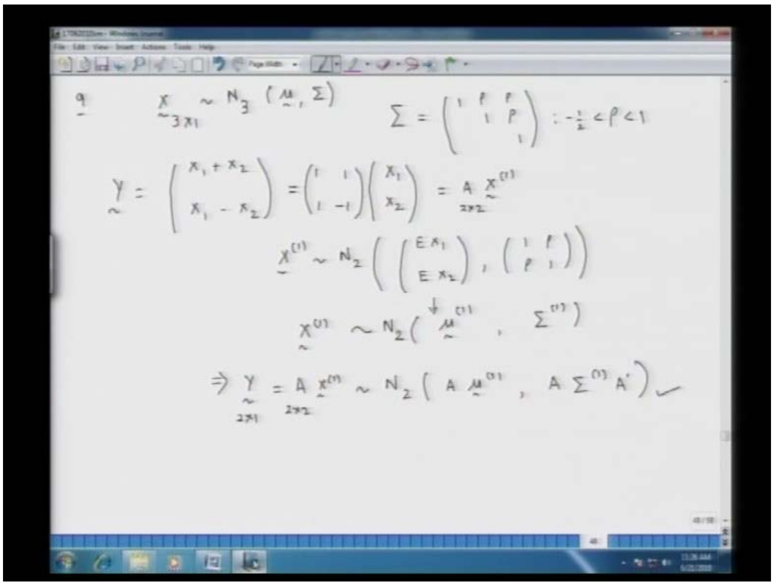

This is problem number 9 we have X the three dimensional vector to have a multivariate normal distribution with the mean vector has mu and a covariance matrix has sigma where sigma is given by let see that is 1 row 1 1 1 and row on the all the half diagonals.

This is with a restriction that minus half less than row is less than 1 this would ensure that this is positive definite and what we are trying to find out is the following the joint distribution of X 1 plus X 2 and X 1 minus X 2.

Let us define this vector Y to be X 1 plus X 2 and this is X 1 minus X 2. Now it is straight forward to see that this can be expressed in this parameter form. If we keep this X 1 X 2 here then by pre-multiplying that with the matrix 1 1 and 1 minus 1. This is what is the quantitive of whose joint distribution we are interested in finding out. This is what this is a sub vector say X 1 where X 1 is having the 2 quantities X 1 and X 2 has the 2 entries.

Now, what is the distribution of X 1 what we can we say from the distribution of this X which is 3 dimensional. Now X 1 is the sub vector which is having the first 2 entries in this particular random vector and this is a sub vector of a multivariate normal distribution. That itself would be having a multivariate normal distribution a bivariate normal distribution here with the mean vector as the corresponding expectations expectation X 1 has its first entry and then expectation of X 2 has its second entry this is expectation of X 2 and then the covariance matrix of X 1 X 2 the X 1 sub vector would be derived from here. That this is 1 row row 1. That is simple.

Let us denote this by mu 1 sub vector and this by sigma 1 matrix. Let us have this particular notation going with n 2. That we will have this X 1 random vector to have this distribution and then we are interested in finding out the distribution of Y and that is to real.

This would imply that this Y which is A X 1 will have now what is the dimension of A . A is the 2 by 2 matrix. This is 2 by 2 matrix and hence this vector bivariate 2 dimensional vector would be having bivariate normal distribution with mean as A mu 1 and the covariance matrix as A sigma 1 A transpose and that is the solution because A is given by this mu 1 is this particular part here and this sigma 1 is given by 1 1 row row .

This is the desired joint distribution of this X 1 plus X 2 has its first entry X 1 minus X 2 has the second entry of this particular random vector. This completes the proof of or rather the solution of 9.

Let us now look at the solution of problem number 10 and 11 which would complete the problems of this particular set what is problem number 10 we have X a multivariate normal 2 dimension with this has its covariance matrix we are trying to find out what is the distribution of this particular quantity.

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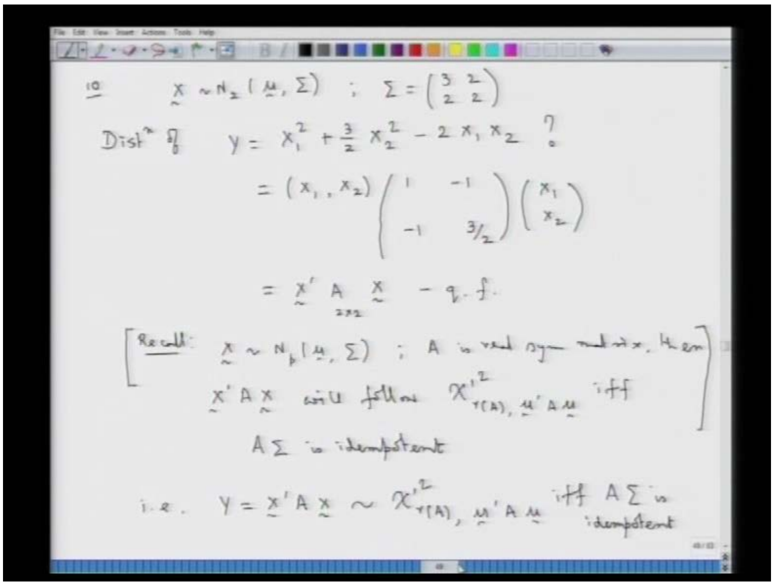

Let me carry forward this information what sigma as its covariance matrix and where sigma is 3 2 2 2 . This is a positive definite matrix and we are trying to find out mu is any vector that may be specified y is X 1 square Y is. We are trying to find out distribution of Y equal to X 1 square plus 3 by 2 X 2 square minus 2 X 1 X 2.

We are trying to find out what is the distribution of this particular random variable. Now note that this is actually quadratic form in the components of this random vector X. We actually would write it in the way quadratic form results in theory where actually presented.

Note that this particular quadratic form can be written compactly in the following form that this is X 1 X 2 which is X vector nothing, but X vector and then the following matrix which is having the square entries corresponding to X 1 is one corresponding to X 2 is 3 by 2 and then the half diagonals are going to be X 1 and X 2 terms.

Now, that is going to be half of each of these terms. That X 1 X 2 product had a coefficient minus 2. That is divided equal into the 2 half diagonals and then this is X 1 X 2. This is nothing, but of the form that it is X transpose A matrix A is a 2 by 2 matrix that multiplied by X. This is a quadratic form the type of quadratic form results for which we have actually done in the theory lectures.

We are trying to find out the distribution of this. Now remember what the type of result that we had actually proved in theory is that I will just put it in bracket to recall the following result that if X follows a multivariate normal with a mean vector mu and covariance matrix sigma and if we have A a real symmetric matrix, then X transpose A X and say that will follow a chi prime square with rank of A has its degrees of freedom and non-centrality parameter has mu prime A mu if and only if A sigma is idempotent .

This result can be used in order to solve this particular problem that is what we had this random variable expressed in the terms of quadratic form that is using this result our Y random variable which is expressed in the form that it is A transpose A X will follow chi prime square with rank of A has it is degrees of freedom and mu prime A mu has its non-centrality parameter if and only if the A that we defined this is our A and this is our sigma if and only if this a sigma is idempotent.

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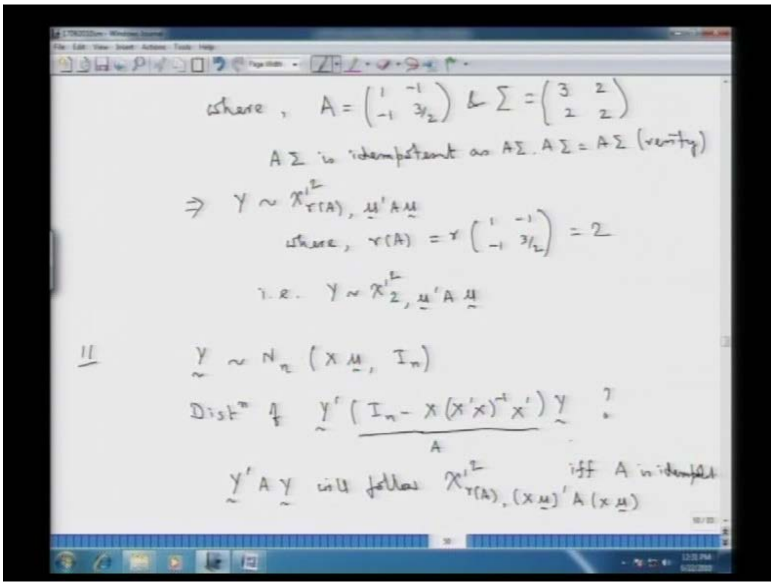

Now, it is easy to see actually that this A where this A is what we had obtained here as 1 3 by 2 minus 1 minus 1 1 3 by 2 minus 1 minus 1 and sigma is the variance covariance matrix which is 3 2 2 2 .It is easy to check that this A sigma that is what we have is idempotent as A sigma into A sigma is equal to A sigma 1 can easily verify this by simple matrix multiplication.

And hence this would imply that our random variable Y which is that quadratic form follows chi prime square with rank of A as it is degrees of freedom and mu prime A mu as it is non-centrality parameter. Now what is rank of A where rank of A is rank of this matrix 1 3 by 2 minus 1 minus 1 it is easy to see that this is of full rank and hence this is of rank 2.

And whatever be the mean vector specified then that would actually lead us to the explicit form of this non-centrality parameter otherwise we have Y this random variable a chi prime square on 2 degrees of freedom and the non-centrality parameter being given by mu prime A mu where a is given by this matrix and mu is the mean vector .

That is solves this particular problem we will look at the next problem which is problem number 11 this is Y following a multivariate normal n dimension with X mu as its mean vector and it is covariance matrix as i n where X is n by p matrix of constants and mu is a p by p vector of constants then we are required to find out the distribution of this quadratic form.

Let us see how to get this problem done. This is what is our given condition. Y is an N dimensional multivariate normal with a mean vector as X mu and I n has its covariance matrix. Now we are trying to find out the distribution of Y prime I n minus X X transpose X inverse X transpose times Y .

The question is to find out what is the distribution of this. Now this problem once again is in line of finding the distribution of quadratic forms because this is a quadratic forms. We can denote this particular matrix by A matrix A and then verify whether the conditions for this quadratic form to follow a chi square distribution are satisfied.

Now, the variance covariance matrix under this setup is I n. This Y prime A Y this Y prime A Y will follow A non-central chi square on rank of A as it is degrees of freedom and now the mean vector corresponding to Y is X mu. That we will have this as X mu transpose A X mu as this as its non-centrality parameter if and only if now since sigma the associated variance covariance matrix is I n we will require the condition that this will follow a non-central chi square if and only if A is idempotent .

Let us see whether that condition is satisfied for this particular quadratic form and what is the rank of A in under such a situation and what happens to this non centrality parameter X mu transpose A X mu.

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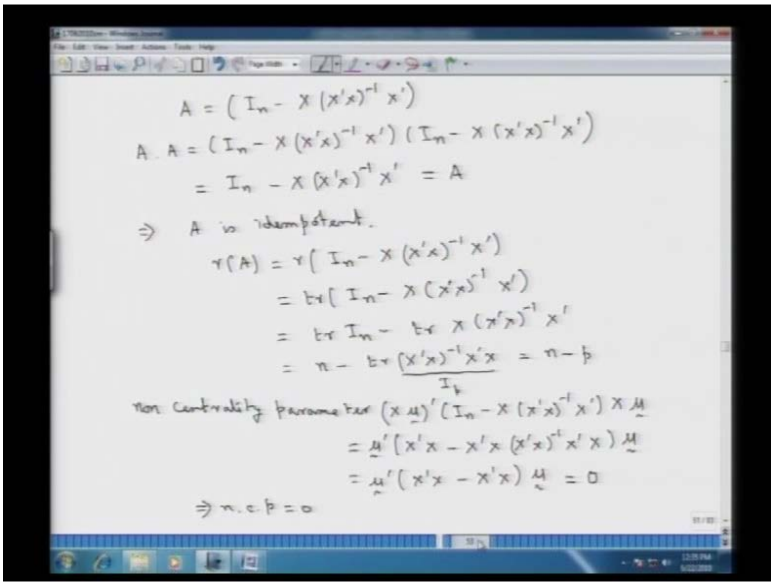

This matrix A in the present situation is our I n minus X X transpose X inverse X transpose. What is A into A that is I n minus X X transpose X inverse X transpose into I n minus X X transpose X inverse. This is basically the projection matrix. This multiplication readily will actually lead us to observe that this is nothing, but I n minus X X transpose X inverse X transpose which is nothing, but A matrix.

This would imply that A is idempotent in our case. This quadratic form which we have here will follow A non-central chi square as of now we will compute what is rank of a and what is this non-centrality parameter in order to complete this particular problem.

Now, A is idempotent. What we have is rank of A is rank of our I n minus X X transpose X inverse X transpose. This rank is equal to trace it is an idempotent matrix. That we will have this as X transpose X inverse X transpose. This is trace of A minus B. It is trace of A minus A trace of B.

We will have this as trace of I n minus trace of X X transpose X inverse X transpose. That this is equal to n minus now trace of this quantity is trace of X transpose X transpose X inverse into X transpose X. This is an I p matrix. That this is equal to n minus p where p actually we assume that X is having full column rank X matrix what we had in this problem was n by p. We assume that X is of full column rank that is rank of X is p and under that condition X transpose X inverse is a p by p matrix which is non-singular and hence this is I p and this is n minus p.

Now, what happens to the non-centrality parameter we were supposed to have the non-centrality parameter as the following non-centrality parameter was this X mu transpose A is our I n minus X X transpose X inverse X transpose X times mu.

Now, take this X transpose from the left hand side and X from the right hand side. Pre-multiplying this particular quantity with X transpose and post multiplying by X what we will be having is the following that this is mu transpose and then this is X transpose X minus X transpose X X transpose X inverse X transpose X that multiplied by this mu vector.

Now, what this is equal to identity matrix of order p. That this is finally, equal to X transpose X minus X transpose X only that multiplied by mu prime and hence this is equal to 0.

The non-centrality parameter of this quadratic form Y transpose A Y is equal to 0 .

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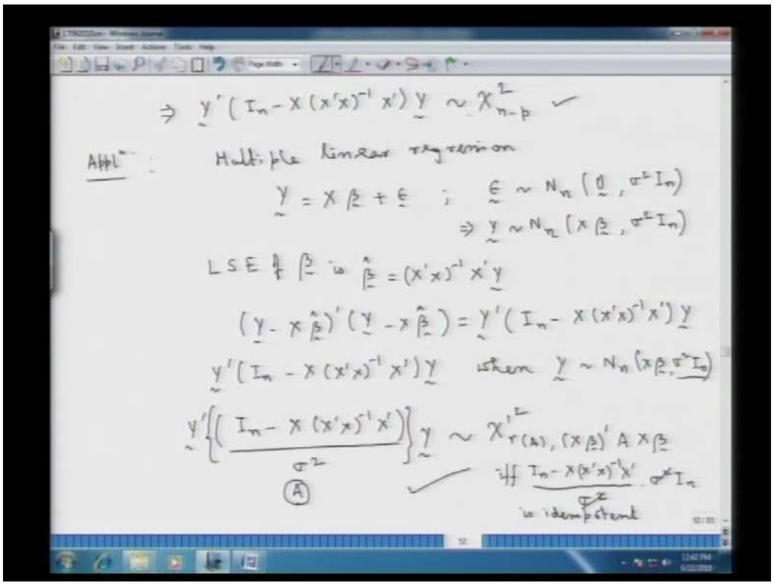

And hence we can finally, write the distribution of the quadratic form in which we were interested in. This Y transpose I n minus X X transpose X inverse X transpose Y this thus follows.

Now, the non-centrality parameter being 0 implies that the non-central chi square is actually a central chi square. That we will have this as the central chi square random variate with degrees of freedom as rank of A and rank of A we had derived as n minus p and the non-centrality parameter vanishes.

This actually is the desired distribution of the quadratic form in which we were interested in now as an application of this particular result we will see that .Suppose we have a multiple linear regression problem setup where we have Y is equal to say X beta plus epsilon with the assumption that epsilon follows and N dimensional multivariate normal with a mean vector as null vector and a covariance matrix sigma square i n this is the standard setup for a multiple linear regression problem.

In such a situation this would imply from the assumption on epsilon that this Y also follows a the multivariate normal distribution n dimension with a mean vector as X beta and a covariance matrix as sigma square I n .

Now, in under such a situation what we usually look at is a following form which is residual sum of squares denoted by R. S. S at the point beta hat that the least square estimated, but before that let me just for completion sake write what is the least square estimator the least square estimator of beta the linear regression parameter vector is given by this beta hat which is X transpose X inverse X transpose Y.

Now, if we look at the quantity which is Y minus X beta hat transpose Y minus X beta hat. This is the residual sum of square at the least square point. We can replace this beta hat by this X transpose X inverse X transpose Y it is easy to see that this form here reduces to this Y transpose I n minus X X transpose X inverse X transpose Y X transpose Y .

If we have this to be the quadratic form we see that we are of course, interested in finding what is the distribution of this residual sum of squares which is given by this it tally is with the quantity of interest in the problem that we had this Y transpose I n. This result actually can the previous problems result can be used to derive the distribution of this.

Now, with just a bit of question that we are looking at this Y transpose I n minus X X transpose X inverse X transpose Y. When Y follows X beta multivariate normal N dimension with mean vector as X beta and a covariance matrix as sigma square I n .

If we are looking at the distribution of this with Y following this the only point at which this derivation of the distribution of this differs from the previous problem that we have just now solved is that the previous problem was solved under the assumption that the covariance matrix of Y is I n and this has got a sigma square term present in it.

In order to find the distribution of this what is usually done is to look at Y prime I n minus X X transpose X inverse X transpose this divided by sigma square Y. Now considering this particular element as the matrix A we will be able to show that this A times sigma which is this sigma square I n. That we will have this to follow a chi prime square on A rank of this A matrix and the non-centrality parameter to be given by this X. Beta transpose this a matrix times X beta if and only if our A times the sigma matrix under this situation which is sigma square I n is idempotent.

If and only if I n minus X X transpose X inverse X transpose by sigma square this is our A now A times sigma which is sigma square I n is idempotent .Now as we see that this sigma square can cancels out and what we have the condition for chi square distribution is that I n minus X X transpose X inverse X transpose is idempotent which we have already proved in the previous problem.

That this really is idempotent and hence this is going to follow this particular term this divided by sigma square is going to follow a chi square distribution the non-centrality parameter once again would vanish and the rank of this A matrix would be rank of this as in the previous case it is going to be n minus rank of X which is if the full column rank is assumed that is going to be n minus p once again. The previous problem result is actually applicable in finding out the distribution of this quadratic form which is the residual sum of squares with beta being replaced by beta hat the ordinary least square estimator the distribution of that to follow a central chi square on n minus rank of X degrees of freedom would follow from that particular previous result .