**Applied Multivariate Analysis**


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**Lecture No. # 04**

# Multivariate Normal Distribution – II

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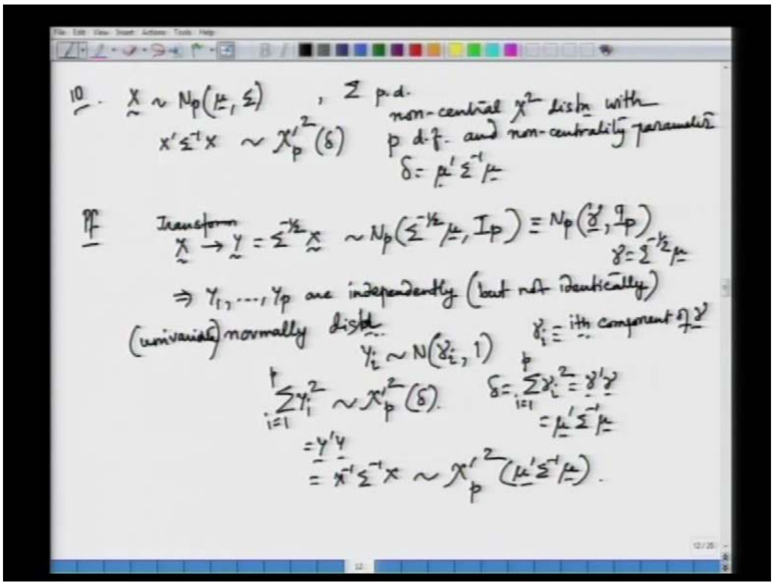


We will considering distributions of quadratic forms involving the random vector X, which follows a multivariate normal distribution p-dimensional with mean vector mu, and covariance matrix sigma. The first such quadratic form, that we had discussed; if incidentally is the exponential part of the multivariate normal p.d.f, and very conveniently, we ended up with central chi square distribution with degrees of freedom equal to dimension of the data, that is X. The next one, that we consider note note that, here we do not make any change in the location, and it is simply in the form of X transpose sigma inverse X. And as expected, this is going to be a non-central chi square distribution, the degrees of freedom will remain the dimension of the data, that is p and the non-centrality parameter delta is going to be mu transpose sigma inverse mu.

Let us look into the proof of this result. If you recall, we had taken a transformation in the earlier case, here also similarly we are going to transform from x to y with a slight change. Now, as expected since we do not have the location change. So, our transformed variable Y is simply the sigma half inverse matrix with X, without that minus mu part. Of course, we have at every step sigma is the positive definite matrix. So, that this matrix is defined. Now, this transformed variable Y is obviously, going to follow up p-dimensional multivariate normal distribution. The mean is going to be sigma minus half mu, and what is going to be the covariance matrix of this, as we can easily see.

Since we can very conveniently write sigma as product of sigma half and sigma half, this is just going to be the identity matrix of order p. So, let us write this as some normal p gamma I p where gamma is nothing but, this mean vector sigma minus half. Now, this implies that, our y 1 to y p which are the components of the y vector. These are independently ofcourse, because look at the covariance matrix that the shear, that this I p. So, they are independently but not identically. Unless we put some extra restrictions on the mean vector, we cannot have the identical situation. So, these are independently but not identically normally distributed.

And this is obviously going to be univariate normal distribution that we are talking of. Infact, we can write that, say any y i of i th component. It follows a normal distribution with gamma i and 1, where gamma i is nothing but the i th component of the gamma vector. Now, if this is so, what we have? We have sum of square of these normal variables sum from 1 to p. This is obviously going to be a non-central chi square distribution. The degree of freedom is going to be the number of terms, we have here and what is going to be the non-centrality parameter? well

If i take this as some delta, delta is nothing but it is sum of these gamma i square i from 1 to p, which is nothing but gamma transpose gamma and that is nothing but, if I Put back gamma equal to sigma minus half mu. I am going to get this as mu transpose sigma inverse mu and what is summation y i square in matrix notation? This is nothing but, y transpose y. And if I again put back the form of y here, this is nothing but x transpose sigma inverse x; which I am going to have as non-central chi square with p degrees of freedom and the non-centrality parameter as mu transpose sigma inverse mu simple. So, this proof all ends here. We are ending up with the non-central chi square distribution.

Unlike the earlier result, where we ended up with as expected; we ended up with a central chi square distribution. Now, note that in both these quadratic forms, the associated matrix that we had was the inverse of the variance covariance matrix of the random vector x, that is sigma inverse. Next, we are going to take up a series of results where we will consider quadratic forms involving x, but they associated matrix will be something else. It is not necessarily the sigma inverse matrix will take up any matrix and try to look at the distribution of the quadratic forms. Obviously, we will need some condition on the matrix that we choose. So, let us look into what are the what are the conditions that we need. So, we are starting with the new sequence of results.

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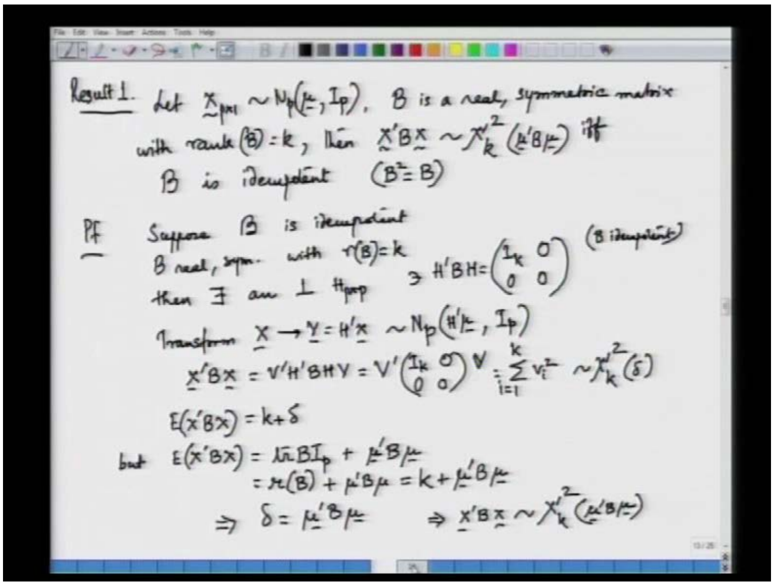

So, let us call this as result 1 now and the first one in this setup is, we consider say the random vector x p dimensional following a multivariate normal mu and I p. So obviously, we are starting with a simpler situation, where the variables are uncorrelated and the covariance matrix is the identity matrix. So, now we take B is a real, symmetric matrix with rank of B equal to some constant we are taking k. Then, the quadratic form involving x with B as the associated matrix; x transpose B x is going to follow a non-central chi square distribution with k degrees of freedom. k, which is the rank of this matrix B with the non-centrality parameter mu transpose B mu and the condition on the matrix that we were mentioning is if and only if, B is idempotent.

So, we need this condition on the B matrix that, B is idempotent; that is, B square is equal to B. This is an if and only if situation. So, the proof of this result, first we take up the if part, sufficiency part; where we assume suppose B is idempotent. So, we are assuming these idempotent and another given setup. We will try to proof that, this quadratic form is actually following a non-central chi square distribution with its degrees of freedom equal to the rank of this B matrix and the non-centrality parameter as mu transpose B mu ok. So, what we do here as a first step is, we have B is real, symmetric; those are already given to me.

So, we have B real, symmetric with rank of B equals to k. Now, this is not a full rank; it is less than p. Otherwise, we we we are going to end up with an identity matrix, an idempotent matrix of full rank. So, this is of rank k, which is less than p. And then, there exists an orthogonal matrix H such that, we are going to have H transpose B H is a matrix which takes this feature. This is because B being an idempotent matrix has eigen values equal to 0 and 1. Moreover B having rank equal to k; there will be k 1's and rest of them, that is the rest of these matrixes, which is this is this is p dimensional. So, p minus k of them will be 0 and hence, we have an identity matrix of k here. These are eigen k. The k eigen values of the B matrix.

So, we have this. So, here at one step we are using B idempotent. So, we are using this since B is idempotent. We have this sort of a breakup and then, we use a transformation. We go from x to say some v, where v is H transpose x ok. So, now what is the distribution of this v? well From our earlier results, we know that this is a p dimensional normal distribution with mean equal to H prime mu. What is the covariance matrix? well We have started with a sigma matrix covariance matrix, which is the identity matrix. So, here we have H transpose H, but then again H is orthogonal; so, I have simply the identity matrix of dimension p here.

So, next is what is the look of this quadratic form that we are considering? So, we have the quadratic form x transpose B x. Using the transformation, this is v transpose H transpose B H v, because x is nothing but H v. So, we are getting this sort of a matrix here and with v transpose, what is our H transpose B H? B being idempotent with rank k, I can have this as I k which we already stated in the beginning and I have a situation like this. So, this simply gives me the sum of the square of v i's i from 1 to k of them now only. So, this is by our result that, v is following a normal with H prime mu and I p.

This sum v i square; this is obviously, going to follow a chi square a non-central chi square distribution with degrees of freedom k as many number of terms, they are are and suppose the non-centrality parameter is delta, we are yet to decide; what is the value of this delta? So, what is the value of delta? well We have expectation of x transpose B x, which follows this non-central chi square distribution. So, this is going to be its degrees of freedom plus the non-centrality parameter. But we also have a result that, expectation of quadratic form like this is actually equal to trace of… well the The general result is that, if you have X following multivariate x is following, x is having mean vector mu and dispersion matrix sigma.

Then, expectation of x transpose B x is nothing but, trace of A sigma plus mu transpose A mu. So, here instead of A, we are using B. The sigma covariance matrix is I p and then, what is our mean vector? Mean vector is nothing but, mu of x; that is, and we do not have A here, but this is some B. So, this is mu transpose B mu. This is trace of B I p, which is trace of B basically and again, we are using the idempotency of the matrix B and this is nothing but, rank of B; since for an idempotent matrix rank of a matrix is equal to its trace.

So, we have rank of B plus mu transpose B mu. But rank of B has been assumed to be equal to k. So, this is k plus mu transpose B mu giving me the non-centrality parameter delta is nothing but, mu transpose B mu. So, I have the quadratic term form x transpose B x is following a non-central chi square with k degrees of freedom. Again, k is the rank of the matrix B and the non-centrality parameter mu prime B mu. mu is the mean vector of x and this is happening, if B is idempotent. We go to the next part, the converse of the proof that is we assumed now.

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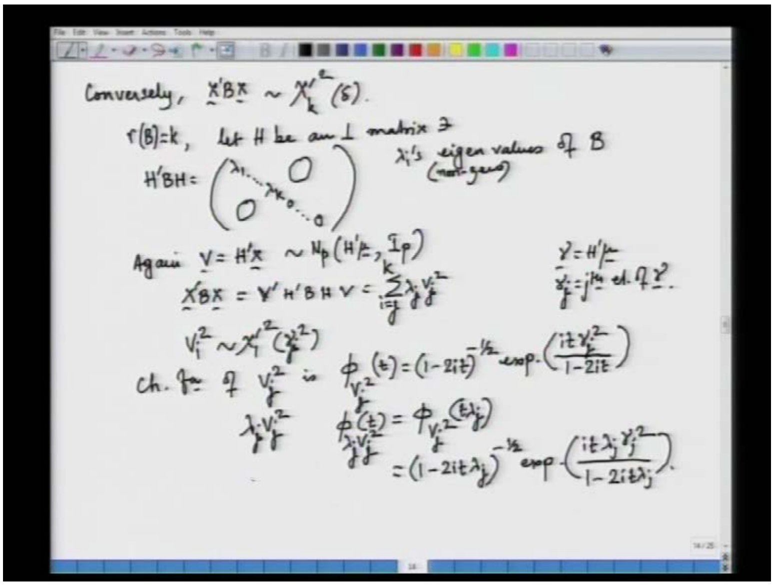


Conversely, we have x transpose B x is now following a non-central chi square with the parameter non-centrality parameter delta. And we start with let not let; because we have been given in the earlier setup that, B is real symmetric with rank of B is equal to k. So, we have rank of B equal to k, but here we do not have the idempotency of B; we have to establish that. And so, we say that in the situation we have, let H be an orthogonal matrix such that, H transpose B H is now a diagonal matrix. But the diagonal elements are say some lambda 1 to lambda k of these and then, rest of them are all zeros and it is a diagonal matrix. So, the all elements below and lower the diagonal all are zero.

So, I have lambda i's are eigen values. are These are essentially the non-zero eigen values, which is obvious non-zero eigen values of B; because it has rank k. So now, this is the form of H transpose B H. Now, as oppose to the earlier one, where we had assumed idempotency; here, we do not assume idempotency. Then, the transformation again we have v is equal to H transpose x. If you recall and this again follows the normal p with H transpose mu and the identity matrix of order p has the covariance matrix. x transpose B x, the quadratic form is nothing but, in this situation again it is as before v transpose then we have H transpose B H v.

And now, the similarity ends here; because our H transpose B H is something different to the earlier case and we have the presence of these lambda i's in it. So, this is again square of v i square, but with the lambda i is before them and for k such terms. So, I have this as lambda i v i square; i from 1 to k. Now, what is v i square, the distribution of v i square? well I am just taking one of them and this is obviously going to be non-central chi square with 1 degrees of freedom. I just have one of it and the non-centrality parameter is nothing but, gamma i square; where again, I have gamma vector is the mu mean vector here and gamma i is nothing but, i th component of the gamma vector. So, square of it is the non-centrality parameter.

Now, we will look into what is the characteristic function of this variable? So, characteristic function of v i square at t is one considering phi of v i square at t. well I am directly using the characteristic function of a non-central chi square distribution and this is given by 1 minus 2 i t to the power minus half and then, we have an exponent term which is i t and then, we have the gamma i square; the non-centrality parameter coming into the picture divided by 1 minus 2 i t. Next, we consider what is the characteristic function of lambda i times v i square? But for this, I can very conveniently use this earlier characteristic function.

Because what I am looking at is characteristic function of lambda i v i square at t. But this is the characteristic function of v i square not at t now, but at t lambda i. So, what we do is essentially replace t by t lambda i in this expression and we get the characteristic function of lambda i v i square. So, this is now equal to 1 minus 2 i t lambda i. One thing we can do here is, instead of writing the summation over i, we can make it as j; because we have this imaginary number i over here. So, let us make all these as j. So, that is a small correction that we are doing all of these are now j.

So, this is now the j th element of gamma and this is actually i, the imaginary number and here, again we are making this as gamma j square. So, this is lambda j v j square, which is the characteristic function of the v j square at t lambda j. And now, this is t replaced by t lambda j in this expression and then, this is nothing but same thing exponent i t lambda j gamma j square by 1 minus 2 i t lambda j. What we need is actually the characteristic function of summation lambda j v j square. We have come up to the stage, where we have the characteristic function of lambda j v j square.

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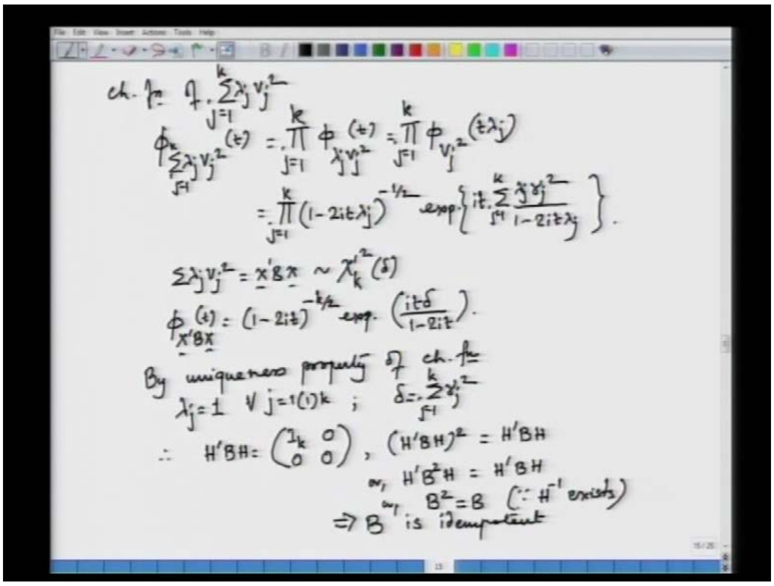

Then, what is the characteristic function of sum of lambda j v j square; j from 1 to k. So, I have this as phi of sum of lambda j v j square at t. Since the v j's are independent, I can very conveniently use that this is nothing but the product of the characteristic functions of lambda j v j square at t, product over j from 1 to k. And we already have the expression for this characteristic function, which is again nothing but phi of v j square at t lambda j. And we will have to take the product over j from 1 to k, that is all we have to do, and so, this nothing but we have the product of the first terms. That is, 1 minus 2 i t lambda j to the power minus half, and for the exponent part the product can be replaced by a sum.

So, I can write here exponent of i t; this is free of j and the rest of them is coming under the summation term and this is lambda j gamma j square by 1 minus 2 i t lambda j. So, the summation actually involves both the numerator and the denominator and sum is over j from 1 to k. So, this is it. So, this is the characteristic function of summation lambda j v j square, which is nothing but the quadratic form x transpose v x. We have lambda j v j square. This is nothing but x transpose B x and this following some non-central chi square with k degrees of freedom and a non-centrality parameter delta has the characteristic function x transpose B x at t.

This is simply 1 minus 2 i t to the power minus k by t k is coming for the degrees of freedom part and we have exponent of i t delta, the non-centrality parameter and 1 minus

2 i t. So, what we have obtained is that, x transpose B x has this characteristic function; also it has this characteristic function. But we use the uniqueness property of the characteristic function and compare these two expressions to get. So, we will say by uniqueness property of characteristic function, we get that lambda j is infact equal to 1 for all j from 1 to k. Not only that, we also have the non-centrality parameter delta is actually equal to summation of gamma j square; j from 1 to k.

If this is the situation, now we have started with H transpose B H as a diagonal matrix, where the diagonal elements k of them one non-zero and those well lambda 1 to lambda k. But we have ended up with the situation, where I have proved those lambda j's are nothing but 1, for all j from 1 to k. So, what can I say now? So therefore, the H transpose B H matrix reduces to I k and the rest of the blocks has null matrixes. Now, we can see that, this H transpose B H matrix is idempotent. So, we have H transpose B H square is actually equal to H transpose B H. Using the orthogonality of the H matrix, we have this side as or we have H transpose B square H as H transpose B H or I have B square is equal to B.

Since well H being an orthogonal matrix, its inverse exists. Since H inverse exists, I can very well write from here that, B square is equal to B implying that, B is idempotent; this is what we had wanted to proof in the necessity or the only if part. So, now we are going to talk about the next extension to this result. A natural improvement on this would be, if you recall that, we had started from the setup; where x was multivariate normal with a mean vector a non-null mean vector. But the covariance matrix was the identity matrix. So, now we we will talk about the correlated case, where we take a general positive definite matrix sigma as the covariance matrix.

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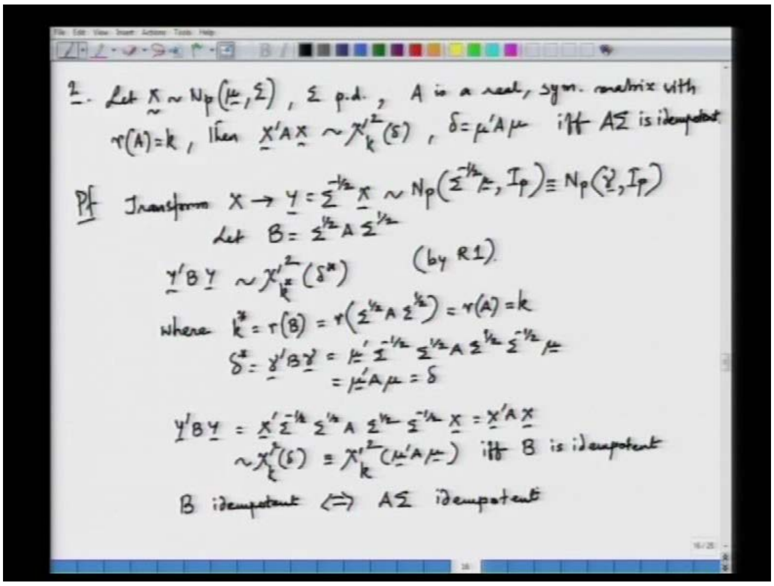


So, next result is in this new series, is we have now x following a p variate normal with mean vector mu and covariance matrix sigma; sigma is positive definite; we have rest of the setup is almost same. So, we have a matrix A is a real, symmetric matrix with rank of A equal to k. Then, obviously now the associated matrix is going to be A. Then, the quadratic forms x transpose A x; this is going to follow a non-central chi square distribution with degrees of freedom equal to the rank of A matrix. And non-centrality parameter delta equal to mu transpose A mu; everything remains same, what will change is the condition that we need here. So, if and only if A sigma is idempotent.

So obviously, we will get back our earlier result, if we put sigma equal to an identity matrix. Now, since we have some sigma; so, this is coming into the picture. It is not enough that, A only is idempotent; we have to have A sigma as an idempotent matrix. As before we start with a new transformation and what we do here is, let us see if we can take the help of the proof of the earlier result, that we proved just now. So, what we do is make some transformation to get back our earlier setup; so, that we can directly use the proof of the earlier result. So, we start the proof with a simple transformation, which was not required in the last case.

Transform x to x and take y is equal to the square root inverse of sigma; sigma half; this is possible, because sigma is positive definite and x. So, this is following a p variate normal with mean sigma minus half mu and covariance matrix as I p. So, by this transformation, if you see that we are going back to our earlier setup, the only change is here in the mean vector part. And what is important is, we have reduced the covariance matrix to an identity matrix. So, we have made the case similar to the uncorrelated case ok. So, this is now I p and we are using some notation for this; for simplicity, we are writing this as the gamma vector and covariance matrix as I p.

Now, if in this situation in this setup, I can directly use the first result and say that, the quadratic form y transpose B y. Before that, I need to say something about this B matrix. What is this B? So, we have used one transformation; we have said that, let y be this and one more step here. Let us have some new B, here we using the same notation as we used as we had used for the earlier result. The associated matrix B is nothing but, sigma half A sigma half. And now, we consider the quadratic form y transpose B y; this is going to follow a non-central chi square distribution say with some degrees of freedom k star. What can I say? By our earlier result, what is k star should be; well it should be the rank of the B matrix.

And some non-centrality parameter delta star, what was it in the earlier situation? well It was mu transpose B mu, but if you see our mu has now changed to gamma. So, this delta and delta star, there will be slight change in this. Let us see, what what is that? So, where k star, by result 1 we have this and k star is nothing but, rank of B; that is also something which our earlier result tells us, but what is B? It is rank of sigma half A sigma half right and both these sigma, this is same matrix. So, this sigma half matrix being a non-singular matrix; so, rank of A does not change when it gets pre multiplied and post multiplied with the non-singular matrix sigma half. And so, this is nothing but, equal to rank of A and which is k. So, this k star is nothing but k.

Let us see, what is now happening to delta star? well delta star, again by our earlier result there it was mu transpose B mu. Here, it is going to be the new mean vector, which is gamma. This is nothing but, gamma transpose B gamma. Now, let us use the original forms of gamma and B in terms of sigma mu A etcetera. So, this is nothing but mu transpose; we are first replacing gamma transpose a symmetric matrix. So, transpose does not make a difference and then, we are putting what we have used for B; sigma half A sigma half and then, again for gamma we have sigma minus half mu sorry this is yeah this is fine.

So, what do you get here? We simply see that, this gets cancelled with these sigma matrixes; they are getting cancelled and we have this turning up nothing but, mu transpose A mu. So, this delta star is also nothing but, delta k star is same as k delta star is same as delta. Now, let us look at the quadratic form. The new quadratic form, that we got by all transformations and our new definition of a matrix y transpose B y. But again we would like to have it back in our original form. And then, this is nothing but x transpose sigma minus half sigma half A sigma half; this is for b.

And then, we have sigma minus half and x giving very (( )) this is nothing but, x transpose A x. So, basically there is no change. We have we are back at the old quadratic form or original quadratic form that we need and this is following chi square a non-central chi square with k degrees of freedom. Now, I can say instead of saying k star; similarly instead of square here; similarly instead of saying delta star non-centrality parameter, I would say it is delta and that is nothing but, a non-central chi square with k degrees of freedom, non-centrality parameter as mu transpose A mu; if and only if B is idempotent.

Now, this result under this setup, where we have the covariance matrix as the identity matrix has already been proved. So, what is there to think about? well We have to see the thing that, now remains is that remains for us to prove is B idempotent. Here, what what would we like to proof? well It is A sigma is idempotent. Now, proving B idempotent; if it is equivalent to proving A sigma idempotent, then we have through. So, let us now look at that going to be very simple. So, we have B is idempotent and what is B? That is all we have to check it is sigma half A sigma half.

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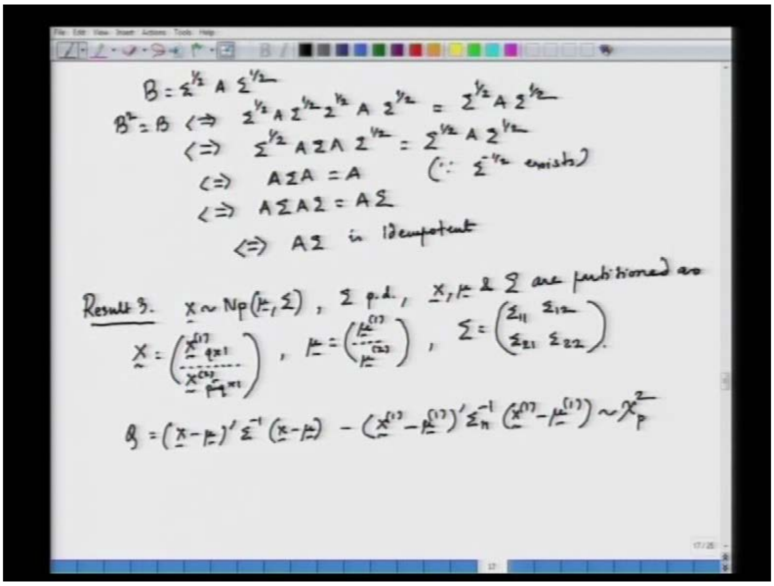

So, B square equal to B implies that, we have sigma half A sigma half again B, which is sigma half A sigma half; this is equal to B matrix, which is sigma half A sigma half. And this is nothing but, sigma half A sigma A sigma half; this is equal to sigma half A sigma half. Now, sigma this being a non-singular matrix, again I can pre and post multiply by its inverse and I get A sigma A equal to A. Since then since this exists, what I do is now simply post multiply by sigma to get A sigma A sigma as A sigma, which proves that A sigma is idempotent. Now, this is equivalent in the sense that, we can trace back to the earlier step in the same way.

And this is nothing but, equivalent to saying that B idempotent is, A sigma is also idempotent. Our next result is a very interesting one. Here, we will consider the difference between two quadratic forms. Each of which is following a chi square distribution and we will see that, this difference is also following a chi square distribution with degrees of freedom being equal to the difference of the individual degrees of freedom. This is the very special feature here. Normally, what we see if we have independent chi square distribution, the sum is going to be chi square.

But here we are having a situation, where the difference is also going to be a chi square distribution. So, this is our third result in this sequence. The setup is now we have, we are now into the correlated situation. So, we have x following multivariate normal with mean mu and covariance matrix sigma. Sigma as always is positive definite. Now, we have the partition situation. We are coming back to that partitioning again and we will have x, then the mean vector mu and the covariance matrix sigma are going to be partitioned as before.

So, x, mu and sigma are partitioned as for x we have, the first one say x 1 q dimensional 1 and the next is taking p minus q elements. Corresponding mean vector mu, similarly partitioned into mu 1 and mu 2 and the covariance matrix comprising a four block matrixes sigma 1 1, sigma 1 2, sigma 2 1 and sigma 2 2. This is all says that, the quadratic form the difference between the two quadratic forms as we have said the two quadratic forms are x minus mu transpose sigma inverse x minus mu; that is, what whatever appears in the p.d.f of the multivariate normal distribution.

This one well we have already consider what is the distribution of this quadratic form. Now, we are going to consider the form, where we are taking out; where we are considering the difference of this with some other quadratic form and that is nothing but, x 1 minus mu 1 transpose sigma 1 1 inverse x 1 minus mu 1. And this is going to follow a central chi square with p minus q degrees of freedom. So, for the proof of this result now, we have sigma positive definite.

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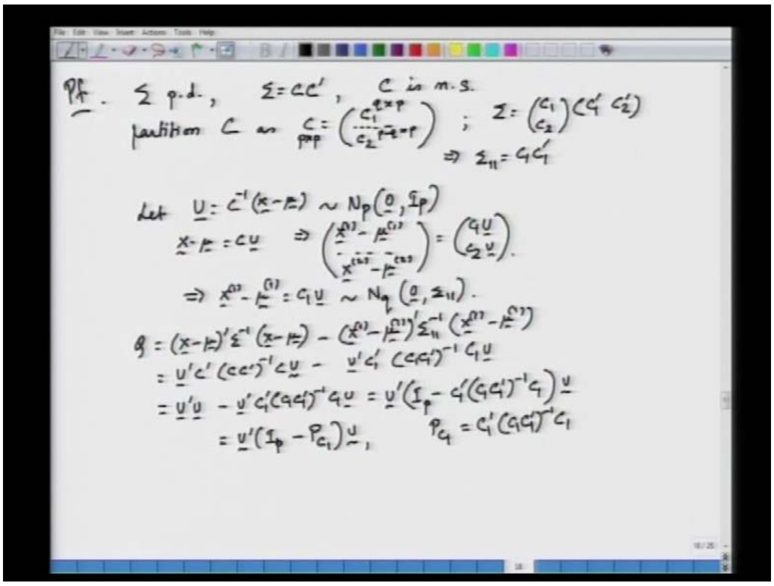

So, I can always consider sigma as some c c transpose, where c is non-singular as sigma right and I can consider a partitioning of this c matrix also. So, let us write partition c as c is nothing but c 1, c 2. Now, this is going to be a p dimensional square matrix as a sigma matrix. So, I am partitioning this means, I have q rows here and p minus q goes here. So, c has been partitioned into c 1 and c 2 in the following in this manner and then, what is the form of … well I have two alternative forms of sigma. In one hand, I am saying that, it is c c transpose; basically which is c 1, c 2 and c 1 transpose, c 2 transpose.

And at the same time, this is sigma is also the given partition that we have. So, this implies that, our sigma 1 1, the first block in the given partition is going to be the same as the first block here, which is c1 c1 transpose. Now, we consider a transformation. Let some u be c inverse x minus mu; no problem in taking this; because c is non singular and I can have the inverse of this; u is c inverse x minus mu. What is the distribution of this? well This is p variate normal with mean 0 and the identity matrix as the covariance matrix. I also have the x minus mu. This is nothing but c u.

So, which basically implies that, I consider the partitioned form in both the sides; so, to get x 1 minus mu 1 and x 2 minus mu 2; this is going to be similarly the partitioned from form from here. That is, by partitioning the c matrix c 1 u and c 2 u. So, I have x 1 minus mu 1, this is c 1 u; whereas, x minus mu is c u and this is following a multivariate normal distribution with dimension as a dimension of x 1. And hence, this is q normal q with mean vector as null and the covariance matrix is going to be well it is going to be c 1. Then, covariance of u, which is nothing but the identity matrix and c 1 transpose. So, what I have is covariance matrixes c 1 c 1 transpose.

But let me use my usual notation for that. So, c 1 c 1 transpose I will simply write this as sigma 1 1. So now, we have to consider the quadratic form over the difference between the two quadratic forms, actually that is Q. Now, let us write this again. This is x minus mu transpose sigma inverse x minus mu, which we have proved; which we have shown to be a chi square distribution with p degrees of freedom and then, we are taking its difference with x 1 minus mu 1 with sigma 1 1 inverse x 1 minus mu 1 a transpose here. And now, let us use all these that we have gained out of these transformation and saying that, these follows normal distribution etcetera.

So, what I can write here is using this u and c, the transformed form. So, we have this quadratic form is nothing but u transpose c transpose. I use c c transpose for sigma. So, this is c c transpose inverse and again for x minus mu, I have c u. Similarly, I have here for this quadratic form, the only difference is that; instead of c, we are going to have the first part of c; that is, c 1 for this quadratic form. So, what I have here is u c1 transpose and then, I have c 1 c 1 transpose inverse. Simply writing c 1 c 1 transpose for sigma 1 1 and then, I have again c 1 u. Now, since c is non-singular, I can breakup this inverse which is occurring here.

And I can very well use that A B inverse is B inverse A inverse, because c is non-singular; c transpose also has to be non-singular obviously. So, I take advantage of that here and this becomes then, simply u transpose u. But note that, no such luck for the second quadratic form; because here forget about non-singularity. The c 1 matrix is not even a square matrix. So, I cannot break open this inverse here. I have to keep it as it is and write rest of it. So, this is as it is c 1 transpose c 1 c 1 transpose inverse c 1 u. What I can do now is, take u transpose coming out from here; then, I have the identity matrix obviously of dimension p.

And then, I can have this c 1 transpose c 1 c 1 transpose inverse c 1 here and take u out from the other side and I use a notation for this matrix. I say this is I p p c 1 matrix; p c 1 is this whole matrix, where p of c 1 is nothing but c 1 transpose c 1 c 1 transpose inverse c 1. Why this particular notation p c 1? because this p c 1, this matrix that we have here involving c 1 is actually projection matrix. So, we are using this p with the subscript c 1 projection on the column space of c 1. So, we have this notation a standard notation for this ok. So, this is what we have Q equal to, that is u transpose I p minus p c 1 and we have proved something for this type of quadratic form; because we have u.

These are independent following normal multivariate normal with mean 0 and covariance matrix identity matrix. So, I can have this very conveniently as a central chi square distribution if and only if, the associated matrix is idempotent. So, all we need to show here is, whether this associated matrix I minus p c 1 is idempotent. If it is, then we are true. We have to check one more thing that is, the degrees of freedom which will be the rank of this associated matrix. And let us go step by step and check whether this matrix is idempotent. well

The projection operator p, this is always idempotent. We have p square c 1 is equal to p c 1. If you take this form of p c 1, what we have here? This is immediately established. Now, if p is idempotent, p c 1 is idempotent; then, I minus p c 1 is obviously idempotent. So, one part is solved. We have the idempotency of the matrix. What we need to show further? well We have this implies that, atleast I have u transpose I p minus p c 1 u. This follows a chi square distribution with degrees of freedom say k star and what is this k star? It has to be the rank of this associated matrix.

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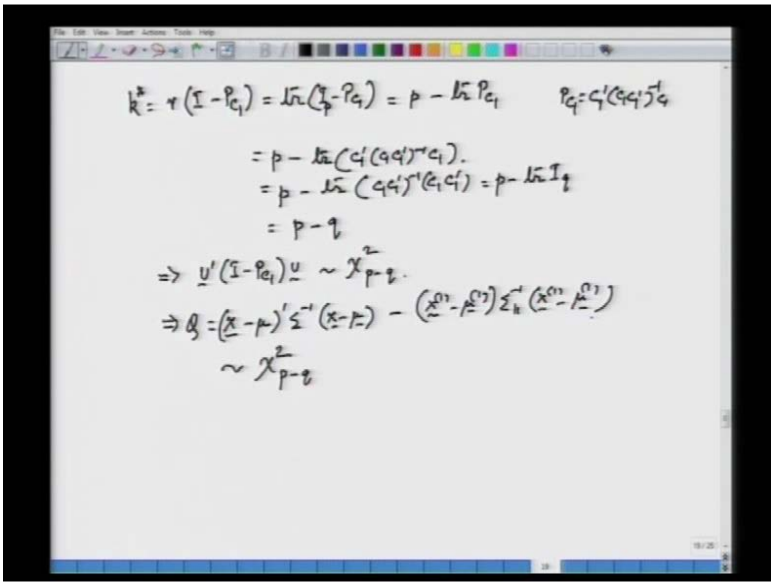

So, k star is nothing but rank of I minus p c 1 and what we use again is, idempotency of this matrix and say that, this is nothing but trace of say matrix I minus p c 1 and I can use the result that, trace of A minus B is trace of A minus trace of B. So, I will write here; this is trace of I minus trace of p c 1. I know trace of I of order p is nothing but p and minus trace of p c 1. Once again, I take the idempotency of p c 1 and I write that, this is nothing but rank of the p c 1 matrix. So, then let us look at this part, what is the rank of this matrix? Which is p minus well I have rank of sorry the trace of I p yes p and then, I have the trace of this matrix.

Let us just recall the form of p c 1. It was c 1 transpose c 1 c 1 transpose inverse c 1 right. So, I have this is c1 transpose c 1 c 1 transpose inverse c 1 and I can use trace of A B is trace of B A. So, this is nothing but, trace of c 1 c 1 transpose inverse c 1 c 1 transpose. So, basically this step is not needed here. Once we are using the rank of the idempotent matrix is trace; after that, we are taking help of the trace factor throughout and this is nothing but, well c 1 is q dimension; q by p matrix and c 1 c 1 transpose is becoming q dimensional square matrix.

So, what we have in the (( )) is trace of I q, so this is basically, I can write here as another step that this is trace of I q, giving me this degrees of freedom k star is actually p minus q. So that, I have u transpose I minus p c 1 u; this is following a central chi square with p minus q, but this u transpose I minus p c 1 u is nothing but our original form Q which is X minus mu transpose sigma inverse x minus mu minus first part of this. We can have similar type of result, where we can involve X 2 instead of X 1, and end up with a chi square distribution with degrees of freedom equal to q.

So, this is following a central chi square with p minus q degrees of freedom. So, difference between two quadratic forms, which are each of them which is chi square distribution, is also turning up as a chi square distribution. And its degrees of freedom is actually, again the difference between the degrees of freedom between these two. So, this completes the proof of this result. The next result, that we are going to take up; we will have an have an important variation from all these results. There we are going to give up the positive definiteness of the covariance matrix sigma. And then, try to show what the distribution of the quadratic form involving x turns out to B.