Complex Analysis

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Module - 4
Further Properties of Analytic Functions
Lecture - 2
Analyticity of Power series

Hello viewers, in this session, we will touch upon Taylor’s theorem, which shows that every analytic function actually has a power series representation around its point of analyticity with some radius of convergence. So firstly, we will start by showing that, power series given a power series it is actually an analytic function, so given power series within its radius of convergence is a analytic function. And then, we will go on to show the Taylors theorem that every analytic function, conversely every analytic function actually has Taylor series, power series representation.

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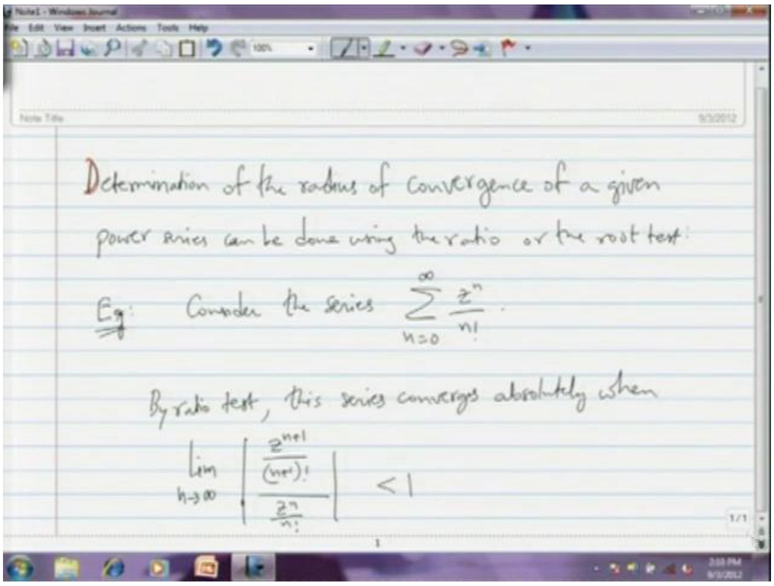

So, firstly you know continuing from the previous session, I want to emphasize that the determination of the radius of convergence of a given power series can actually be done by using either the root test, the ratio test or what I will show you is called the Cauchy Hadamard formula. The determination of the radius of convergence of a given over series can be done using the ratio or the root test.

So, I mean those are two tests at least you can use to find the radius of convergence, so and in case of you are not able to determine the radius of convergence using this two, there is a Cauchy Hadamard formula, which I will show in a moment. So, this is you are aware, so here is an example of using the ratio test to determine the radius of convergence. So, it is an example, so considered the series the power series, so sigma z power n by n factorial and from 0 to infinity, okay.

So, by ratio test this series converges, when converges absolutely, actually absolutely when the limit as and tends to infinity of the modules of the n plus 1 th term. So, the n plus 1 th term, in the series is z power n plus 1 by n plus 1 factorial divided by z power n by n factorial the n-th term. The absolute value of the n plus 1 th term divided by the absolute value of the n th term is strictly less than 1.

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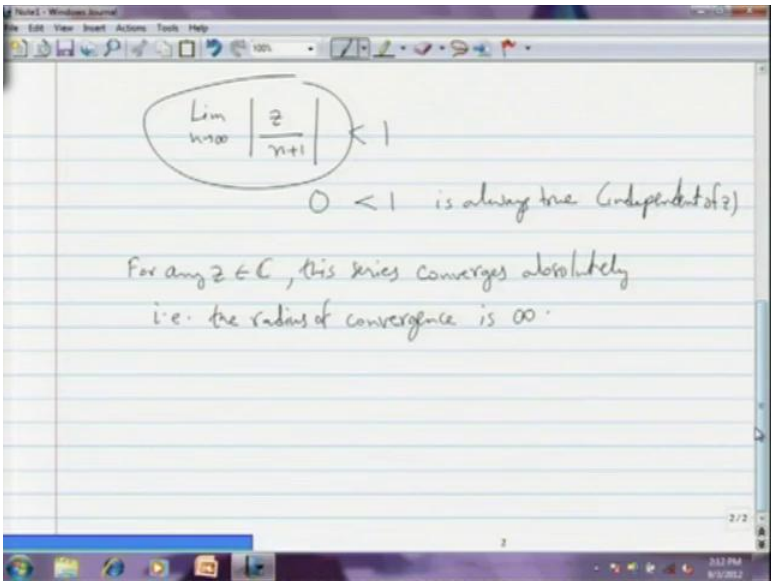

But notice that, the left hand side of this inequality is limit as an n goes to infinity of mod z by… So, z power n cancels z power n plus 1 for 1 power of z and then, you have n plus 1 in the denominator due to the n plus 1 factorial cancelling with n factorial for an n plus 1.

So, this convergence, the series convergence absolutely when the limit of the this quantity is strictly less than 1, but notice that the limit of the quantity 0, no matter what z is. So, 0 is strictly less than 1 is always true independent of z point, so independent of z 0 is less than 1. So, what you have is that for any z belongs to c this series the given series convergence, absolutely i.e the radius of convergence, the radius of convergence is infinity.

So, here we have actually used the ratio test to determine, the radius of convergence of a given series. So, we will come back for this example for a different reason, but for now we can use ratio test to determine the radius of convergence, so also we can use the root test whenever, appropriate to determine the radius of convergence.

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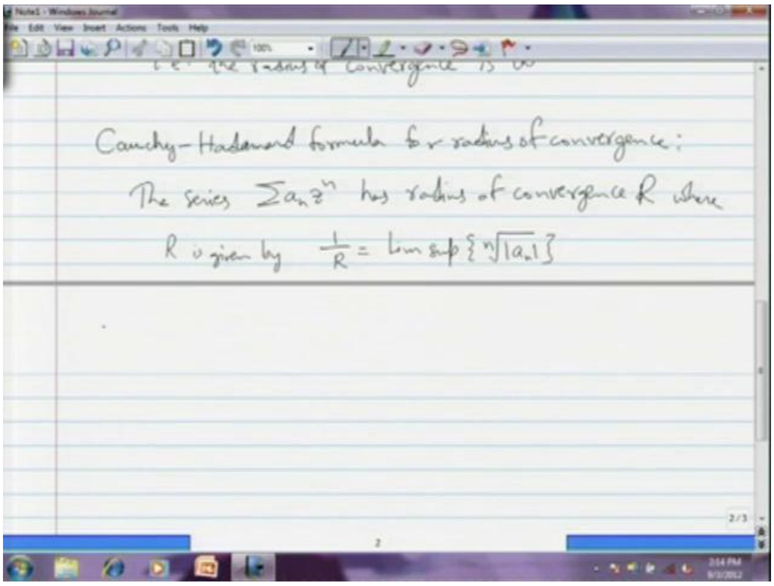

Here is another way; I will provide the Cauchy’s or rather the Cauchy Hadamard formula without proof for radius of convergence. So, by slight modification of the proof of Cauchy’s root test, to determine convergence of series, we can actually give a proof for Cauchy Hadamard formula for radius of convergence.

So, what it says is the series sigma a n z power n has radius of convergence R capital R, where R is given by, R is given by 1 by R. So, it is given by 1 by R is equal to the lim sup of the n eth root of absolute value, the modules of a n. So, 1 by R is equal to the lim sup of entropy, so the viewer most be aware of lim sup is from calculus of one real variable, so if not I will remind what is.

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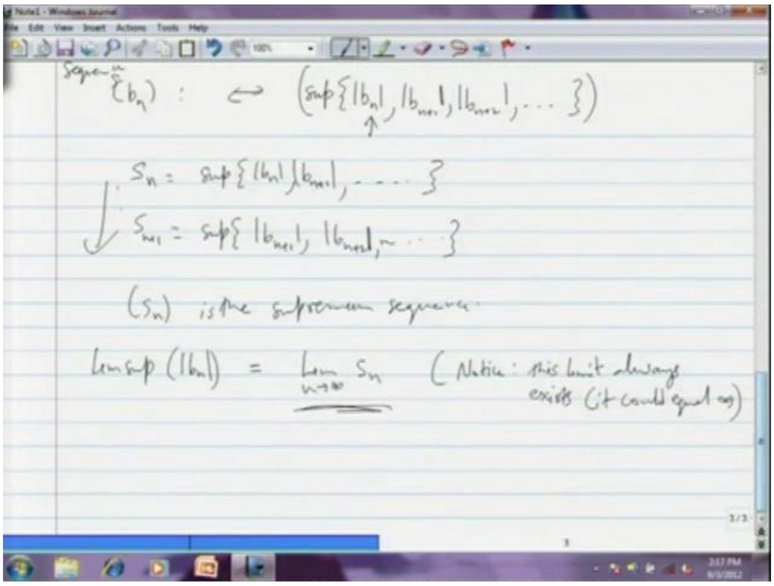

So, if you have a sequence a n, so in this case complex sequence a n then, you have corresponding supreme sequence, supreme of the set modules of a n, so I will use a different symbol here. So, if you have complex equal's b n, let's say then you have the supreme sequence modules of b n plus of b n plus n etcetera modules of b n plus 2 so on.

So, you consider supremum of the set of the modulie of b n, where n greater than or equal to little n, where j greater than or equal to little n and then, so the supreme as this n varies that forms a sequence. So, what I mean by that is, so the supreme sequence s n is equal to the supreme of modules of b n or absolute value of b n b n plus 1 etcetera, this is n s n plus 1 is equal to supreme of modules b n plus 1 modules of b n plus 2 etcetera.

So, this sequence of s n, so this sequence s n is the supreme sequence. So, the lin sup of modules of b n, let us say the lin sup of modules of b n is actually the limit as n tends to infinity of s n, it is easy to see well notice that notice this limit always exists it could equal infinity some times, what that means is the supremum? The suprema are constantly infinity, so it could equal infinity equals sometimes, but nevertheless this limit always exists and that limit of this essence here it is called the lim sup of the modules of b n.

So, likewise given this sequence a n, we can construct the h or we can find the lin sup of the sequence n eth root of modules of a n and that is equal to 1 by the radius of convergence.

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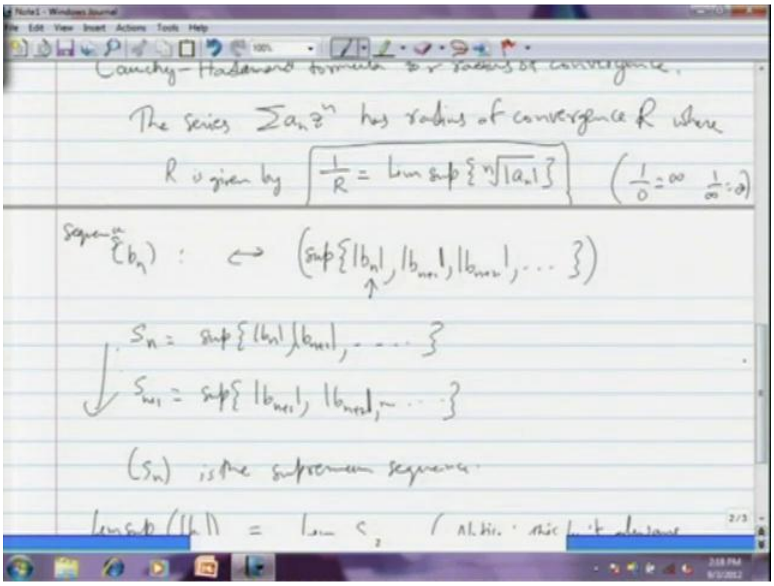

So, here we have the usual agreement that 1 by 0 is actually infinity and actually 1 by infinity is actually 0. So, with that agreement with that agreement on this 2 radius of convergence, we have a formula for radius of convergence given by, you know this equation that is called the Cauchy Hadamard formula, I am not going to give a proof of this here.So, that is about the determination of the radius of convergence of power series.

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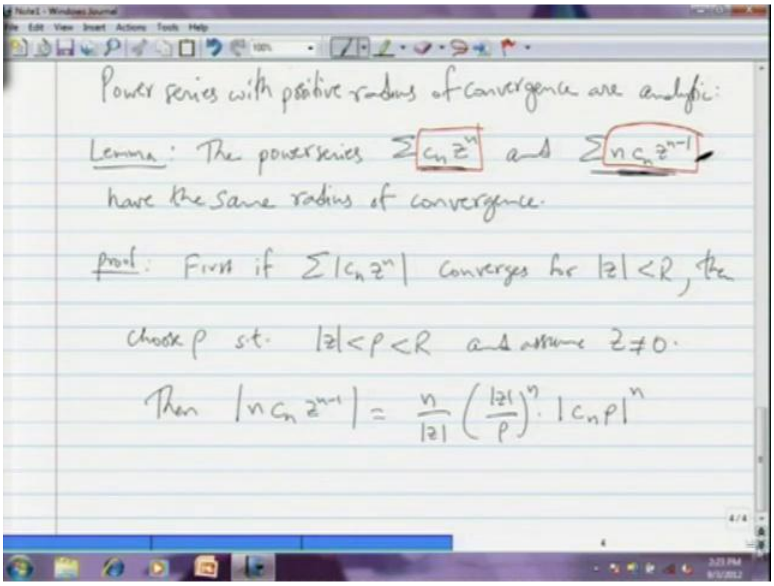

Now, let us actually show that, power series are actually analytic within their radius of convergence. So, in order to show that, first I am going to need little lemma, so we will start showing that power series with positive radius of convergence are analytic are analytic all right.

So, first we will need the following lemma. So, first I will show that the power series sigma c n z power n and sigma n c, n z power n minus 1 have the same radius of convergence. So, firstly note that given note that, I am dealing with series of type 1 power series of type one with 0 as the center of this series. So, this are power theories around 0 and also note that it would be, I mean a candidate for the derivative of a power series is actually differentiating, this power series term by term. What I mean by that is you consider the terms c and z power n its differentiation is n c n z power minus n.

So, it would be really convenient, if this term by term differentiation of the terms in the series is actually equal to the derivative of the power series itself and what we are going to show is that, that is indeed true within the radius of convergence for a given power series. So, with that as a goal, we first start by showing that the term by term differentiation of the power series. So, the series obtained by term by term differentiation of the terms in the power series and the power series itself has the same radius of convergence. So, that is statement of this lemma this c and z power n sigma and n c, n z power minus 1 in within the sigma this 2 have the same radius of convergence.

So, once again to emphasize, this is the derivative of this term so, let us prove this. So, proof of this is as follows, first if this convergence absolutely. So, which is the within the radius of convergence, if this converges for modules of z less than R, then choose rho such that modules of z less than rho less than R.

So, you are choosing a rho between the modules of z and the radius of convergence and assume z naught equal to 0, because I am going to divide by z. So, then the modules of the n th term here, so this is the n th term in this series in this series. So, the modules of this is less is equal to n by mod z times mod z by rho power n times modules of c n rho power n, so I am just rewriting the term within the modules in that fashion. The point here is that, modules of z by rho is actually strictly less than 1, so I can use that to consider a certain geometric series.

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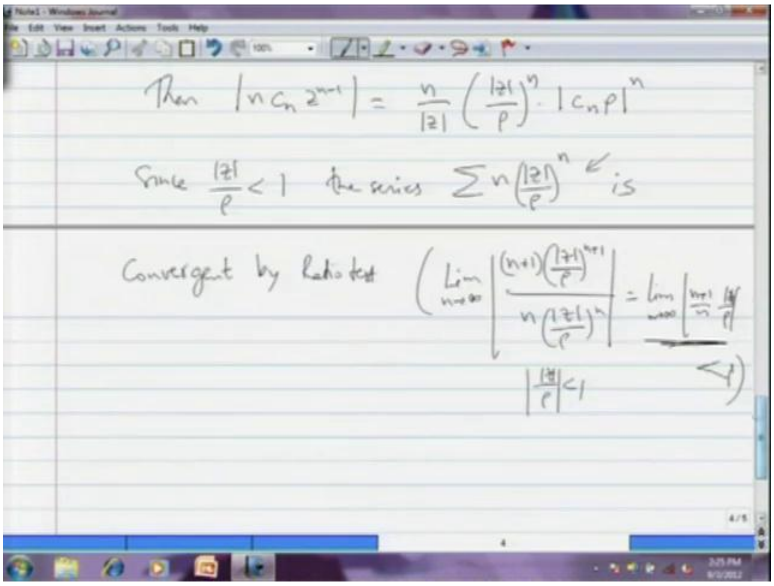

So, since modules of z by rho is actually less than 1, the series sigma n modules of z by rho power n is not exactly a geometric series, but by comparison with by ratio test for example, you can show that this is convergent. So, sigma n mod z by rho power n is convergent absolutely convergent, actually. So, is convergent by ratio test for example, right, because ratio test tells you that the limit as n goes to infinity of the n plus 1 th term here, which is n plus 1 modules of z by rho power n plus 1 divided by n times mod z by rho power n is actually equal to limit as n goes to infinity of n plus 1 by n times mod z by rho and mod z by rho is strictly less than 1.

So, well the modules of this or the absolute value of this. So, the absolute value of this and the absolute value of this, so the modules of z by rho is less than 1 modules of z by rho is less than 1 and then limit as n goes to infinity of n, n plus 1 by n is equal to 1. So, all in this limit is strictly less than 1 and so by ratio test, this converges this series converges, so that series converges. So, we will use the properties which we saw in the last session, that if a series converges absolutely then every term in that series is actually bounded by some constant.

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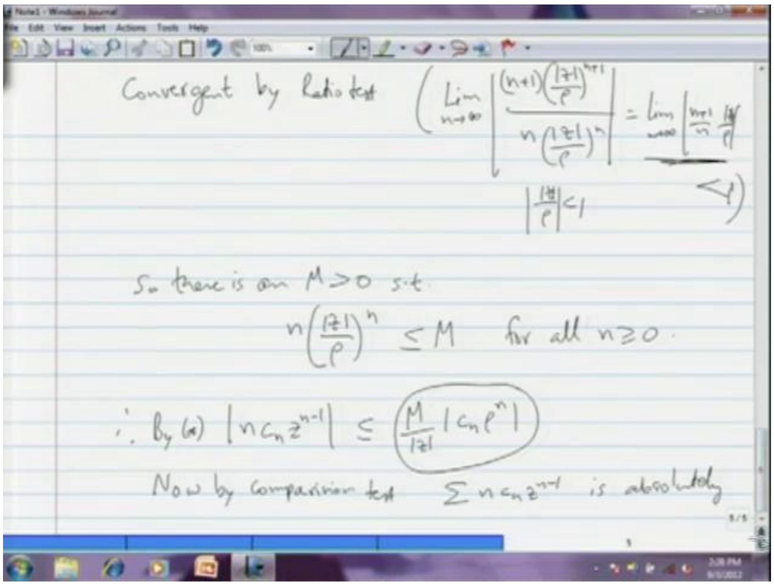

So, there is an m positive such that sigma n or rub this each term in this c. So, m mod z by rho power is less than or equal to m for all m for all positive integers for all n greater than or equal to 0 n say. So therefore, by the above by the above I will call this star, so by star what we can say is that, the modules of the modules of n c n z power n minus 1, let me go back to star. So, that is, so I have clubbed this terms I have clubbed this terms. So, the modules of n c, n z power n minus 1 is less than or equal to m times or n by mod z times modules of c and rho power n, let me make sure c n rho power n or c n rho. I apologize in this star this should be a rho power n here not c n rho whole power n, so this c n rho power n.

So, now notice that comparison test, firstly this are n eth terms of a convergent series, because rho is within the radius of convergence, rho is strictly less than capital R and so m by mod z c and rho power n is absolutely convergent. Now, by comparison test sigma n c n z power n minus 1 is absolutely convergent.

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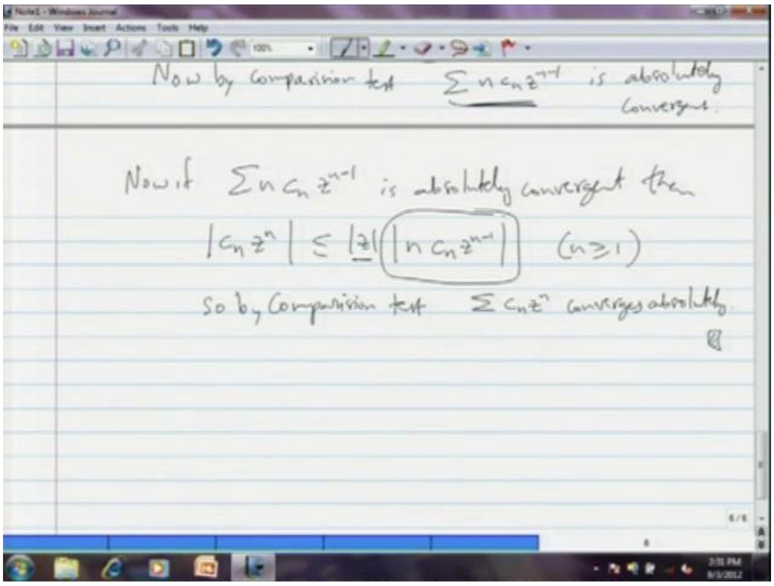

This term is less than or equal to, I mean in absolute value less than or equal to the absolute value of a term which belongs to absolute convergent series. So, by comparison test this itself is this series is absolutely convergent. So, we have shown that if sigma c n z power n is absolutely convergent, then the derived series sigma n c, n z power n minus 1 is also absolutely convergent. Now, in the other direction also its easy now, if sigma n c n z power n minus 1 is absolutely convergent, we want to show that sigma c and z power n converges. So, then the modules of c n z power n, we will estimate this. The n term in the original series, this is less than or equal to modules of z times the modules of n c n z power n minus 1 for n greater than or equal to 1, let us say.

So, by comparison test, so by comparison test, this is the n eth term of a convergent series and then you are multiplying it by mod z, which is a fixed number for a given z. So, by comparison test (( )) sigma c n z power n converges absolutely, that is easy. So, that completes the proof of this lemma notice, that you cannot you could not have proved this lemma directly by using some kind of limit comparison test, so that that kind of test fails.

So, you have to actually go through this kind of proof for this lemma. Now, we know that the derived series n c n z power minus 1 of sigma c n z power n has the same radius of converges. So, next what I will do is, I will show using this, that power series are actually analytic within their radius of convergent. So, then we can announce that power series are examples of analytic functions.

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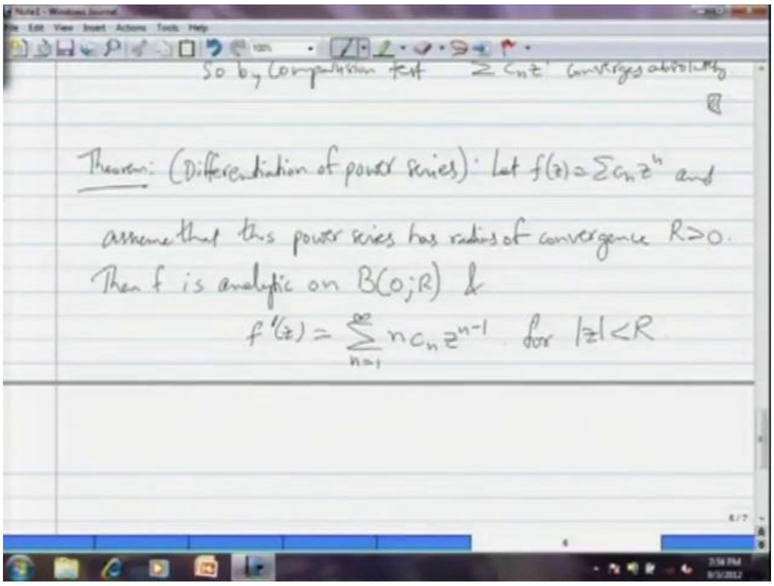

So here is the theorem, so theorem differentiation of power series. So, let f of z equals sigma c n z power n and assume, that this power series has radius of convergence are strictly greater than. So, we want our series with a positive radius of convergence, so then f is analytic b 0 R that is the conclusion and more over and the differentiation of f as a definite formula it is equal to sigma n equals 1 through infinity of n c n z power n minus 1.

So, term by term differentiation works and that is the derivative of f, where f is a power series within the radius of convergence. So, for this is true for modules of z strictly less than R I, in the radius of notice that this statement. Once again, is statement about series of a power series of type one and very similar a parallel statements holds for power series of type two. So, you can modify the statement just by putting z by minus a and assuming that, the instead of z and assuming that the series, power series about a point of analyticity or is about a point a, in the complex plane. So, a similar theorem can be stated for series of type two nevertheless, we are going to prove this version.

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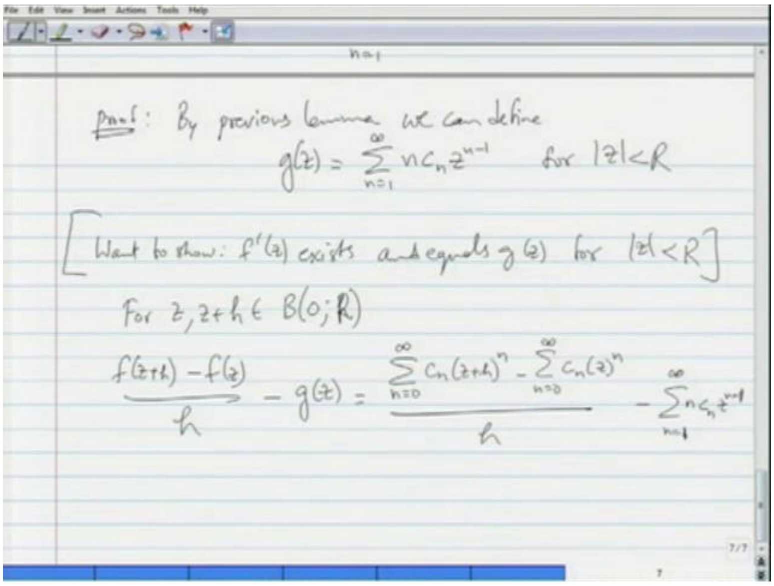

So, proof firstly by previous lemma it, since the derived series, this sigma n c n z power minus 1 and this series itself sigma c n z power n have the same radius of convergence. So, what we can do by previous lemma? What we can do is, we can define g of z is equal to sigma n equals 1 through infinity of n c n z power n minus 1, for modules of z strictly less than R. So, this definition make sense by previous lemma star, we can define because this series converges and it converges with a same radius of converges. So, then now, what we want to show? I will put this parenthesis, what we want to show is that, this g of z is indeed f prime of z. So, we want to show that f prime of z exists and equals g of z for or within the radius or disc of converges.

So, then what we need to do is, actually estimate the different quotient of f and the difference quotient of f and the function g of z. So, for z comma z plus h within the radius of convergence or the disc of convergence, wherever we will consider the expression of z plus h minus f of z by h, which is the difference quotient of f at the point z and then its difference with g of z. So, this subtraction gives you, what does it give you? It gives you well, what is f of z plus h it is sigma n equals 0 through infinity. I will just write everything c n z plus h power n minus sigma n equals 0 through infinity c n z power n divided by h minus and then you have g of z which is sigma n equals 1 through infinity of n c n z power n minus 1.

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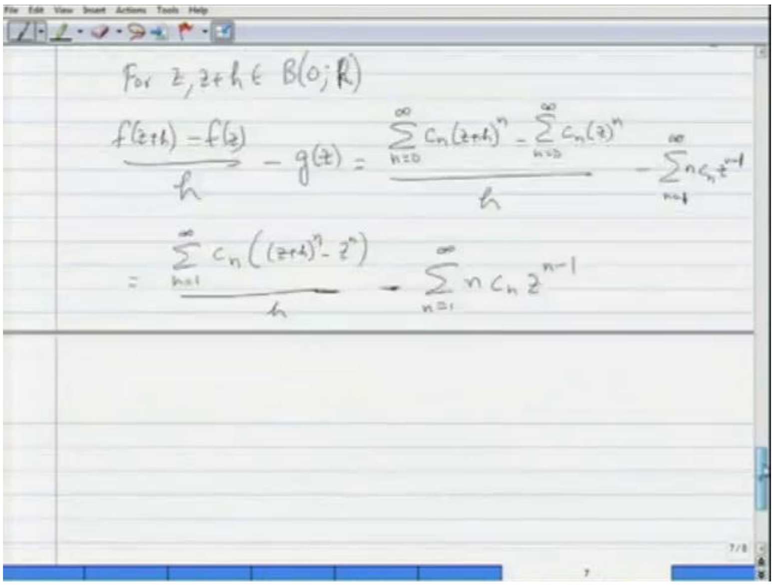

So, this is intern equal to well, observe that in the numerator further a difference quotient the 0 terms are c 0 and c 0 respectively. So, both those cancel each j other, so you can actually begin the difference from the index n equals to 1 3. So, this becomes sigma n equals 1 through infinity of c n times, z power z plus h power n minus z power n and then divided it, h minus all this minus sigma n equals 1 through infinity and c n z power n minus 1, okay?

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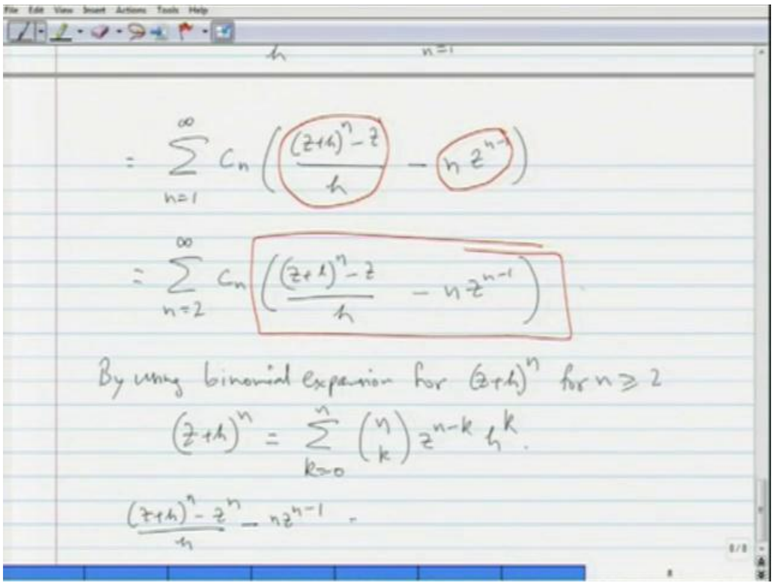

So, this intern is equal to sigma n equals 1 through infinity, I can take division by h into the series. So, what I have is c n times also notice that, now both the series here begin with the index and equals to 1. So, I will take the submission, since both them are convergent series, I can take the submission all together at ones and then factor out a c n. So, what I have is z plus h power n minus z divided by h and then minus n c n (( )) z power n minus 1.

So, that is what is (( )) and then further, so continuing this what I have is sigma n equals this is equals to n equals through infinity c n times z plus h power n minus z by h minus n z power n minus 1, because when n equals 1, what I have is h by h on for this term, okay?.So, I have h by h for this term, which is 1 and then I have n equal's n. So, this is one times z power 0, which is 1. So, 1 minus 1 0, so the index n equals 1 does not really give me any, it gives 0. So, I can start the index from n equals to, we will see, we will use this later, okay?

So, we will store this estimate for the difference between, the difference quotient of f and g of z. Now, we will use the binomial expansion. So, by using binomial expansion for z plus h power n for n greater than or equal to 2, z plus h power n is equal to sigma k equals z 3 infinity or 0 through n rather n choose k z power n minus k times h power k, okay?

So, please note that, we haven't actually proved that binomial expansion works for complex numbers, but once again notice that the binomial expansion is, actually the an arithmetic property, which work even for complex numbers like, it work for binomial expansion for of real numbers or real binomials. So, one can, if the viewer is a interested the, one can actually prove this for complex numbers like, one has done for real numbers or real binomials, okay?

So, it does hold for complex number as well and you can expand the z plus h power n, like that using binomial expansion. So, we will use this form and what we will get is z plus h power n minus z power n divided by h minus n z power n minus 1, which occurs here. In our estimate which occur in, our estimate this is equal to this is equal to z power n for lack of space this is equal to z power n plus n choose, 1 z power n minus 1 h etcetera plus so on.

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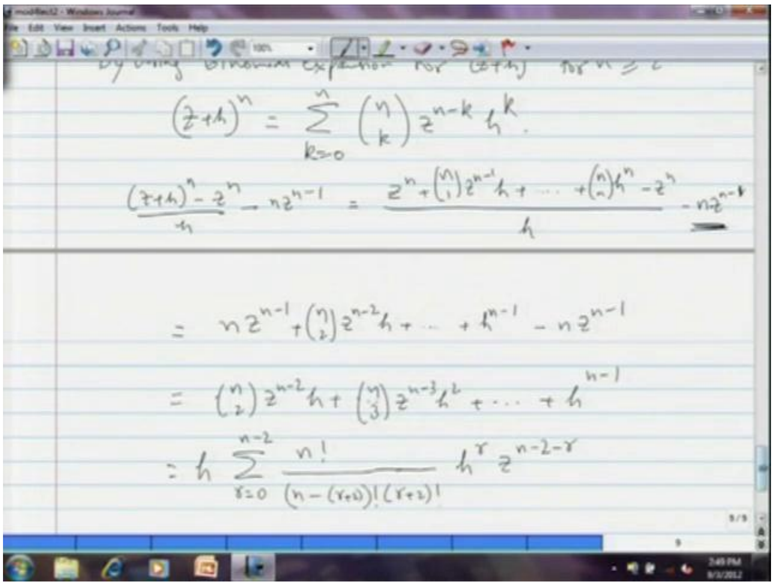

Until n choose n h power n minus there is a z power n, in the numerator divided by h minus n z power n minus 1. So, this equal to the z power n cancel and then I can also divide the expression in the numerator by h, I cancel one factor of h is not equal to 0. So, what I have is n choose 1, so I have n z power n minus 1 plus, plus n choose 2 z power n minus 2 times h plus so on and then I have n choose n, which is 1 h power n minus 1 and then I have a minus n z power n minus 1, which is here.

So, now once again the n z power n minus 1 cancels and what I have is and this equals well. The n z power n minus 1 cancels and this equals n choose two, z power n minus 2 h plus so on. So, I will write one more term n, choose three z power n minus 3 h square, so on until h power n minus 1. So, I can factor out h and what I have is h plus sigma, I will write in terms of sigma this is R starts from (( )) I will have index r, r starts from 0 and goes until n minus 2 of n factorial divided by n minus r plus 2 factorial and R plus teo factorial times h power r and z power n minus r, r actually n minus 2 minus r okay.

So, for example, when r equals to 0, you get n factorial by n minus 2 factorial times r plus 2 factorial, which is your n choose or 0 plus 2 factorial, which is 2 factorial. So, you get n factorial by n minus 2 factorial times two factorial which is n choose two and then h power 0, z power n minus 2. So, you get you get back your expression here and you can write the estimate in this form or term within the estimate of this form.

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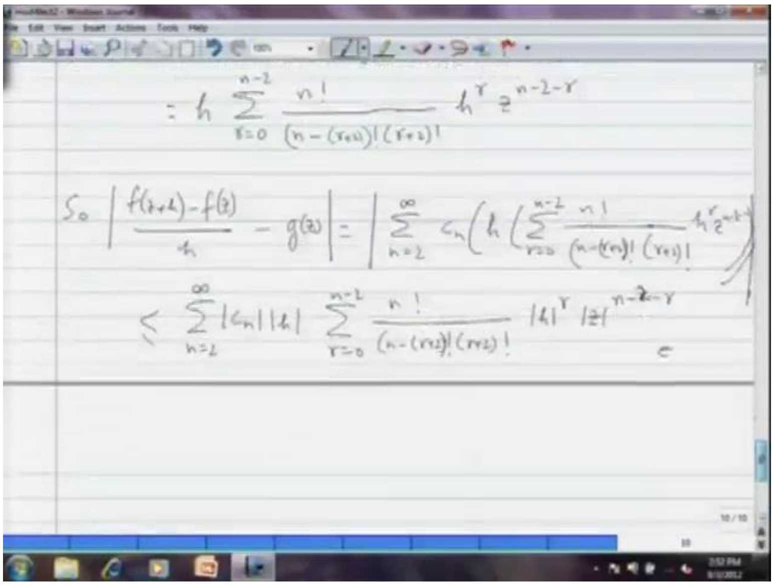

So, now the estimate of the difference quotient f of z plus h minus f of z divided by h minus g of z in modules, this is equal to the modules of sigma n equal 2 through infinity of c n times h times. So, everything in the parenthesis now, in a h times sigma r equal 0 through n minus 2 of n factorial divided by n minus r plus 2 factorial times r plus 2 factorial times h power r times z power n minus 2 minus r please bare with me. If there is parenthesis their and then an additional parenthesis and absolute values of sin, okay?

So, this is now this can be this, can be less than or equal to sigma n equals to through infinity of the modules of c n and modules of h times sigma r equals 0, through n minus 2 of n factorial divided by n minus r plus 2, factorial times r plus 2 factorial and modules of h power r modules of z power n minus 2 minus r, okay. So, from here, from the previous step to this step, what I have used is actually, an infinite a version of triangle inequality.

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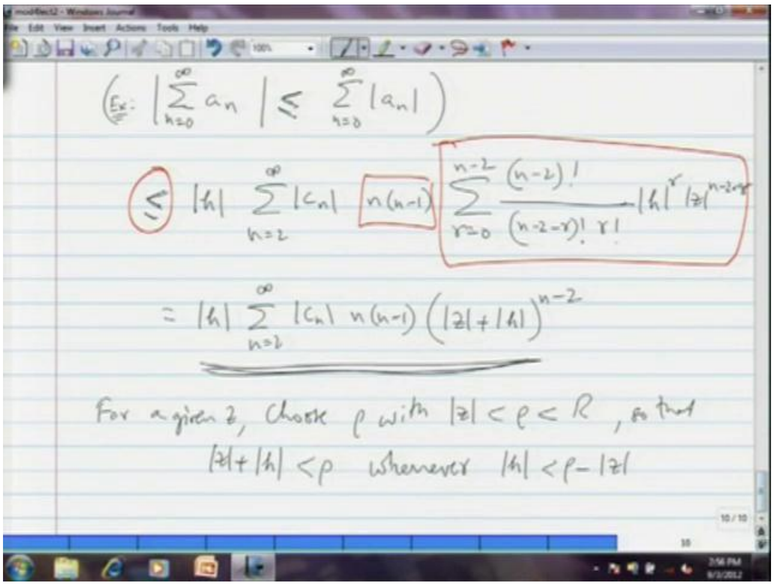

So, it is an exercise to the viewer to actually prove, that whenever the series a n is convergent sigma a n is convergent sigma a n, n equals 1 through infinity or 0 through infinity is less than or equal to sigma of absolute value or the modules of n equals 0 to infinity sorry or sigma the modules of a n, where a sigma runs from n equals 0 to infinity. So, using this form of infinite version of triangle inequality using partial sums one can prove this.

So, this an exercise using this, we can go from the previous step to this step and further this, this is the expression is less than or equal to modules of a h, as I will pull out a modules of h, from this sigma and equals to through infinity of modules of c n times then again I have n times n minus 1, I am pulling out n times n minus 1 this n factorial sorry n factorial in the numerator, so I am pulling out an n times n minus 1.

So, what I will be left with is n minus 2 factorial. So, here is n minus 2 factorial within the sigma r equals to n minus 2 and minus 2 factorial times. I will rewrite the denominator as of n minus r plus 2 factorial as n minus 2 minus r factorial and since I have less than or equal to here. So, emphasis I have a less than or equal to, so what I will do is I am actually less than the denominator. So, I will write r factorial here instead of r plus 2 factorial.

So, then I can write mod h for r here and modules of n minus 2 minus r here. So, r plus 2 factorial is greater than r factorial. So, 1 by r plus 2 factorial is less than or equal to 1 by r factorial. So, I am using that to come to the this step here and then this intern is equal to the modules of times, the modules n equals two through infinity modules of c n times n times n minus 1.

What I have here? What I have within this sigma? You will observe a is modules of z plus modules of h power n minus 2 by binomial expansion. So, this is nothing, but the modules of z plus modules of h power n minus 2 for a given for a given z choose rho with mod z strictly less than r. So, that modules of z plus h also states within that rho. Modules of z plus modules of h is strictly less than r rho sorry, whenever modules of h is less than rho minus mod z.

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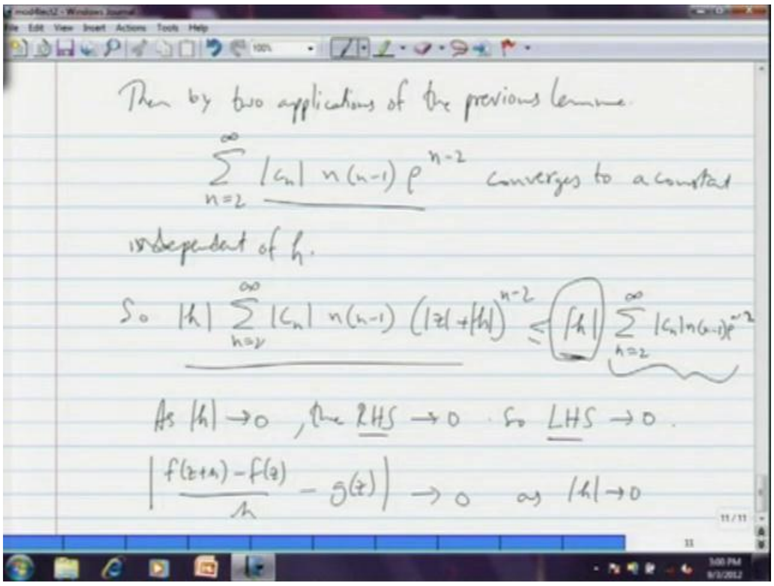

So, then by two applications f previous lemma of the lemma, that we have proved of the previous lemma sigma n equals 2, through infinity modules of c n and n times n minus 1 rho power n minus 1, has the same radius of convergence. Convergence to a constant and that and it is important that that constant is Independent of a h a constant independent of h.

So, the previous lemma tells you that, if modules c and z power n are sigma c and z power n converges, then the derived series sigma n c n z power n minus 1 also converges within the radius of convergence. So, since rho the number, the real number rho occurs within the radius of convergence, it is less than r c n rho power n sigma of that converges. So, by the previous lemma n c n rho power n minus 1 and once again using that lemma n times n minus 1 c n rho power n minus 2 also converge both those converge within the radius of convergence, when rho is actually less than strictly less than r they converge absolutely.

They converge independent of this h, because there is no appearance of h in this series set of. So, in the series sigma modules of c n, n times n minus 1 rho power n minus 2, so that is independent of h, so that is important, so, the above. So, modules of h sigma n equals 2 through infinity of modules of c n, n times n minus 1 modules of z plus modules of h power n minus 2, which is our estimate of? What we want is less than or equal to modules of h sigma n equals through infinity modules of c n, n times n minus 1 rho power n minus 2 and since we have a multiplication by mod h of a convergent series of a convergent series, whose convergence does not depend on h or whose value is independent of h and since modules of, okay?

So, what I can conclude is as modules of h tend to 0 the r h s tends to 0. So, l h s tends to 0 as modules of h tends to 0, so the l h s tends to 0 l h s of this inequality, so by r h s and l h s, I mean the sides of this inequality above. So, since the l h s stands to 0, which is the estimate of modules of f of z plus h minus f of z by h minus g of z, since this is I mean the above is the estimate of this tends to 0's as modules of h tends to 0.

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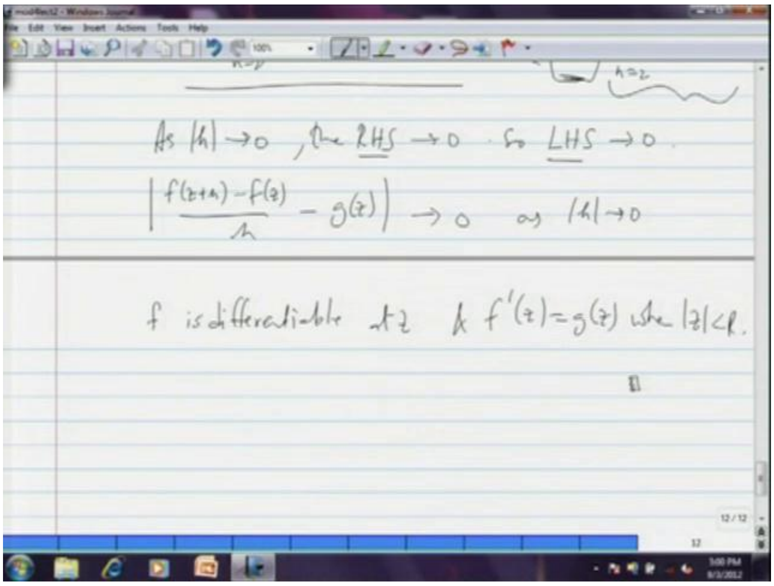

So, we conclude that f is differentiable at z and f prime of z is equal to g of z, when the modulus of z are strictly less than r. So, that proves this theorem and that shows that power series are analytic within their radius of convergence and their differentiation is precisely the term-by-term differentiation of the terms within the series, okay? So, and then in the next session, we are actually going to prove that every analytic function actually, has a local expression as a power series that is the Taylors theorem and we will cover that in the next session.