Introduction to Fuzzy Sets Arithmetic and Logic
Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology - Delhi

Module - 3
Lecture - 7
Fuzzy Sets, Arithmetic and Logic

Welcome students, to the seventh lecture on Fuzzy Sets, Arithmetic and Logic. As I mentioned in the last class that, from today I will start Fuzzy Arithmetic.

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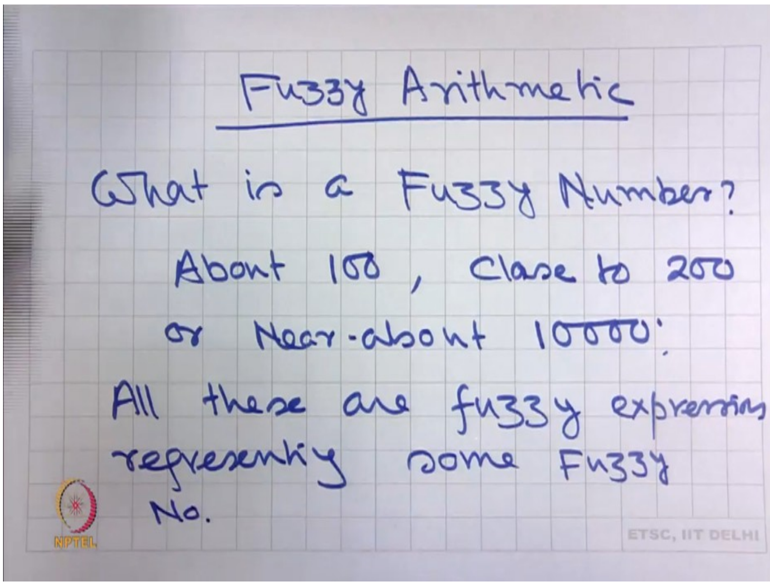

In my very first lecture, I have introduced you to the concept of fuzzy arithmetic.

It is very relevant for solving many practical problems.

For example: Suppose for a job, one needs say labourers and each labourer will be paid, say .

Now, if one has to budget, then one has to multiply one fuzzy number, into another fuzzy number which is .

How can you compute those values if those values are not precise?

In those cases, you will have to resort to fuzzy arithmetic. In particular, for this problem, you will have to resort to fuzzy multiplication.

So, in today's lecture and in next few, I will be discussing mostly on fuzzy arithmetic. As a first step, we need to define fuzzy number.

So, what is a fuzzy number?

From my earlier discussions, you understand that expressions like, , or -; all these are fuzzy expressions representing some fuzzy number.

So, what is a fuzzy number?

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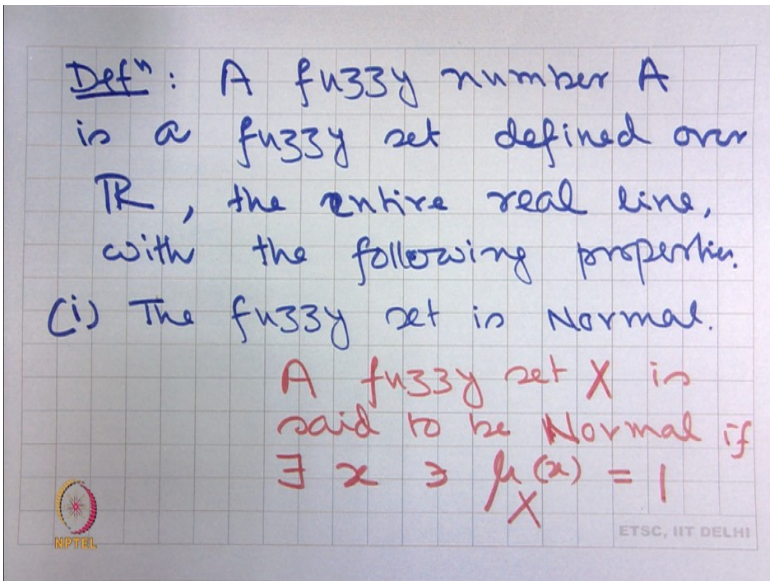

Definition:

A fuzzy number is a fuzzy set defined over , that is the entire real line, with the following properties.

1) The fuzzy set is normal

A fuzzy set, say is said to be normal if there exists such that, membership of to the fuzzy set is equal to , that is . That means that, there has to be at least one real whose membership to the set is . Or it has a full membership to the fuzzy set .

The intuition is that;

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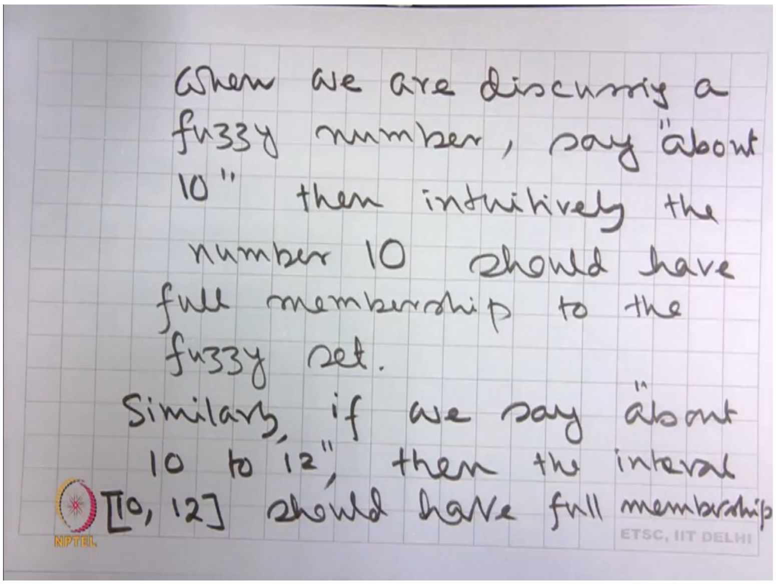

When we are discussing a fuzzy number, say . Then, intuitively the number should have full membership to the fuzzy set.

Similarly, if we say , then the interval should have full membership. Hence, intuitively we can feel there will be certain values whose membership to the fuzzy set has to be . Or in other words, the fuzzy set is going to be normal.

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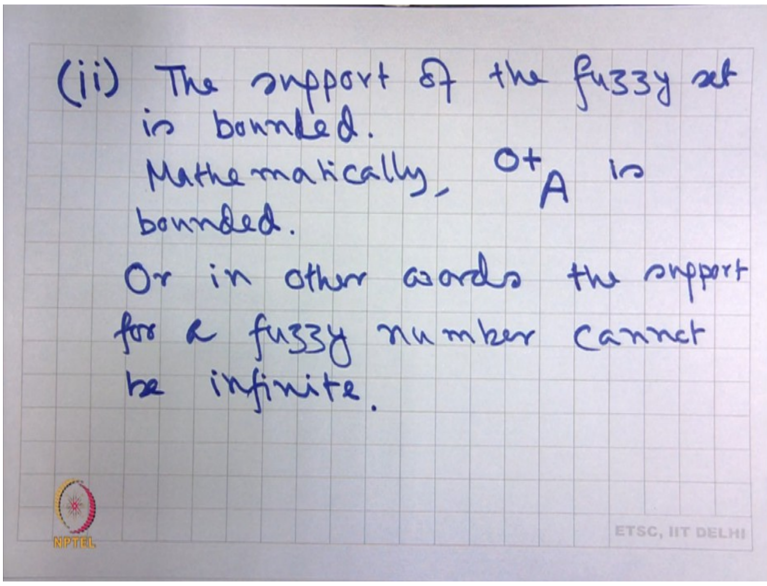

2) The support of the fuzzy set is bounded.

Mathematically,   is bounded.

Or in other words, the support for a fuzzy number cannot be infinite.

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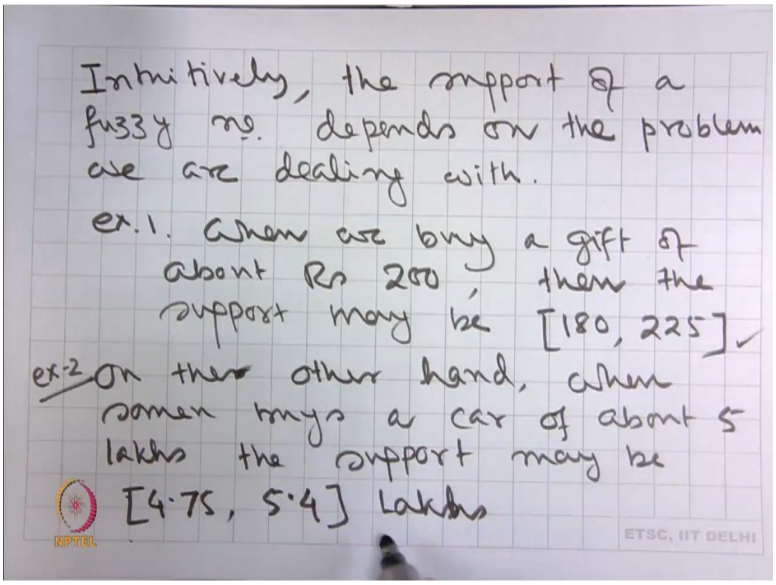

Intuitively, the support of a fuzzy number depends on the problem we are dealing with. Example:

When we buy a gift of ; then, as we discussed earlier, the support may be, say .

On the other hand, when someone buys a car of about lakhs, the support may be, say to lakhs.

So, the fuzzy set, if you look at real values, is actually, has a much wider support than the one that we have discussed with respect to Example 1.

So, in this example, we find a much bigger support.

We can like that go further and we can feel that, as the values that we are dealing with are bigger and bigger, the support also can be wider and wider.

But it cannot be as wide as . So, the support for a fuzzy number is bounded.

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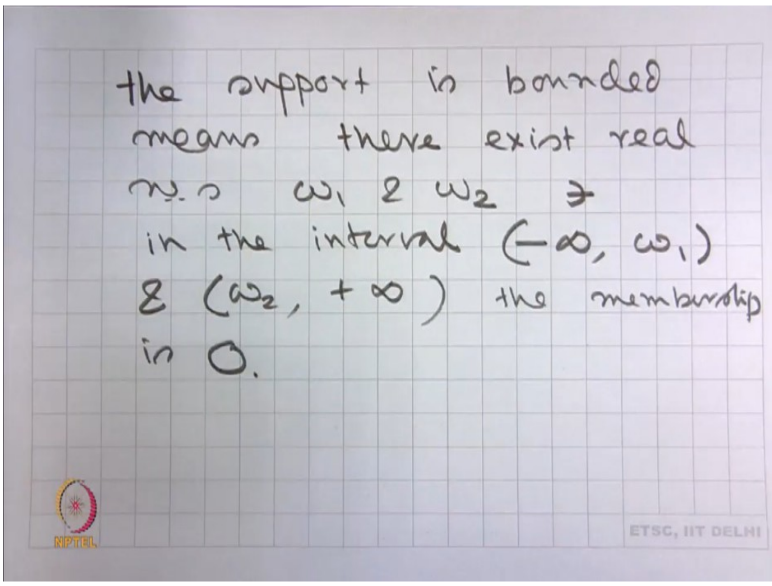

The support is bounded means, there exist real numbers and , such that in the interval and the membership is 0. And the third property is:

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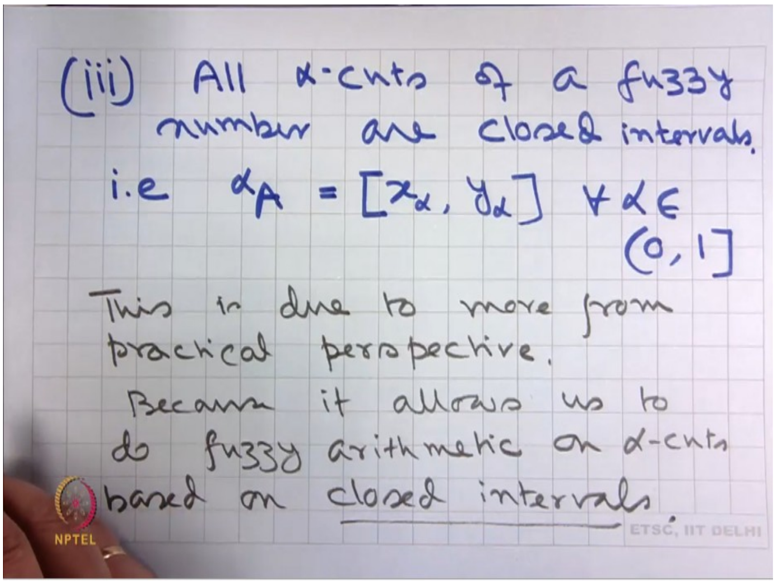

3) All -cuts of a fuzzy number are closed intervals.

   That means, for all

This comes due to more from practical perspective, because it allows us to do fuzzy arithmetic on alpha-cuts based on closed intervals.

And, this is a fairly established technique in mathematics. As we will see later that, instead of working on functions, if we work on alpha-cuts, our calculations for different arithmetic operations like plus, minus, multiplication, division, exponentiation becomes simpler. The question is:

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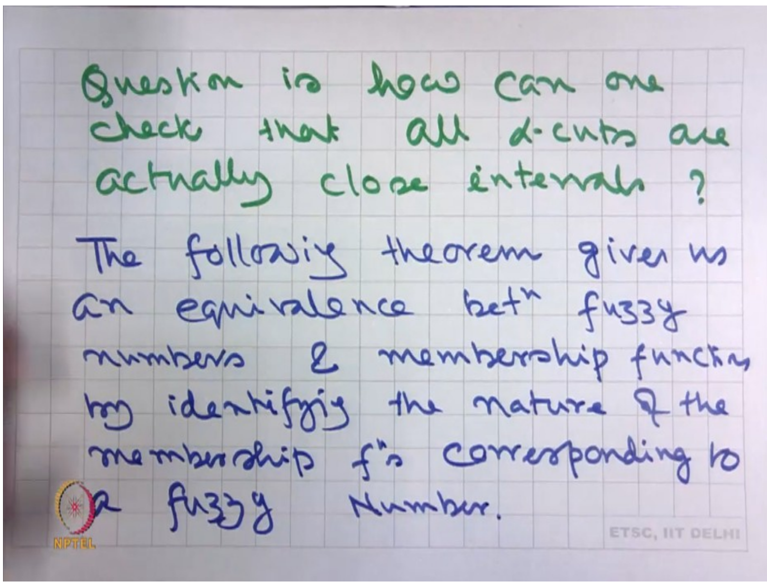

How can one check that all -cuts are actually closed intervals?

As there are uncountably many alphas even in the small interval , it is not practically possible to check for all alphas, whether the corresponding -cut of a fuzzy number is indeed a closed interval.

So, the following theorem gives us an equivalence between fuzzy numbers and membership functions by identifying the nature of the membership functions corresponding to a fuzzy number. So, what is the theorem?

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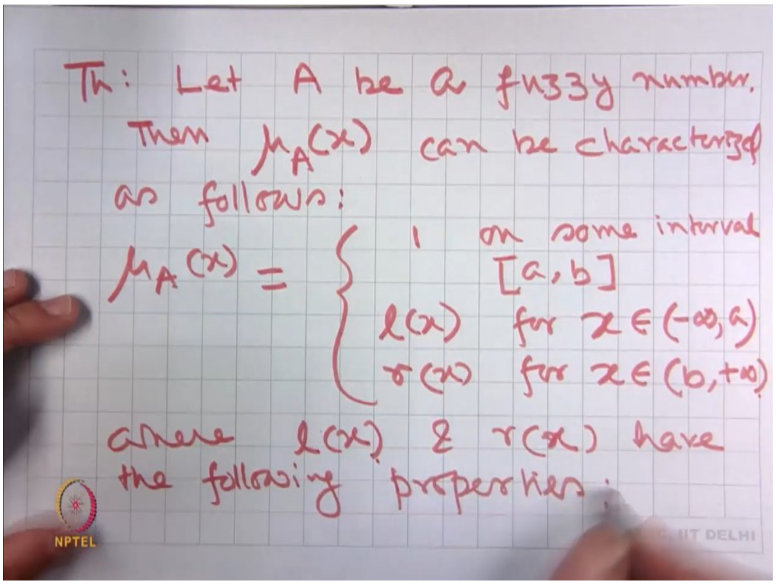

Theorem:

Let be a fuzzy number. Then, can be characterized as follows:

,   where and have the following properties.

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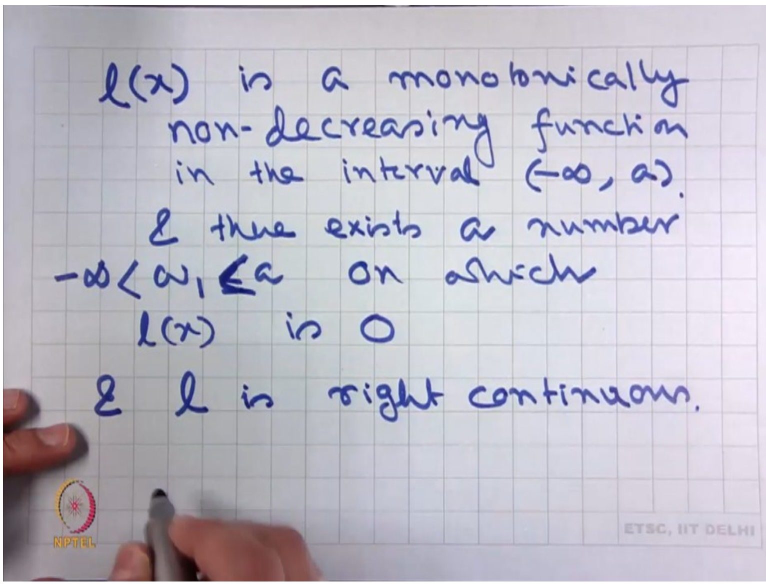

is a monotonically non-decreasing function in the interval . And there exists a number , such that , on which is and is right continuous.
So diagrammatically, if we look at it, it is the following.
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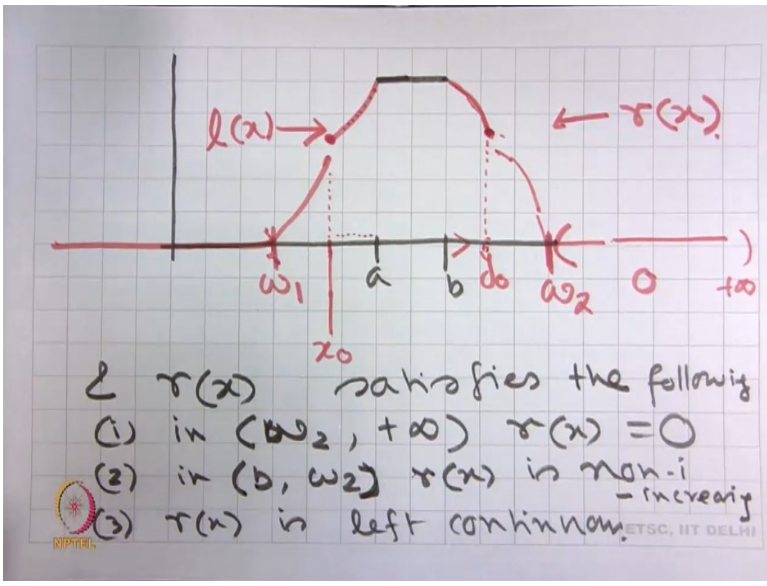

So, there exists an interval on which the membership function is . There exists a point , such that, from to , the membership function is 0. In between, it is a monotonically non-decreasing function. That function need not be continuous. There may be discontinuity. Say for example, the function may be something like this, when there is a discontinuity at this point.
But, in that case, the function has to be right continuous.
Suppose, this point of discontinuity is called ; then, if we take any sequence of points in the interval, coming to from the right-hand side; then, the limit of those values is going to be the value of the function at .
So, this is the characterization of .
What are these?
It is a monotonically non-decreasing function.
It has to be 0 in the interval .
And between to , it is monotonically non-decreasing, but it is right continuous.

In a similar way, we can characterize the function .
So, there will be a value such that, in this interval to , the value of the function is . Between to , it is monotonically non-increasing function. But it is going to be left continuous. It need not be continuous, but if there is a point of discontinuity, suppose is a point of discontinuity, then it is left continuous at .

That means, if you are approaching from the left side, we will find that the limit of that sequence is same as the value at .

So, this side is called .

So, characterization of is as follows satisfies the following:

i. In to ,.
ii. In to , is non-increasing.
iii. is left continuous.

I will now give you a proof which establishes the equivalence between the definition that we have given and the new definition in terms of the membership functions; and to show that they are equivalent.

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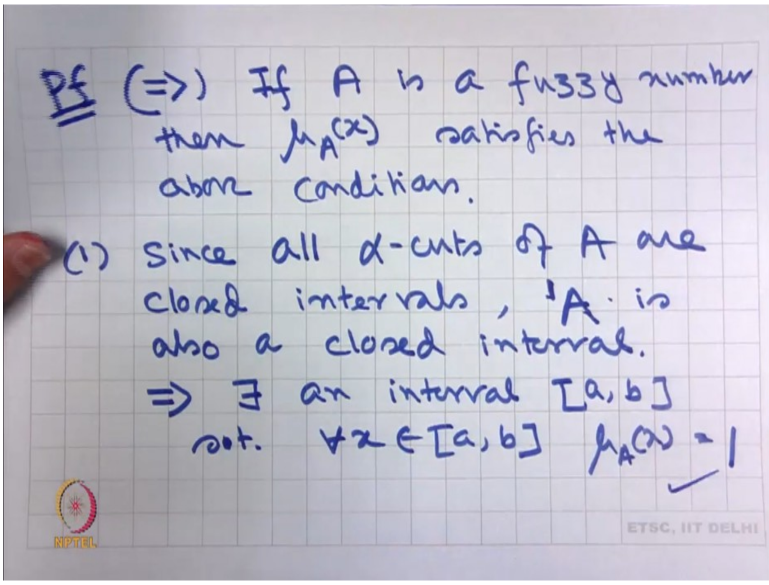

So, first we are proving in forward direction.

That is, if is a fuzzy number, then satisfies the above conditions.

Since all -cuts of are closed interval, is also a closed interval.

 there exists an interval such that, for all

So, this part of the characterization of the membership function is proved. Or in other words, we will get such an interval on which the value is . And that it has to be , comes from the normality.

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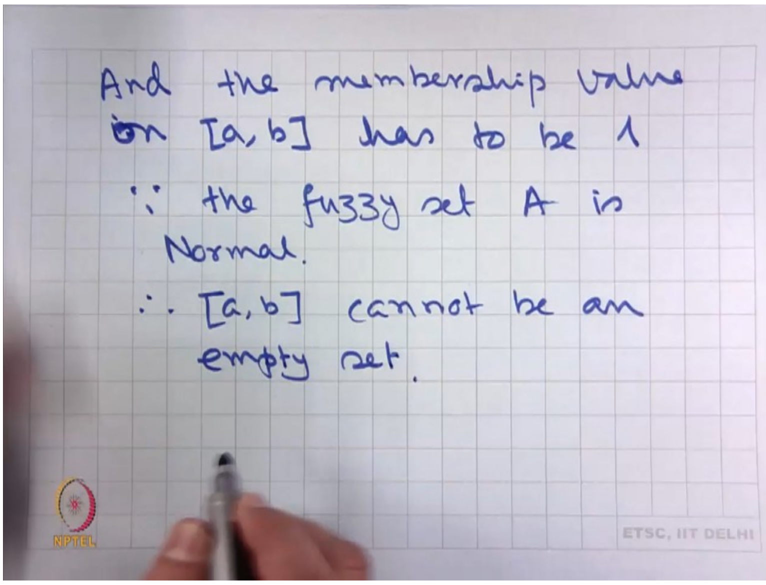

And the membership value on has to be since, the fuzzy number is normal.

Therefore, cannot be an empty set.

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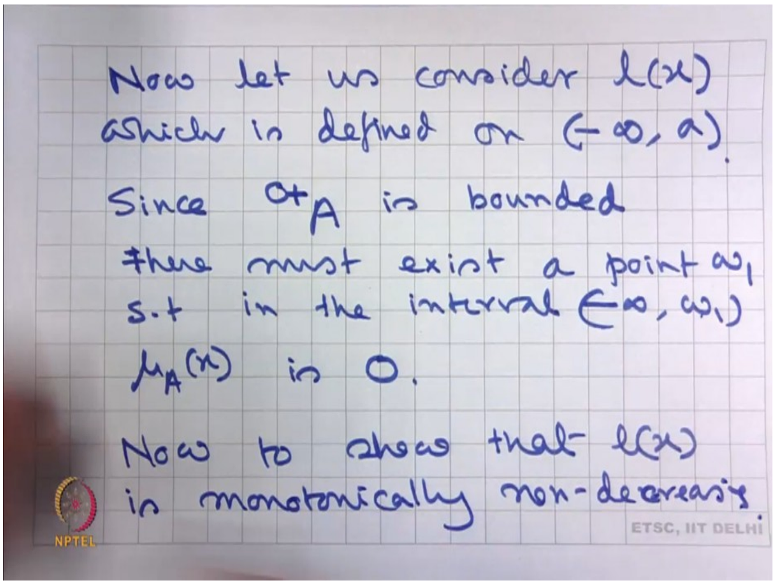

Now, let us consider which is defined on .

Since the support of   is bounded, there must exist a point such that, in the interval, , is .

Because, if no such exists, then the support becomes unbounded.

Now to show that is monotonically non-decreasing. This comes in the following way.

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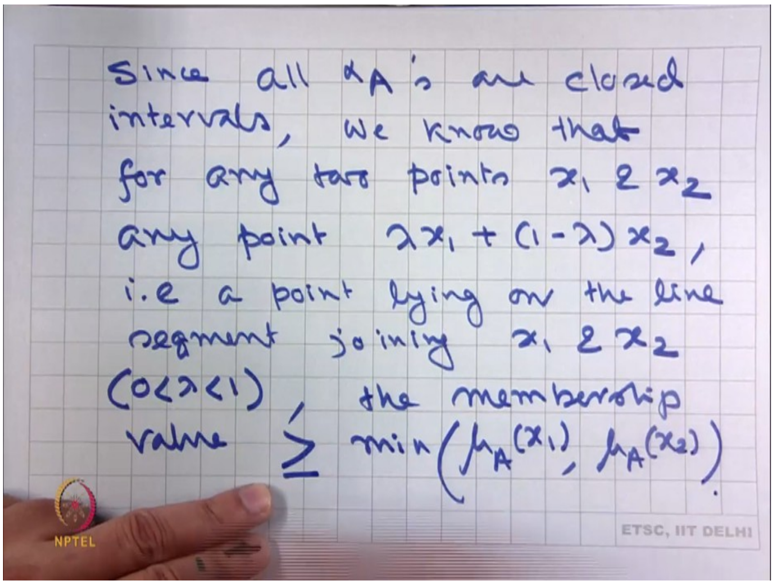

Since all are closed intervals.

We know that, for any two points and ; any point ; that is a point lying on line segment joining and ; which happens when .

So, for any point lying on that line segment, the membership value has to be greater than equal to .

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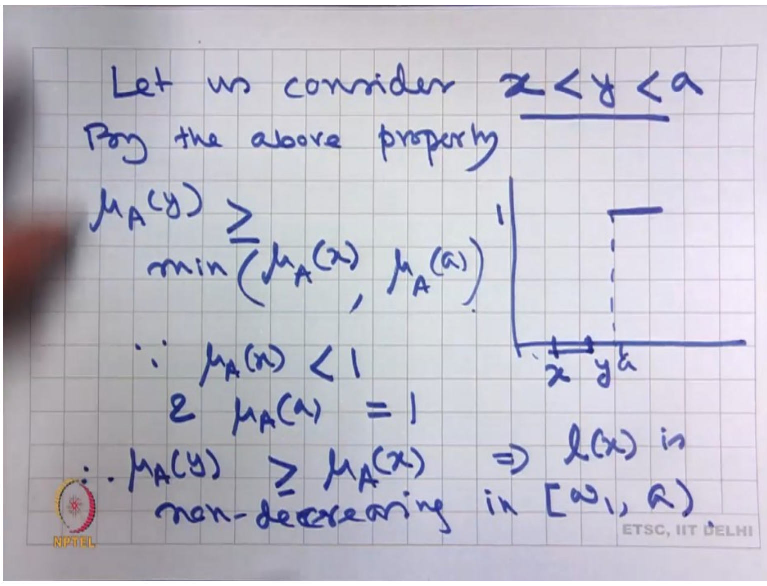

So, let us consider , where if you remember, at , the membership value has been . And I am looking at two points and , such that lies in between and .

Therefore, by the above property,

Since and .

Therefore,

 as I am going from to , the function is non-decreasing in to

The question is:

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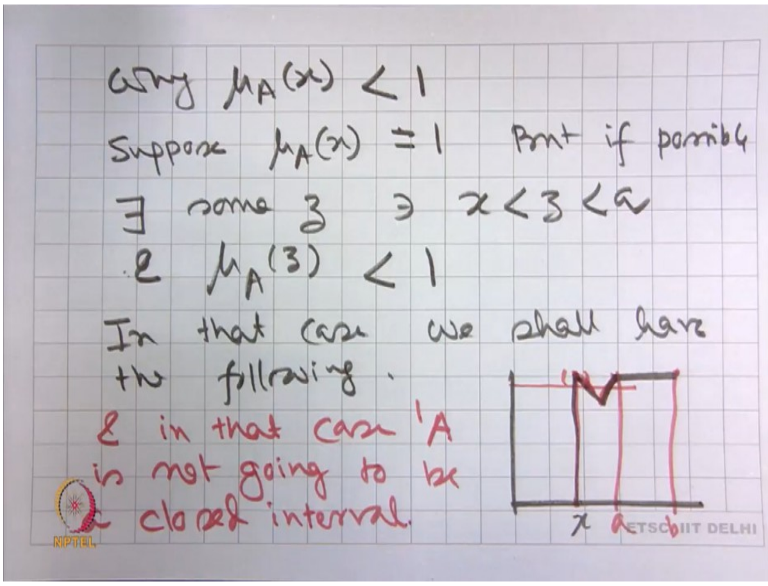

Why ?

Suppose , but if possible, there exists some , such that and . In that case, we shall have the following situation.

for and at it is again .

But in between, the membership function has come below , but again it picked the value .

And therefore, we shall have such a situation.

And in that case, is not going to be a closed interval.

Because, the interval will come in . At the same time, the value or maybe some interval around , will also be in .

And therefore, is going to be at least union of two different things, which is not a closed interval. Therefore, what we have seen the following:

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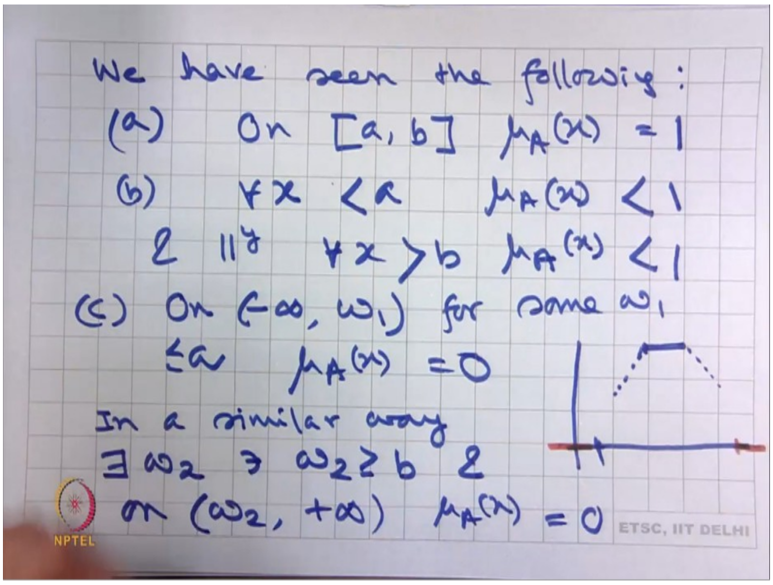

(a) On ,.

(b) For all , . And similarly, for all ,

Therefore, as we go to from right-hand side, the membership value is less than . Similarly, on left side also, the membership value is less than .

(c) We have seen that, on , for some . In a similar way, there exist , such that . And on , .

That is, after some point it is And from here it is .

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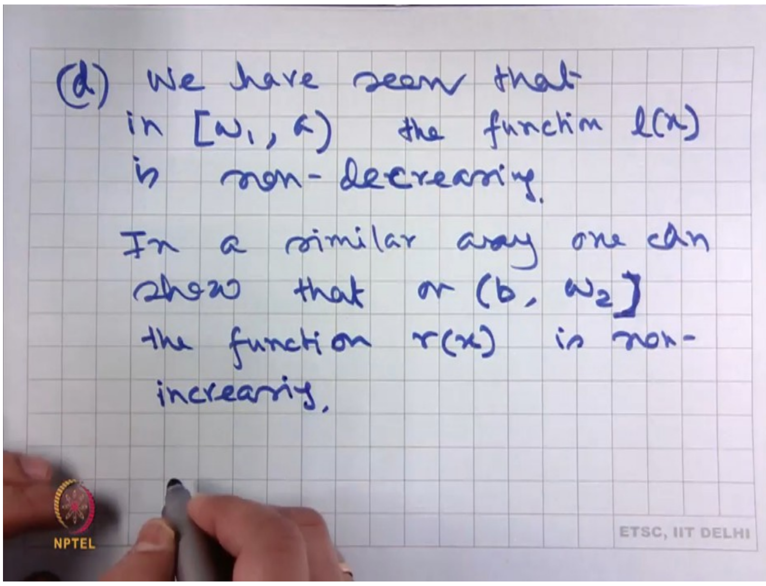

(d) We have seen that, in , the function is non-decreasing.

In a similar way, one can show that, on , the function is non-increasing.

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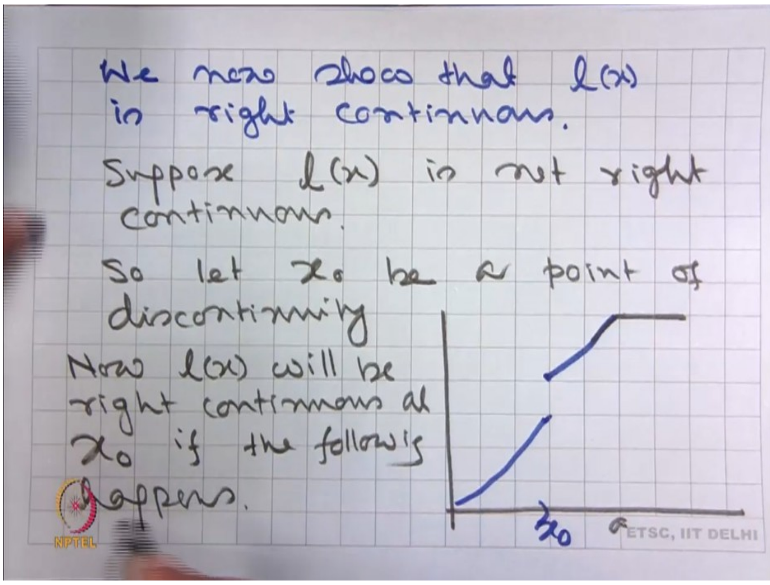

So, now we show that is right continuous.

Suppose is not right continuous.

So, let be a point of discontinuity. So, let us consider the situation.

This is , is coming like this. Suppose is a point of discontinuity and we have the function like that.

Now, will be right continuous at , if the following happens:

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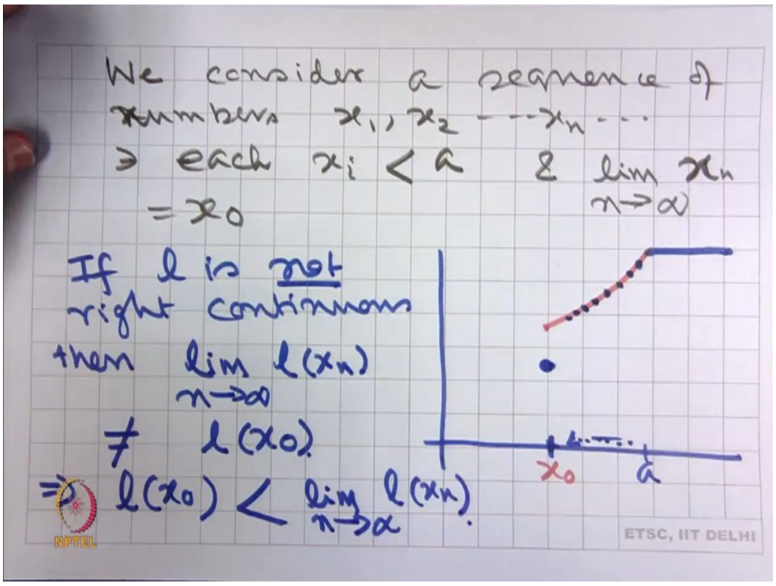

We consider a sequence of numbers , such that each and

So, we have here; we have here; and we are looking at points which are limiting to from the right-hand side. And these are their corresponding values of .

So, if is not right continuous, then, for such a situation, .

Because, if the sequence is converging to and the limit of their functional values is converging to the ,then it is right continuous. But we are assuming that it is not right continuous; therefore, this limit is not equal to .

And, since is monotonically non-decreasing, therefore the value at is going to be somewhat less.

Implies that,

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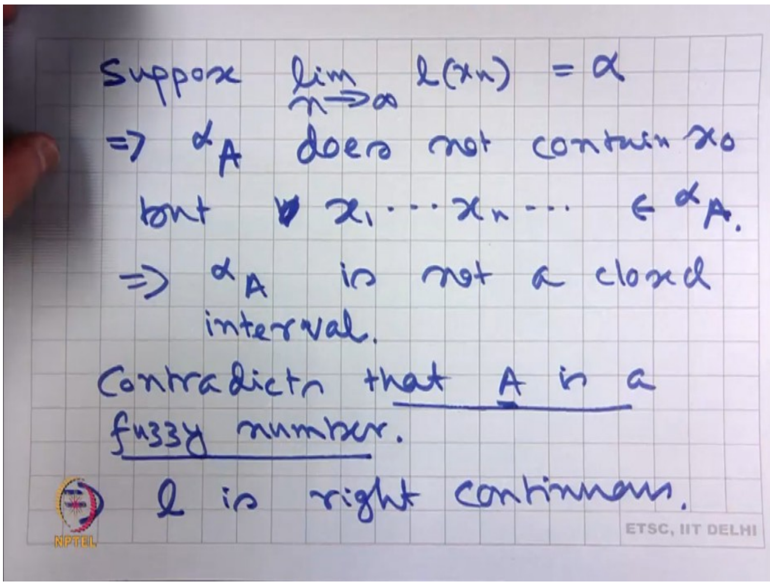

Suppose .

  does not contain , but all .

  is not a closed interval

Then it contradicts that is a fuzzy number, because we started with is a fuzzy number. And we wanted to show that all the characteristics of the membership function should hold.

And if the right continuity does not hold, then we contradict the fact that is a fuzzy number. is right continuous.

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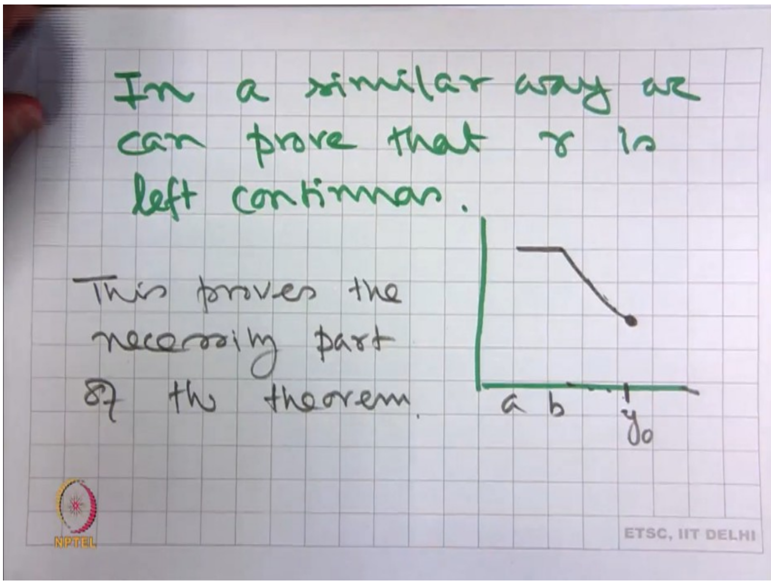

In a similar way, we can prove that is left continuous.

That is, if we have such a situation; this ; and if there is a point of discontinuity, say . Then, we will find that, for any sequence converging to , the limiting value of is going to be . Therefore, is going to be left continuous.

So, this proves the necessity part of the theorem.

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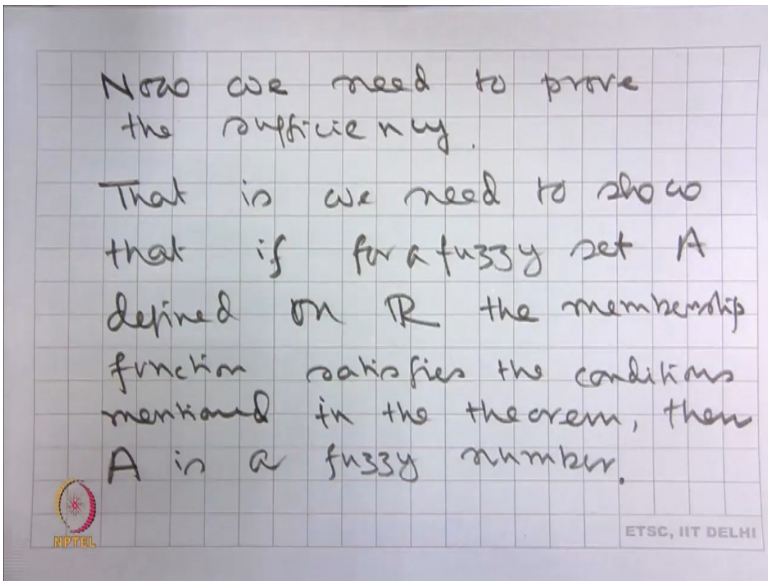

Now, we need to prove the sufficiency.

That is, we need to show that, if for a fuzzy set defined on , the membership function satisfies the conditions mentioned in the theorem. Then, it is a fuzzy number.

Okay students, I shall prove this sufficiency in my next lecture. Till then, you try to go through the proof again and try to understand the implications of these properties.

How the simple three properties of a fuzzy number lead to this analytical property of the membership function for a fuzzy number. So, in the next class, I will prove the sufficiency. And also, I will start the basics of interval arithmetic so that, in my subsequent lectures, when I do arithmetic on fuzzy numbers, you can understand how they are carried out to get, to solve numerical problems for involving fuzzy numbers.

Thank You.