Introduction to Fuzzy Set Theory, Arithmetic and Logic
Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology Delhi

Lecture 05
Fuzzy Sets Arithmetic and Logic

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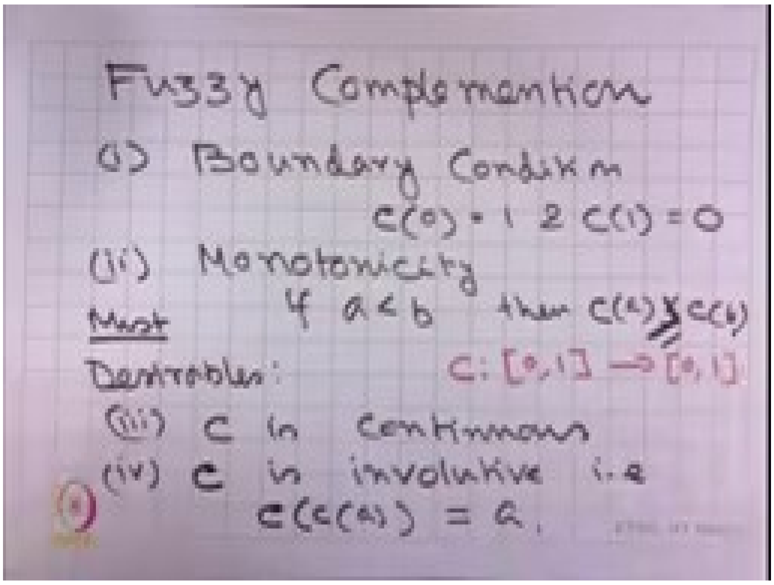

Welcome students to the 5th lecture on fuzzy sets arithmetic and logic.

In the last class we talked about fuzzy complementation and we have identified four properties. Two of them are must; that is

i. Boundary condition

and

ii. Monotonicity

If then,

These two are must. Also, desirables are

iii. is continuous

iv. is involutive

In the whole discussion, is a function from to .

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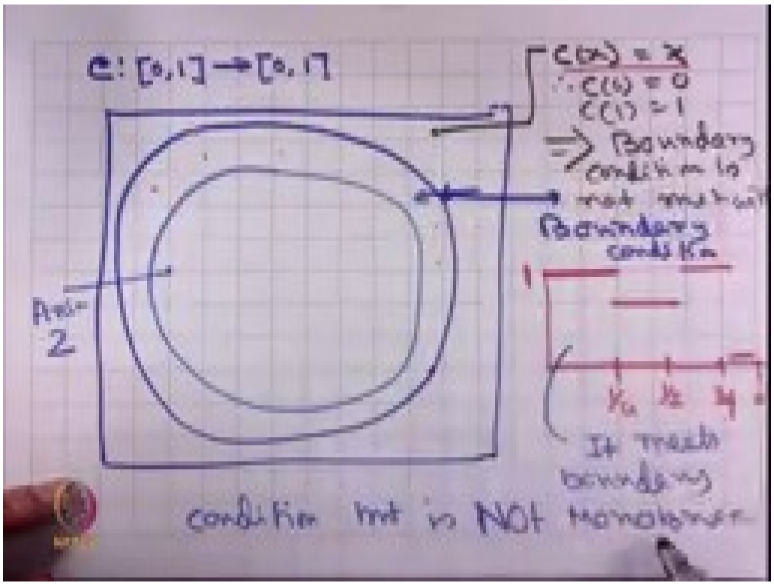

Now let us see a picture.

Let us consider all the functions such that

Consider a function .

Therefore, , implies boundary condition is not met with.

Therefore, there exists function something like this that do not follow the first boundary condition.

So, within let us look at that class of functions which follow boundary condition.

Consider this function:

If we consider such a function, we see that it meets boundary condition but is not monotonic. So, these are the functions that satisfy axiom 1.

Within that therefore, we look for functions which satisfy axiom 2 that is monotonicity.

This example shows us that this space is non-empty.

There are functions which meet boundary condition but do not follow monotonicity.

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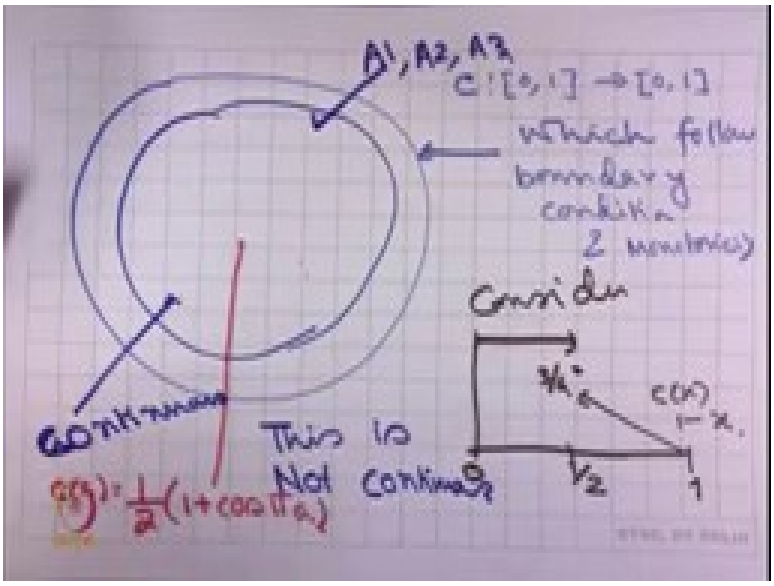

Now let us consider the set of functions and let this be the set which follow boundary condition and monotonicity.

Now consider this function

Obviously, it follows the first two but we can easily see that this is not continuous.

Therefore, within this we have a subclass of functions which are also continuous.

That means, these follow axiom 1, axiom 2 and axiom 3 and continuous.

Consider the following function:

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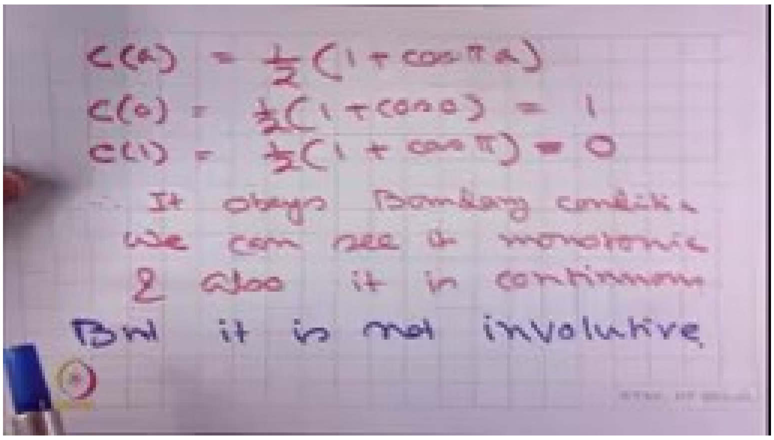

So let us examine the function.

Therefore, and .

Therefore, it obeys boundary condition we can see it is monotonic and also it is continuous but we can see that it is not involutive.

Because if it is involutive then, from this definition we should get that . But since is a trigonometric function and we are not using any , we can easily see that it is not going to be true for all values of .

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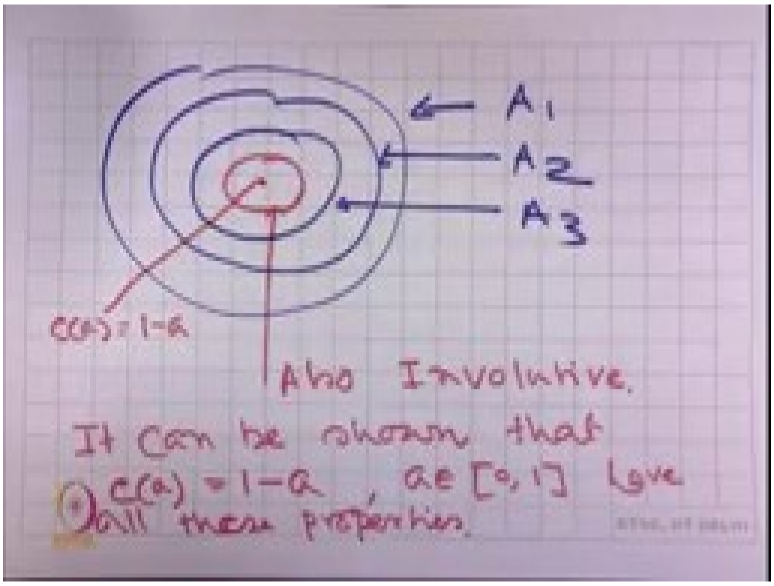

Therefore, we have this which follow , this which follow , this which follow that is continuous among them there is a subclass which are also involutive.

It can be shown and I want you to verify that, , have all these properties. So, is a function that lies here.

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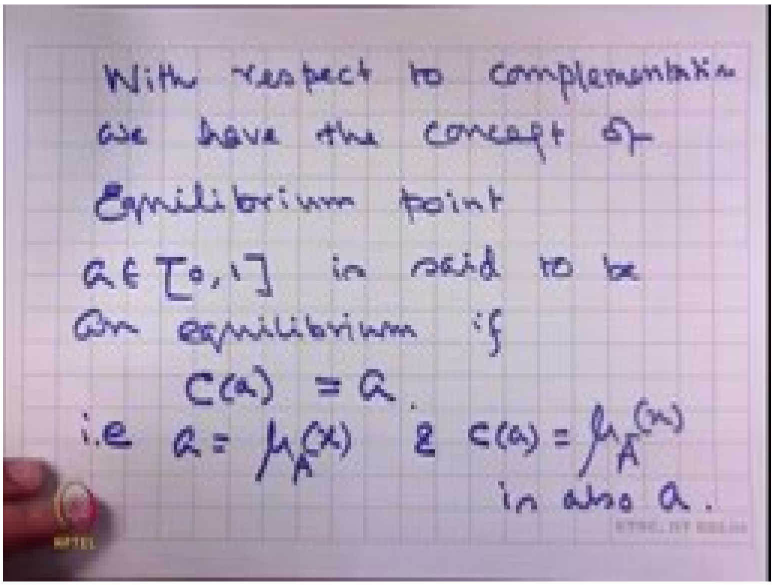

Now, with respect to complementation we have a concept of equilibrium point.

 is said to be an equilibrium if , when we interpret in terms of membership functions, if then is equal to is also .

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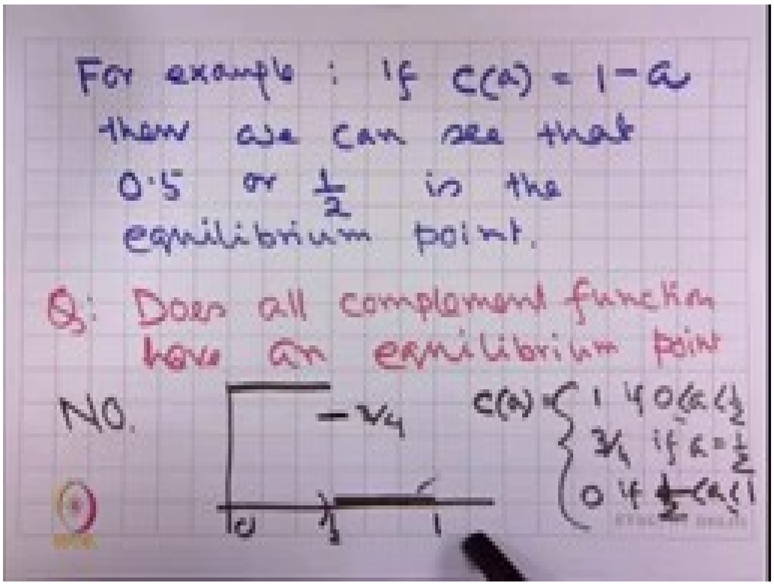

For example:

If then, we can see that is the equilibrium point.

Question, does all complement function have an equilibrium point?

Answer, is no.

In fact, the function

Therefore, you can see that there does not exist any point such that .

Therefore, but it satisfies all the condition of a complementation function.

Therefore, we find the complementation function which does not have an equilibrium point.

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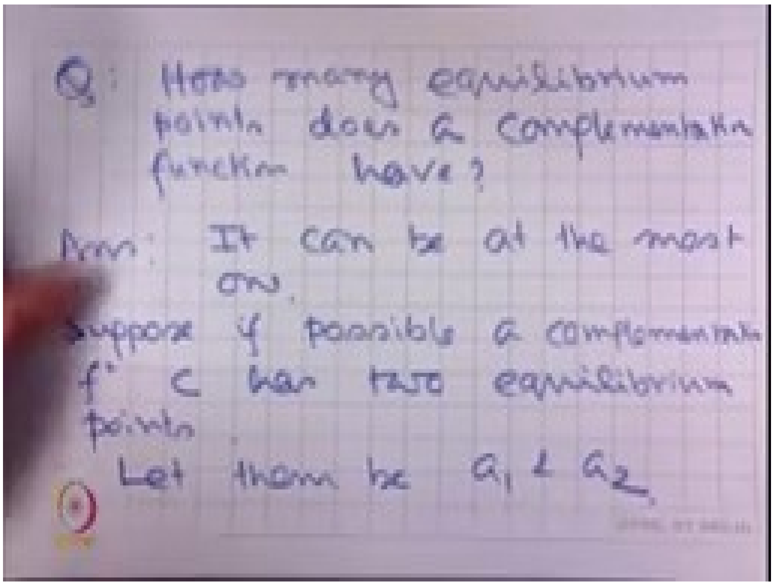

The natural question is how many equilibrium points does a complementation function have? Answer is it can be at the most one.

Why?

Suppose if possible a complementation function has two equilibrium points.

Let them be and .

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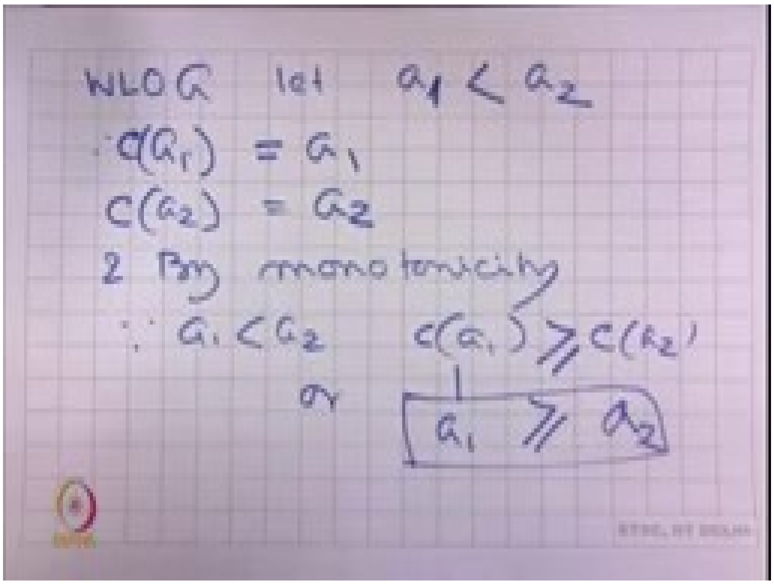

Without loss of generality

Let

Therefore, and

and by monotonicity

or .

Therefore, contradiction.

Therefore, there cannot exist two equilibriums.

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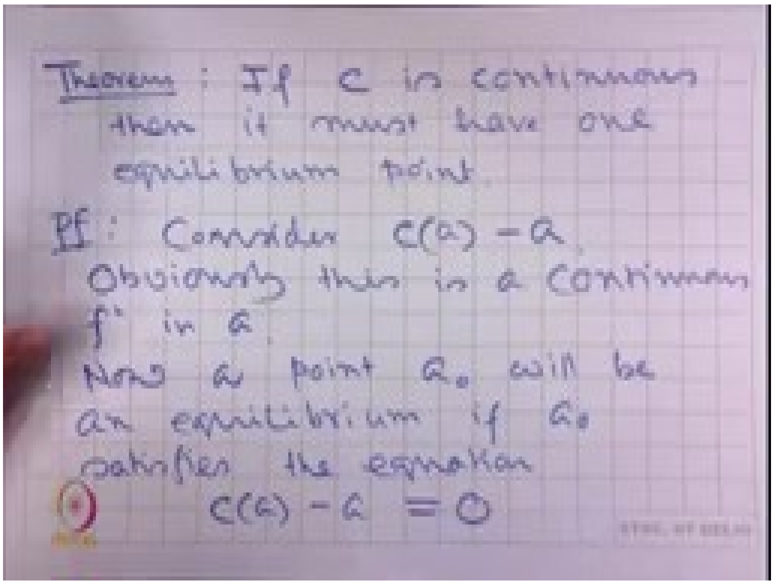

Theorem:

If is continuous then, it must have one equilibrium point.

Proof:

Consider . Obviously this is a continuous function in .

Now a point will be an equilibrium if satisfies the equation

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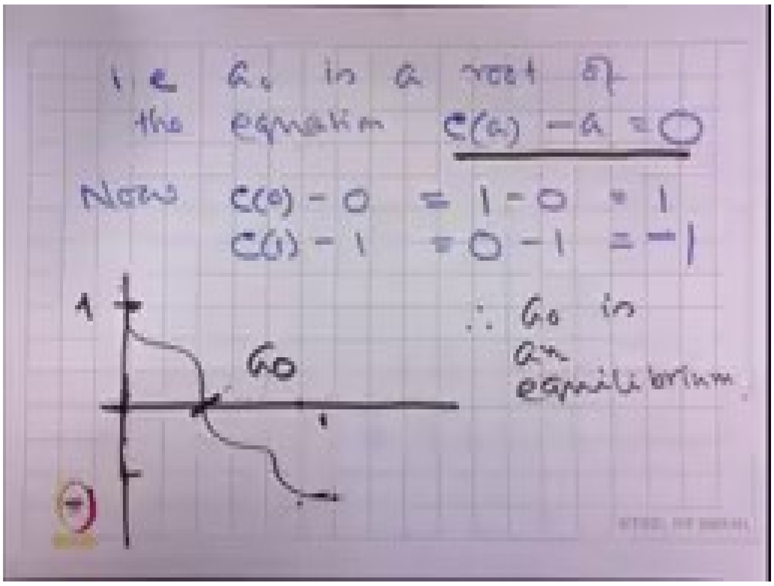

That is, is a root of the equation .

Now, and

Therefore, the function . At 0 it takes , at it takes the value.

Therefore, whatever may be the shape of the function it is going to intersect with the axis at some point because this is a continuous function.

So, let us call that point to be .

Thus, is going to be the root of the equation and therefore is an equilibrium.

We shall discuss more about complementation and we will look in depth in some of my subsequent lectures.

For the time being let me just here and discuss the next set operation which is fuzzy intersection.

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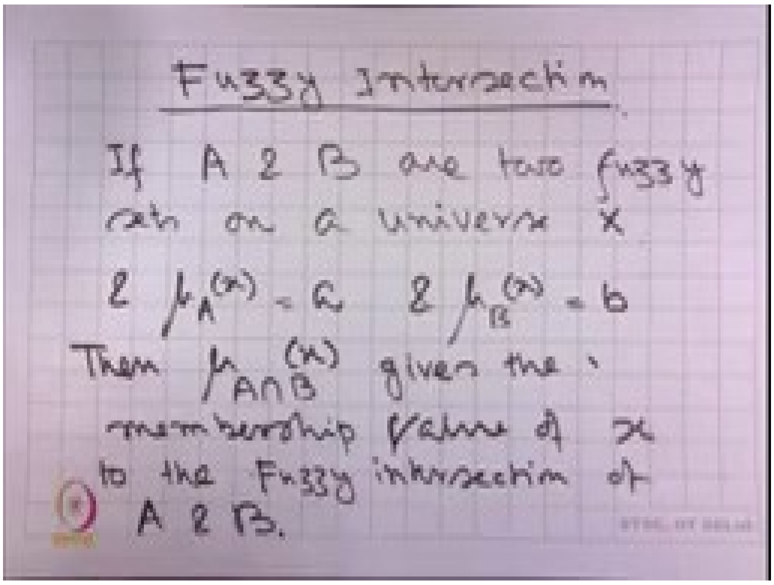

What it is?

If and are two fuzzy sets on some universe and if and .

Then gives the value of to the fuzzy intersection of and .

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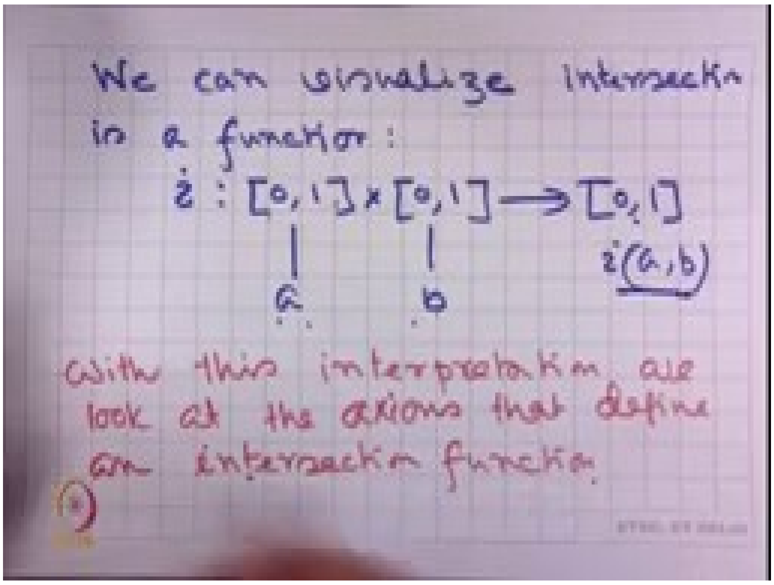

Therefore, we can visualize intersection is a function such that

That means it accepts two values and from the interval and produces a value , which also belongs to and which gives the intersection of these two.

With this interpretation we look at the (properties) axioms that define an intersection function.

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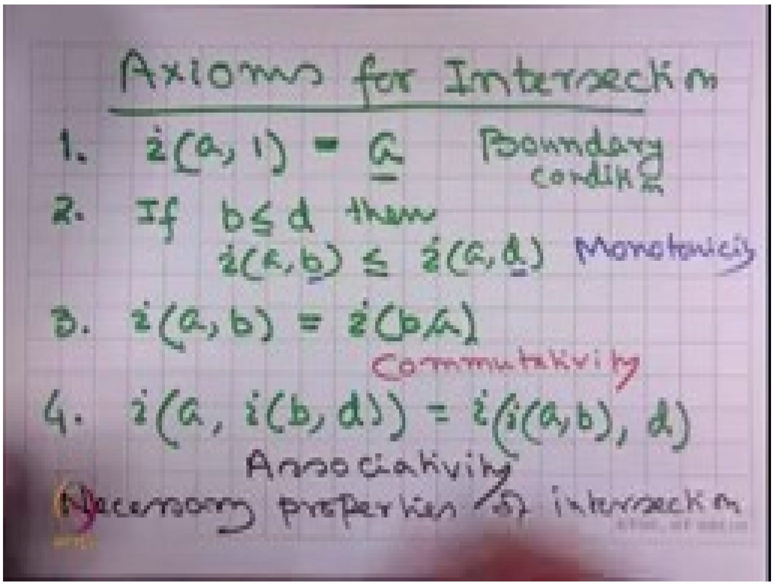

So, what are the axioms for intersection?

1) , this is the boundary condition

That means if but, then, its membership value to should be .

2) If then, . So, this is called monotonicity.

That means as the second parameter increases then the value of the function should ideally be non-decreasing.

3) This is commutativity.

And it is very clear. It does not matter in which order we keep the sets and, the intersection value should depend only on the respective membership functions and not in which order they have been put.

4) . This is the associativity.

So, these are the necessary properties of intersection.

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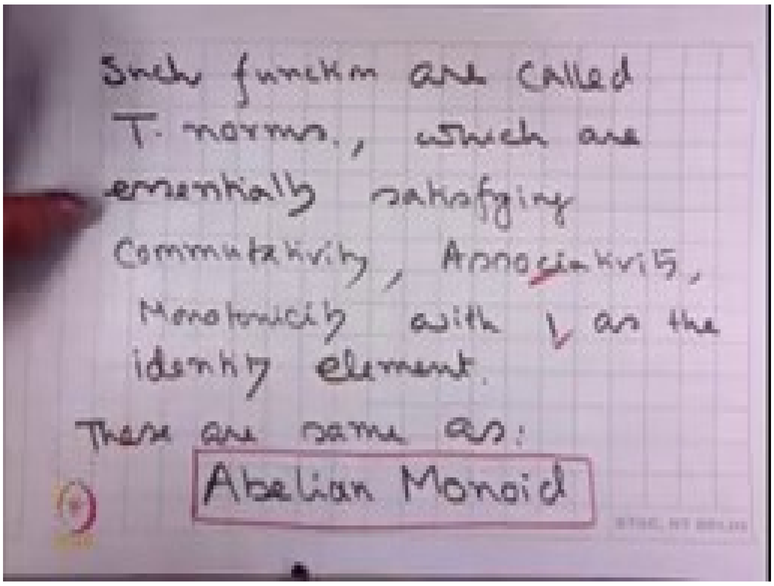

Such functions are called T-norms, which are essentially satisfying commutativity, associativity, monotonicity with as the identity element.

In fact, these are same as Abelian Monoid.

Those who know algebra and have that concepts of monoid, we know that a monoid needs an identity element which in this case is and they are associative and the commutativity gives it the Abelian property.

We shall often refer to these as T-norms and also I may refer to these as intersection functions interchangeably.

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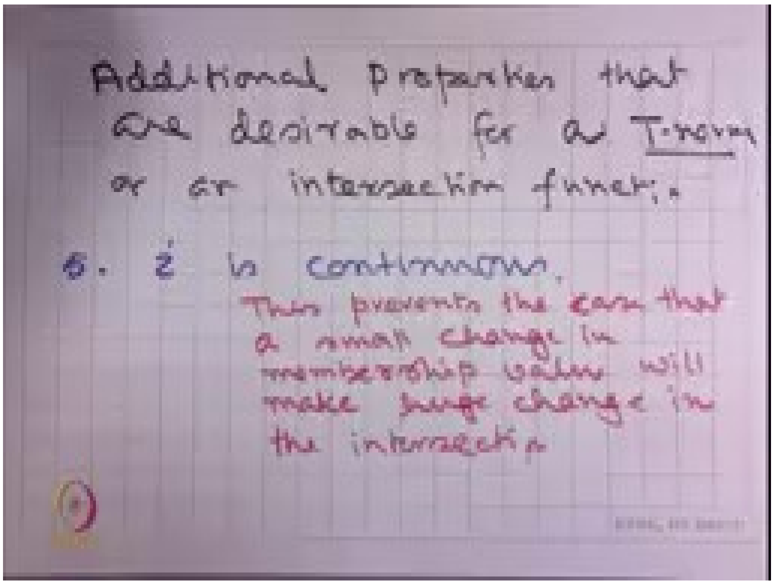

Apart from the above four, additional properties that are desirable for a T-norm or an intersection function.

What are these?

5) is continuous.

This is understandable because this prevents the case that a small change in membership values will make huge change in the intersection.

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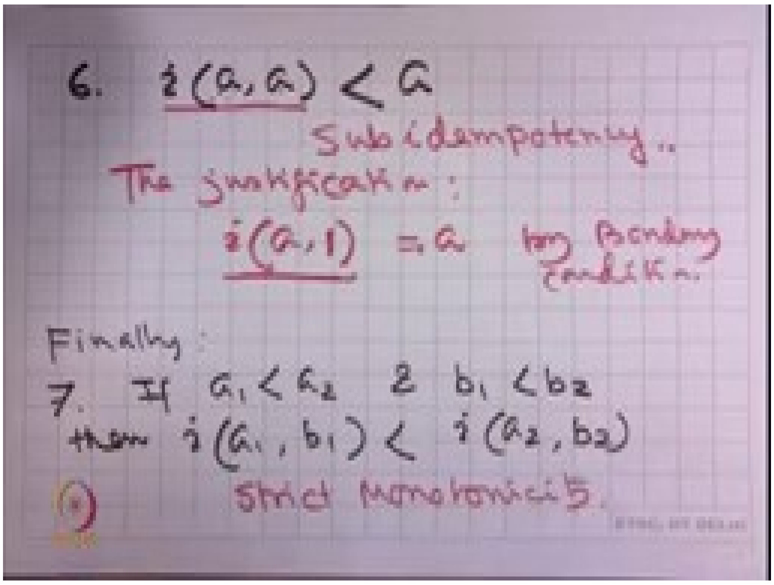

6) This is called sub-idempotency

We know that if then, we call it an idempotent function. But what is desirable is that is actually less than . Because then it is sub-idempotent. Why?

The justification is that we know by boundary condition. Therefore, if , it is expected that the value of the intersection of with should be less than intersection of with which is equal to

Therefore, should be less than .

7) if and then, . This is called strict monotonicity.

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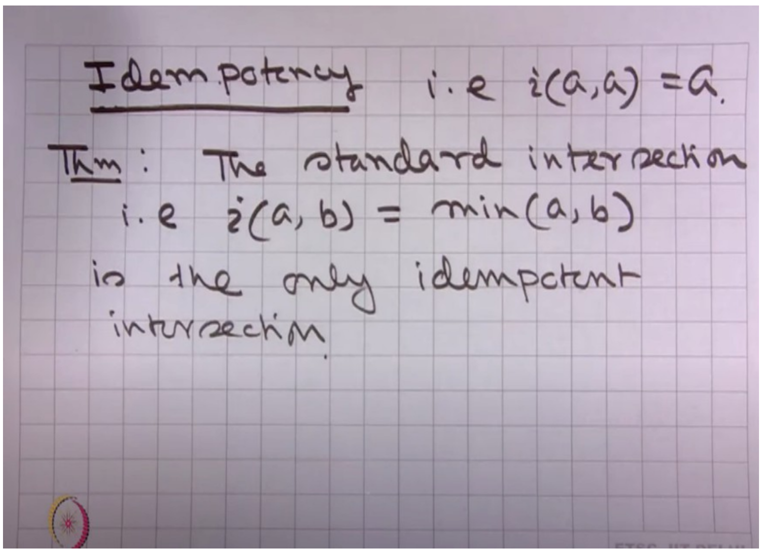

Let us now come back to idempotency that is

Theorem:

The standard intersection is the only idempotent intersection.

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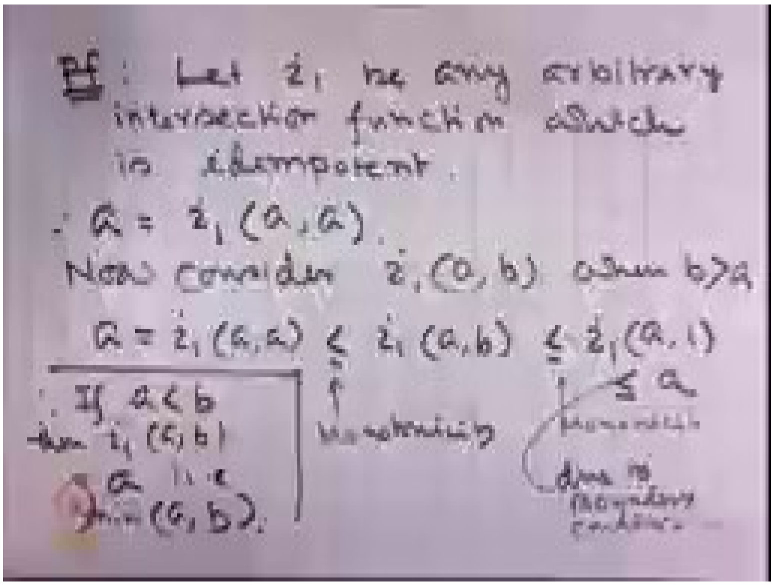

Proof: Let be any arbitrary intersection function which is idempotent.

Therefore,

Now consider when .

Therefore, because of monotonicity, again due to monotonicity and because due to boundary condition.

Therefore, what we get?

If then   that is .

By a similar logic, we can show that if then which is again the

Therefore, what we find that if is an idempotent intersection or an idempotent T-norm then effectively it returns the minimum of the two parameters.

Therefore, results to the standard intersection, .

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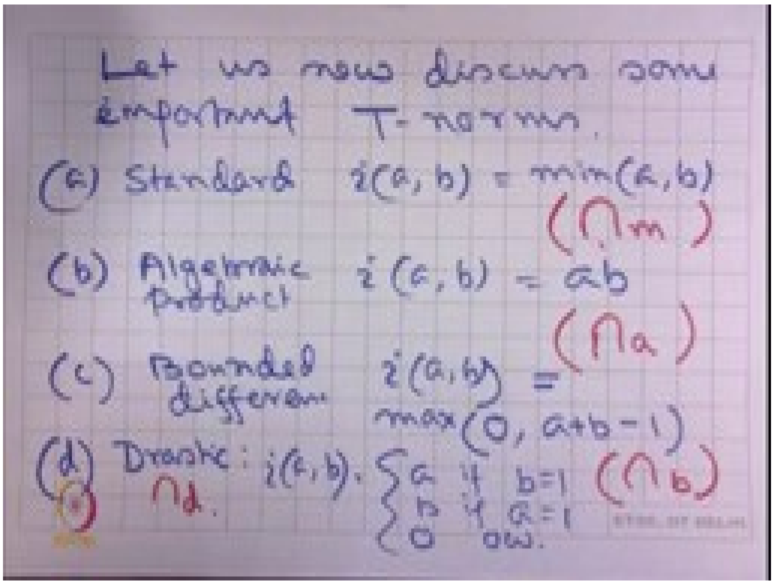

Let us now discuss some important T-norms

(a) Standard. We shall denote it as

(b) Algebraic product, , that is the product of the two elements. We shall denote it as

(c) Bounded difference. This we shall denote as

(d) Drastic .This we shall denote as

In the tutorial sheets, when you find problems with these symbols you understand that these are standard, algebraic product, bounded difference, and drastic respectively.

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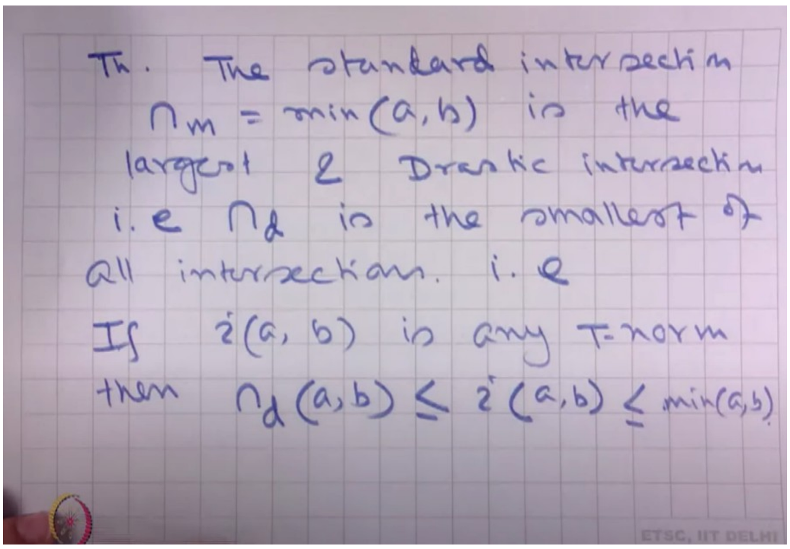

Now there is an interesting property, which is that:

Theorem:

The standard intersection is the largest and drastic intersection is the smallest of all intersections

If is any T-norm then

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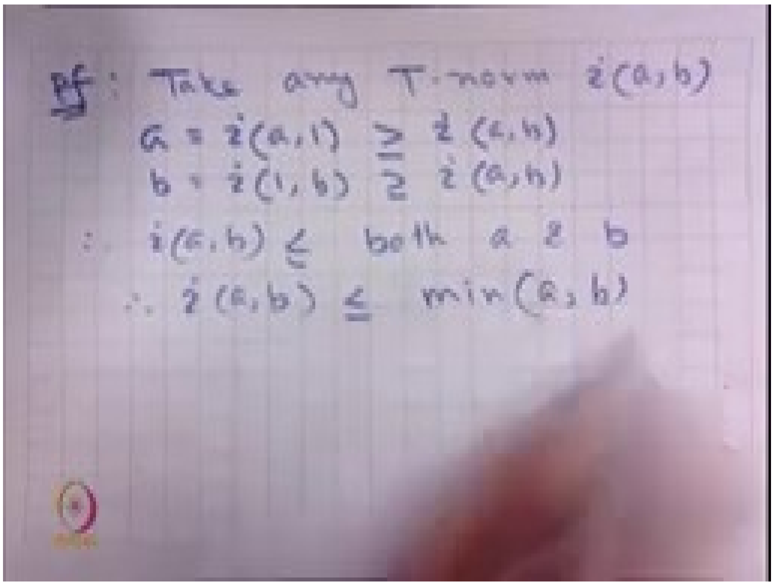

Proof:

Take any T-norm

Therefore, both and . Therefore,

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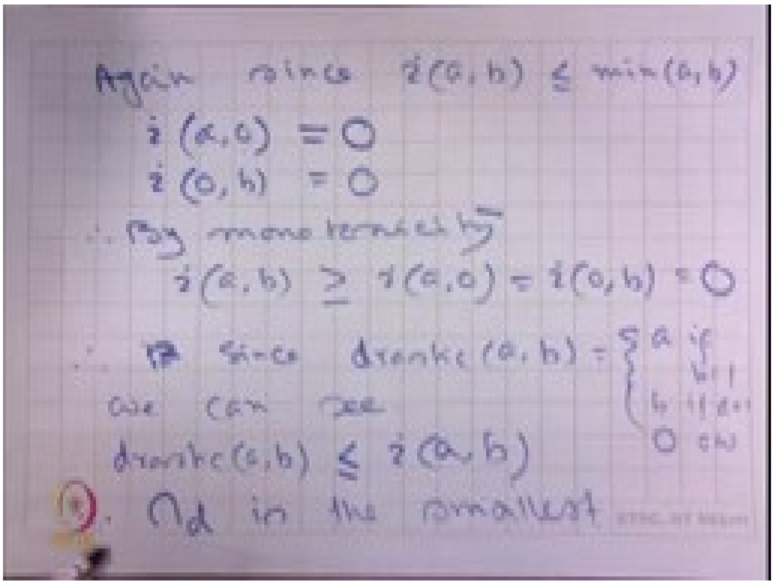

Again, since

and

Therefore, by monotonicity

Therefore, since

We can see . Therefore, intersection with drastic is the smallest.

Okay students with that I stop here today.

In this class we have seen some examples of complement functions.

Also we have seen intersection function and its properties which are necessary, which are desirable.

And also we have seen some examples and properties of intersection functions which are also called T-norms.

In the next class I shall be talking about Fuzzy Union function which are called T-conorms. We'll investigate their properties and also some examples of fuzzy unions. And the treatment of T -conorms or unions is very similar to T-norm, so certain things I may leave for you to verify and instead I will study the properties of combining more than one fuzzy operators.

In particular, our focus will be on De Morgan's laws and how they work with respect to Fuzzy intersection, Fuzzy Union and Fuzzy complements.

Okay students, thank you so much.