Introduction to Fuzzy Set Theory, Arithmetic and Logic
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Lecture 04
Fuzzy Sets Arithmetic and Logic

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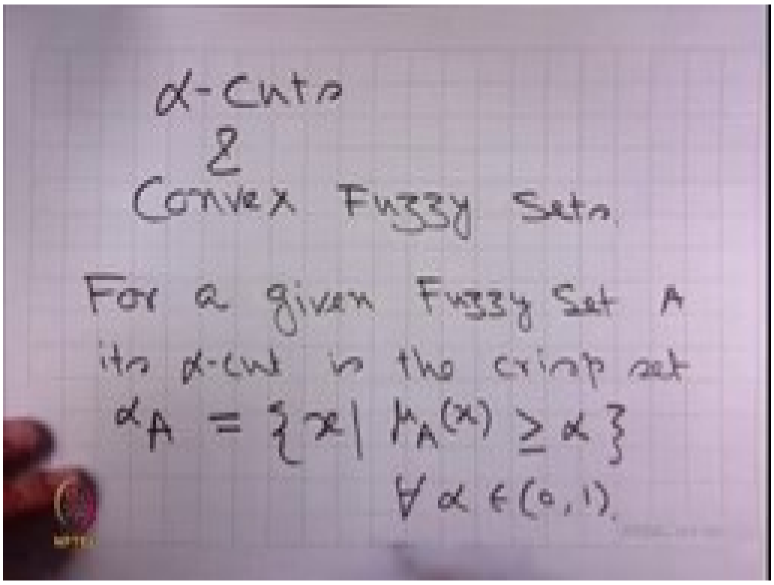

Welcome students to the fourth lecture on fuzzy sets arithmetic at logic.

In the last lecture we discussed two important topics namely α-cuts and convex fuzzy sets.

For a given fuzzy set , its -cut is the crisp set denoted as

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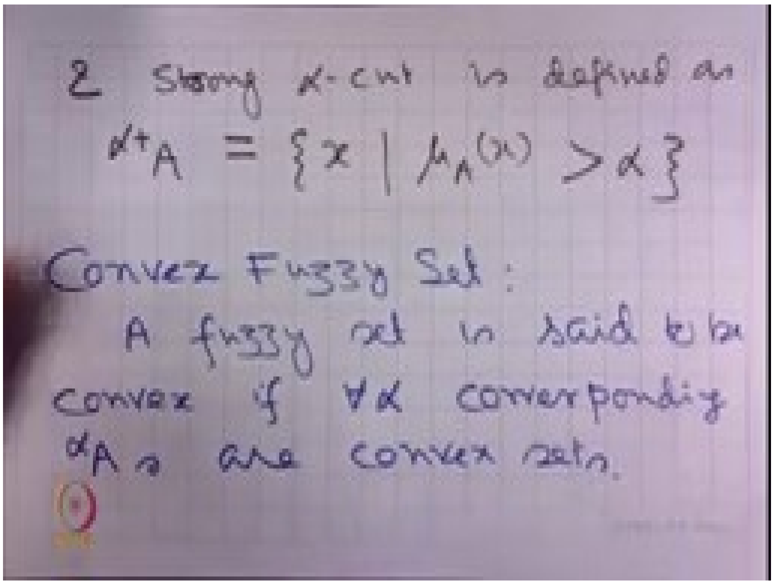

And Strong alpha-cut is denoted as

So, the difference is that in this case the inequality is a strict inequality. In the definition of alpha-cut we have used greater than equal to.

And what is the Convex Fuzzy Set?

A fuzzy set is said to be convex if for all corresponding α-cuts are convex sets.

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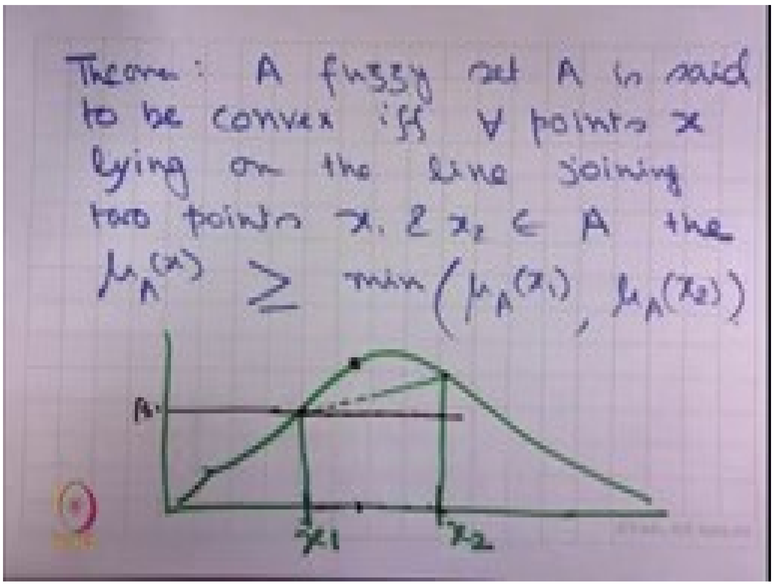

And a fuzzy set is said to be convex if and only if points lying on the line joining two points and   the

I have proved the theorem.

Now let me show you with a diagram.

Suppose this is a convex fuzzy set. Now consider the points and . Now if we look at any point on the line joining and and we look at its membership value . You find that the membership value is more than the minimum of the membership value of and membership value of .

Since in this case membership of is smaller than membership of . We will look at all the points and we can easily see that their membership values are more than the membership of to A which is equal to say this point let us call it say .

So, for all these intermediate points the membership value is greater than .

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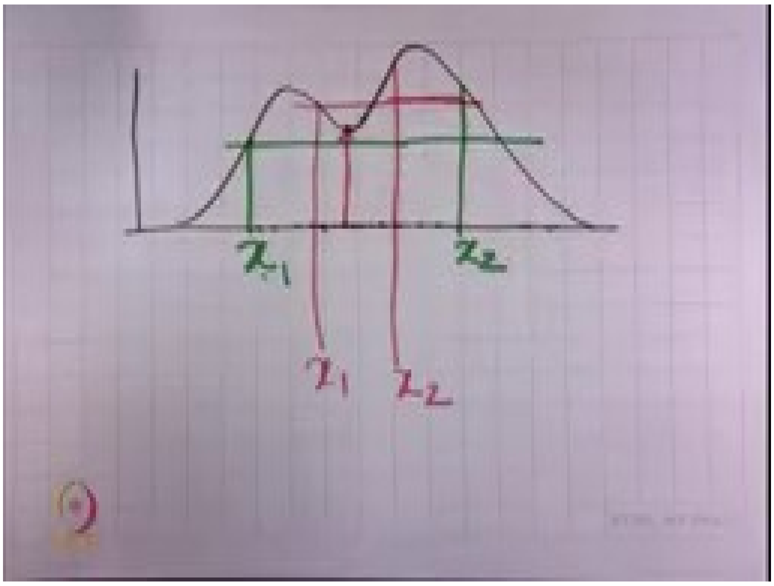

This however is not true if the fuzzy set is not convex. For illustrations consider a fuzzy set with this type of membership.

Consider this to be and this to be then we can see that for all the points we have the membership values actually bigger than the . Therefore, it can hold for some of the lying on the line.

But suppose I consider this and this , so these are the set of points lying on the line joining and .

And we can see that there are intermediate points whose membership is

In fact, you can see that it is smaller than that.

This happens because this is not a convex fuzzy set.

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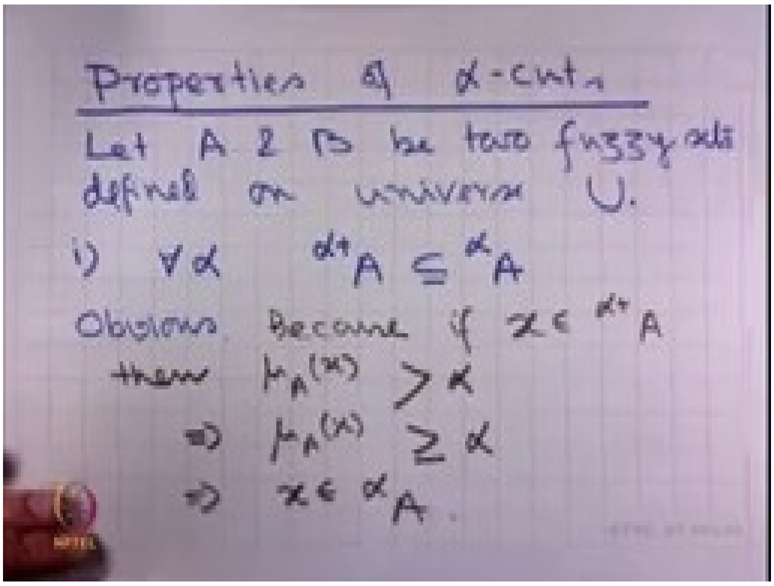

With that background, let us now study some properties of α-cuts.

So, let and be two fuzzy sets defined on universe .

i.

This is obvious. Why?

Because if

Then,

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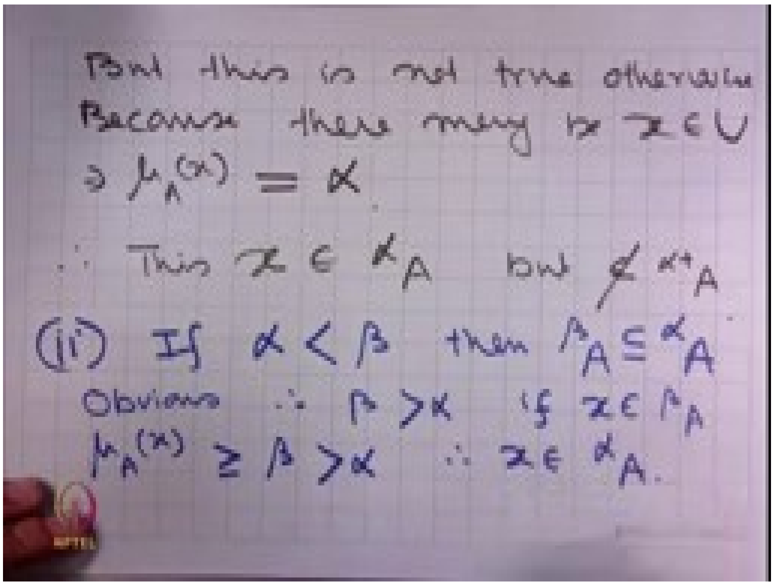

But this is not true otherwise.

Because there may be such that .

Therefore, this , but   .

ii. If then, .

This is obvious.

.

If , then

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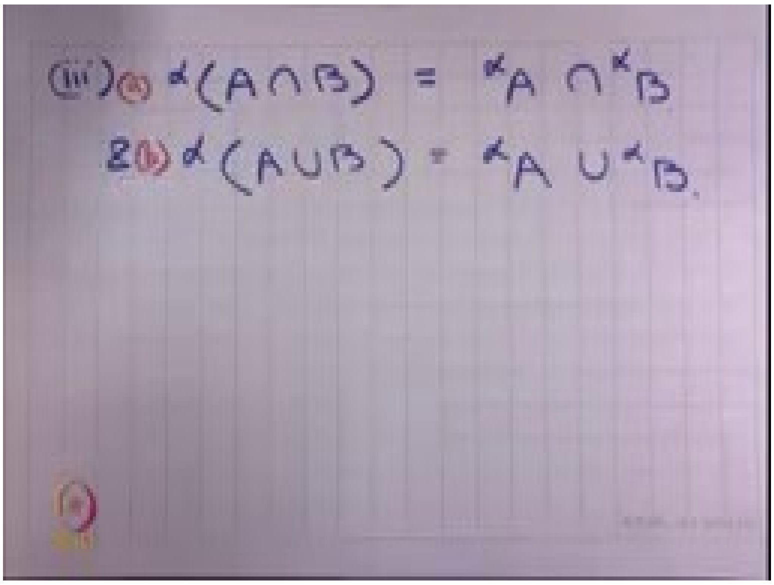

iii. (a)

(b)

So, let me prove them.

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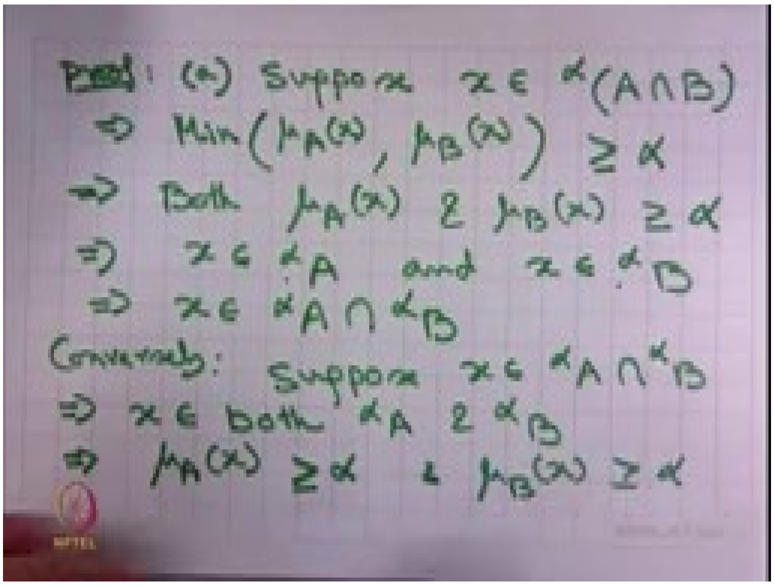

So, proof.

Let me first prove the (a) part.

Suppose

Both and

and

Conversely, suppose

both and

and

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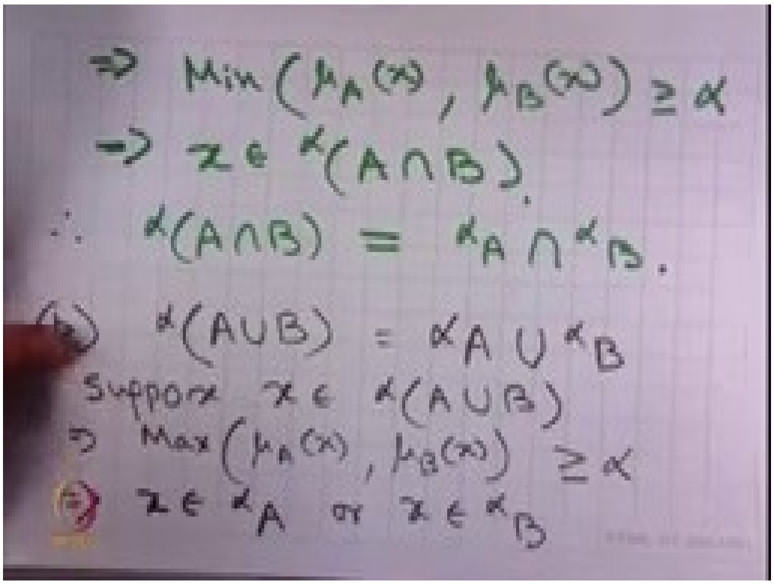

Now proof of (b)

Suppose

or

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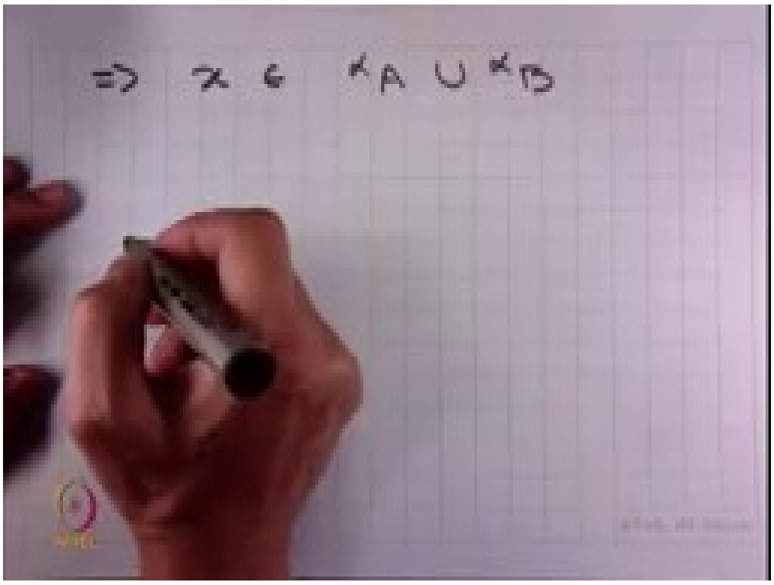

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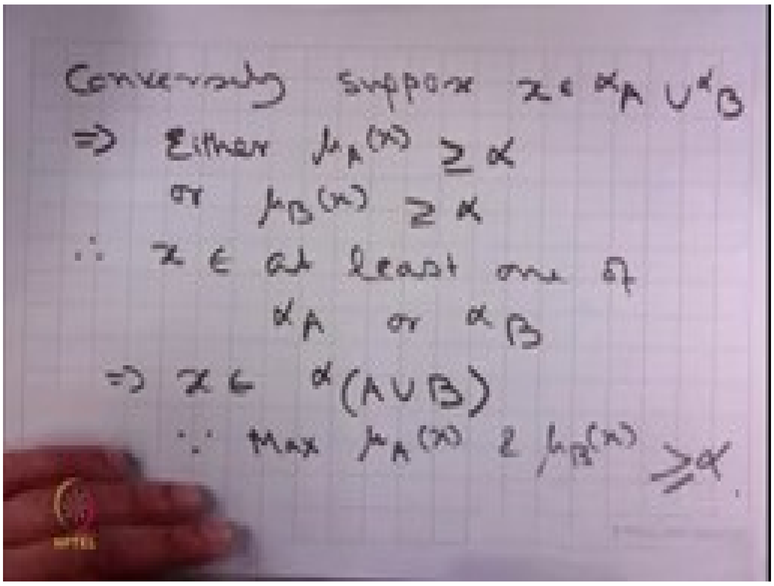

Conversely, suppose

Either or

at least one of or

Very similar properties can be proved with respect to -cuts also.

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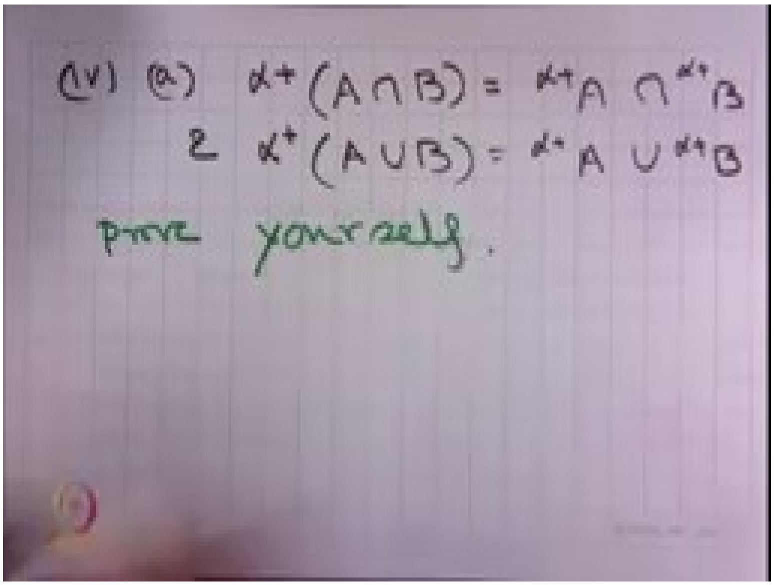

iv. (a)

(b)

Prove yourself.

Since the proof is in a very similar line I want you to prove them yourself.

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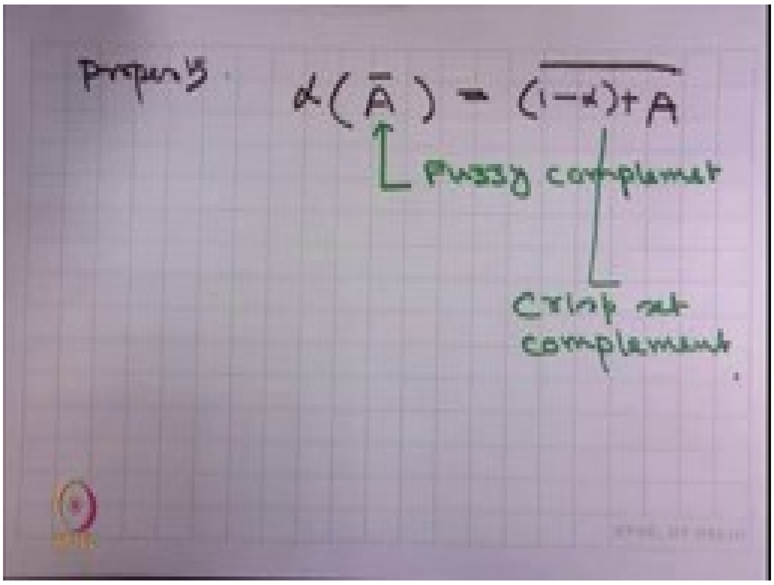

Another property is

v.

You have to understand that this is fuzzy complement, is a fuzzy set I am looking at which is fuzzy complement. And then I am looking at α-cut of that.

But, is a crisp set complement, is a crisp set and I am looking at its complementation.

So, you have to remember this let me give the proof.

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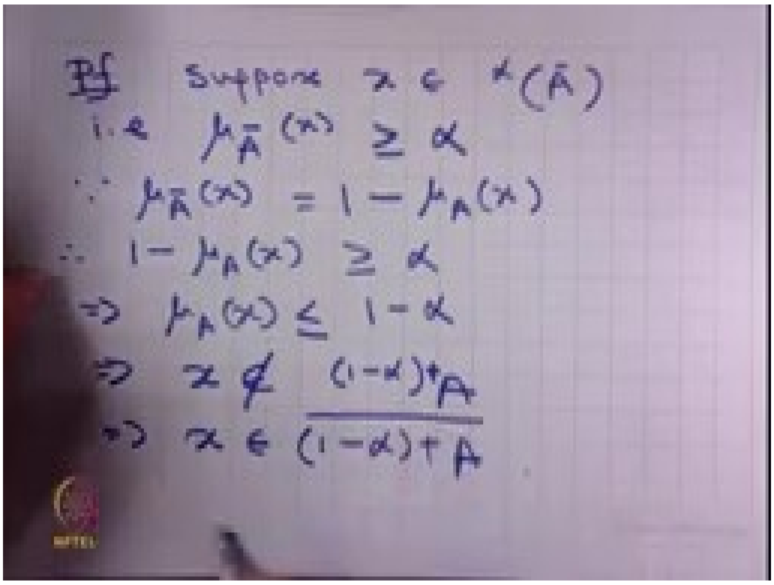

Suppose

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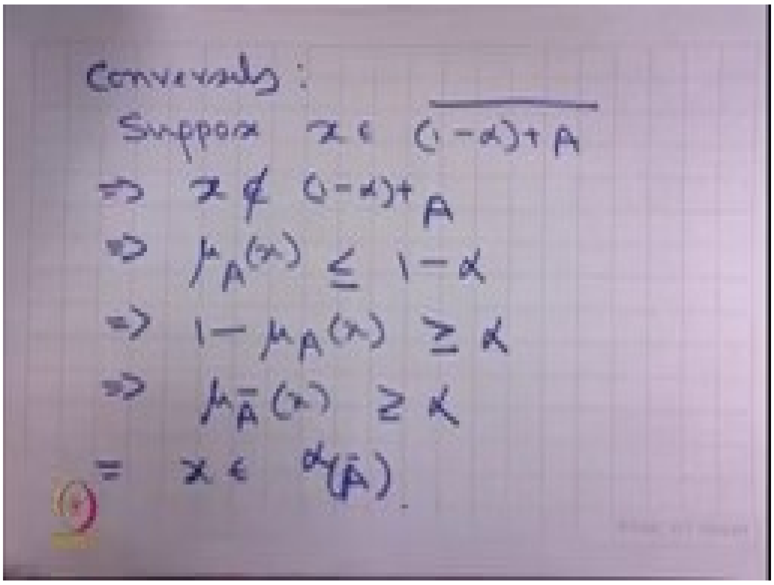

Conversely, suppose

These are some interesting properties with respect to alpha-cuts.

In some of my subsequent lectures, I shall deal with these properties in a slightly more generalized way.

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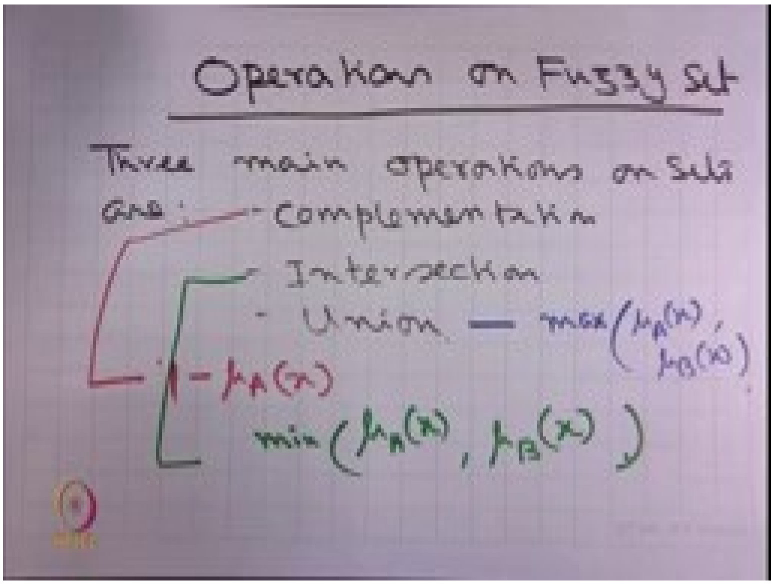

In the remaining part of this lecture, I shall start studying operations on fuzzy sets.

We have already seen that the three main operations on sets are

- complementation
- intersection
- union.

We have already seen some methods for doing it.

For example:

Complementation, we have done

Intersection, we have done using

And Union we have done using .

However, why should these be the only way of operating on fuzzy sets.

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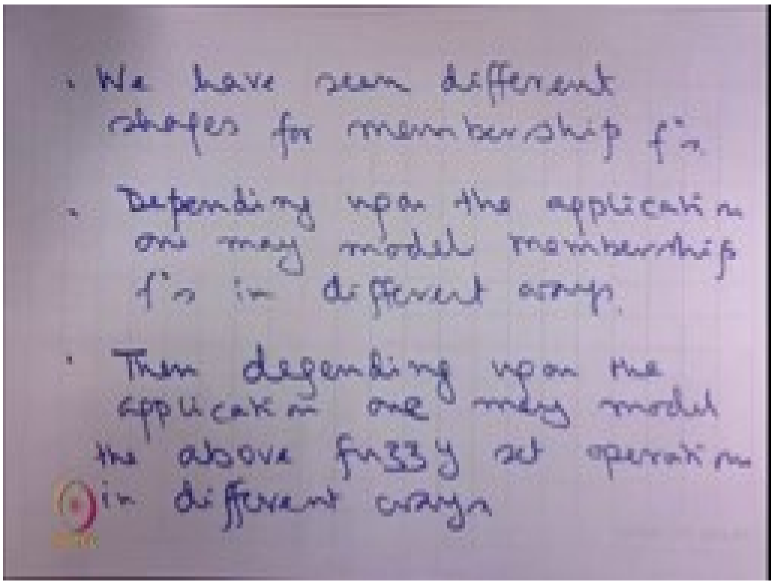

We have seen different shapes for membership functions and we have discussed that depending upon the application, one may model membership functions in different ways.

In fact, in one of my earlier lectures we have seen different types of membership functions. Then depending upon the applications, one may model the above set operations, or say fuzzy set operations in different ways.

So, this led to more generalization of these set operations.

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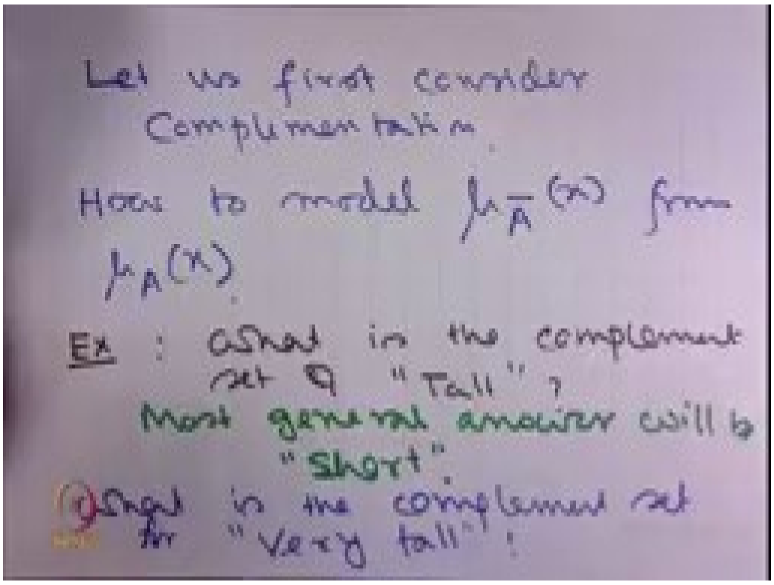

In particular, let us first consider complementation.

So, how to model from .

Actually we will see that sometimes complementation may have different interpretations. Example: What is the complement set of say ‘’?

If I ask you the question I am sure most general answer will be ‘’. Then, what is the complement class, (complement set) for ‘’ ?

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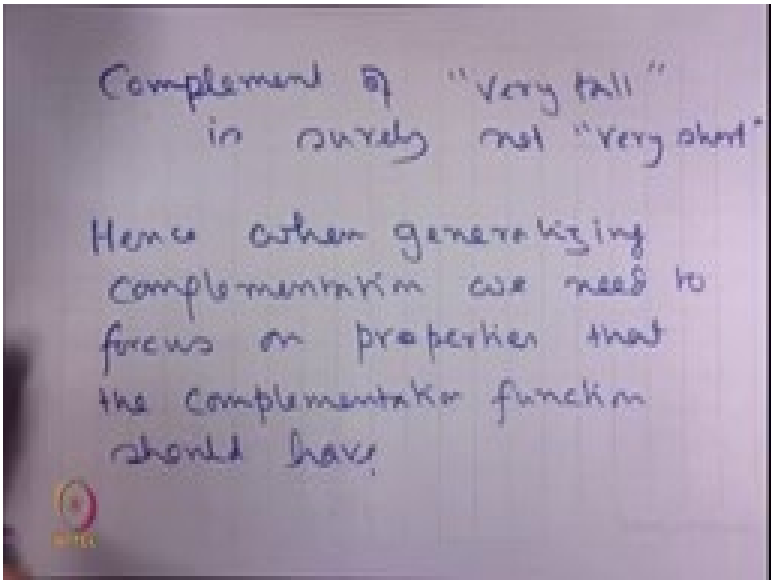

Complement of ‘’ is surely not ‘’. So, common intuition sometimes fails. hence when generalizing complementation, we need to focus on properties that the complementation function should have.

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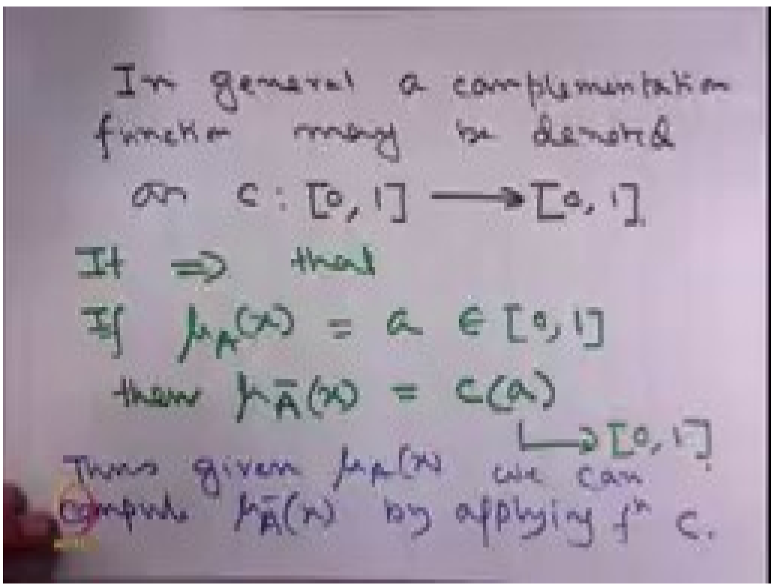

So, in general a complementation function maybe denoted as such that

What does it mean?

It means that if then which again lies in the interval .

Thus, given we can compute by applying function .

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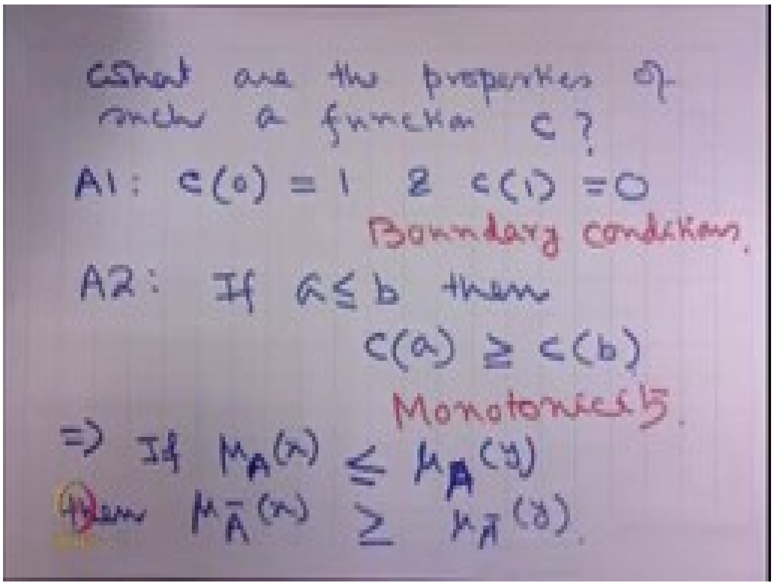

Question is what are the properties of such a function ?

A1.   and .

So, these are called boundary conditions. What does it mean?

It means that if membership of to a fuzzy set is then the membership of to its complement is .

This comes naturally from the concept of crisp set.

If membership is that means does not belong to the set A. Therefore, belongs to the complement. In a similar way, if membership of in is . That means if is a member of the set , its membership to the complement set is .

A2. If then,

This is called monotonicity.

That means if then, , which is understandable.

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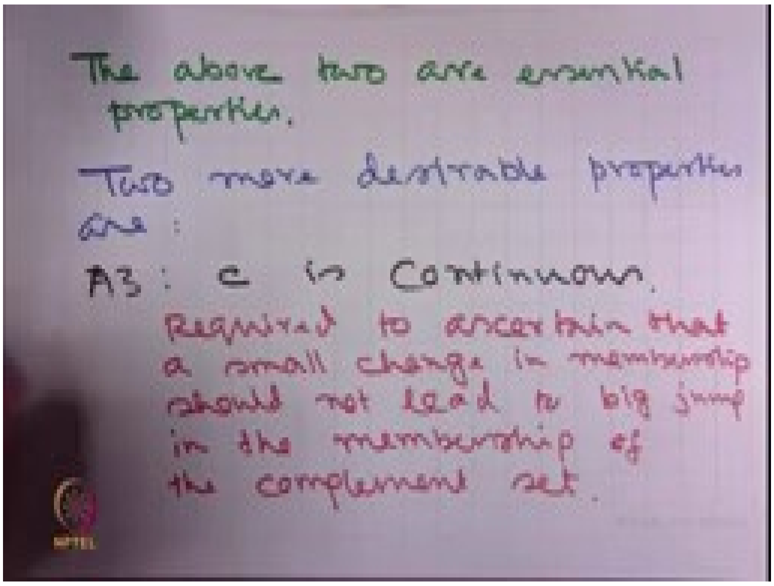

These two are essential properties.

Two more desirable properties are:

A3. is continuous

This is required to ascertain that a small change in membership should not lead to big jump in the membership of the complement set.

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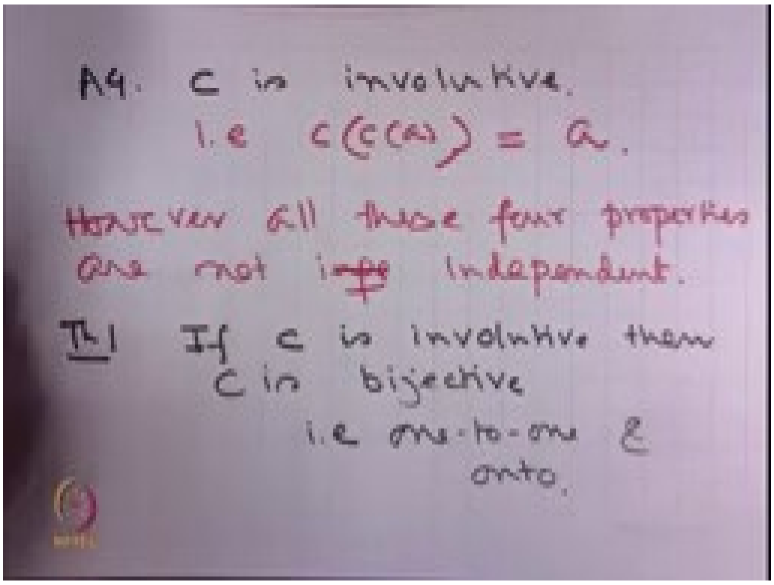

And there is a fourth property which is

A4. is involutive

That is complement of complement of is . However, all these four properties are not independent.

Let me give you a few results:

If is involutive then, is bijective one-to-one and onto.

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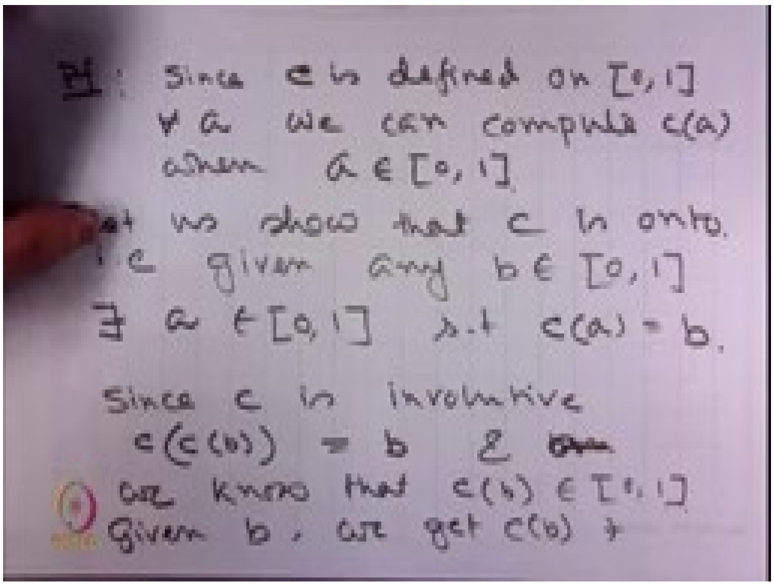

Proof:

Since is defined on ,   we can compute when .

Let us show that is onto , given any there exists such that .

Since is involutive, and by above observation and we know that .

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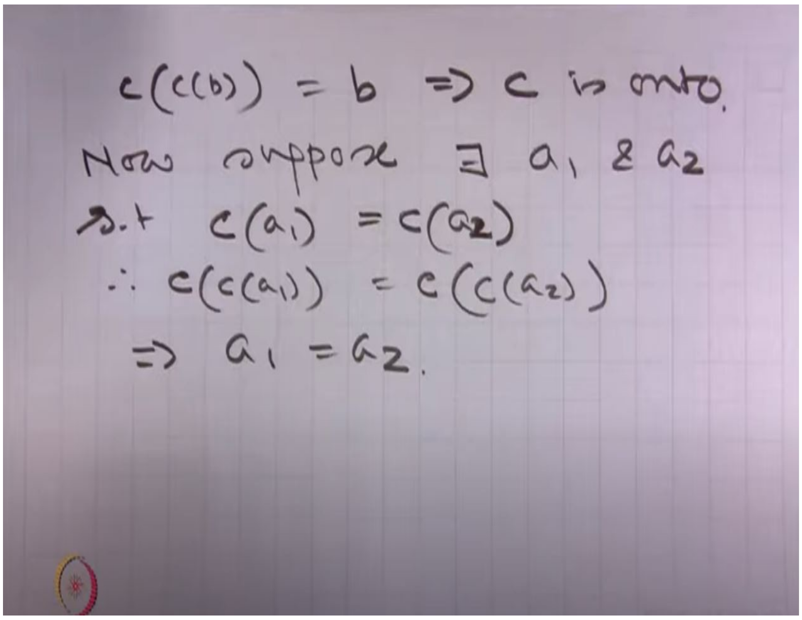

Therefore, given we get such that is onto.

Now, suppose there exists and such that

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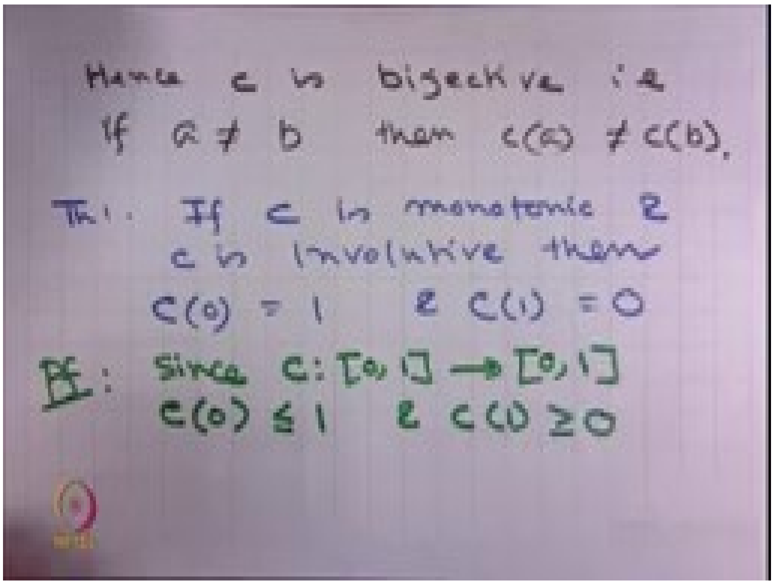

Hence is bijective. if then,

Theorem: If is monotonic and is involutive then and

Proof:

Since

 and

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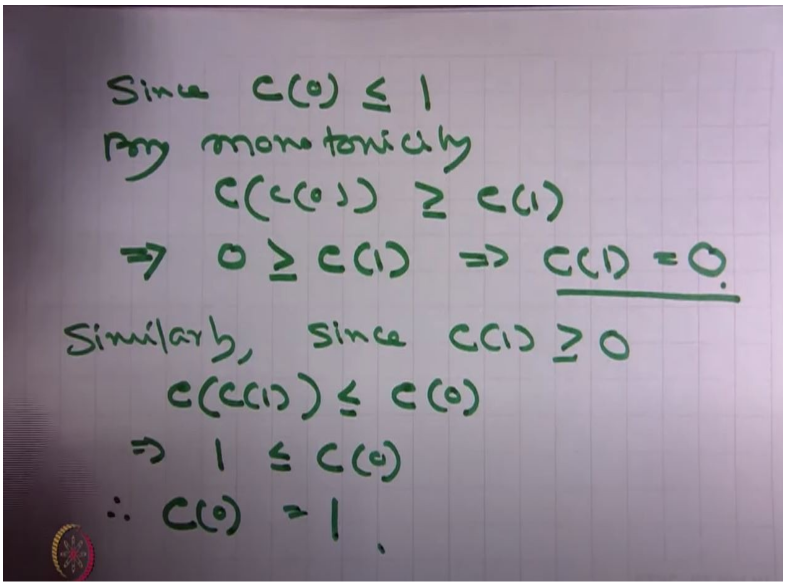

Since

By monotonicity,

Similarly, since ,

Therefore,

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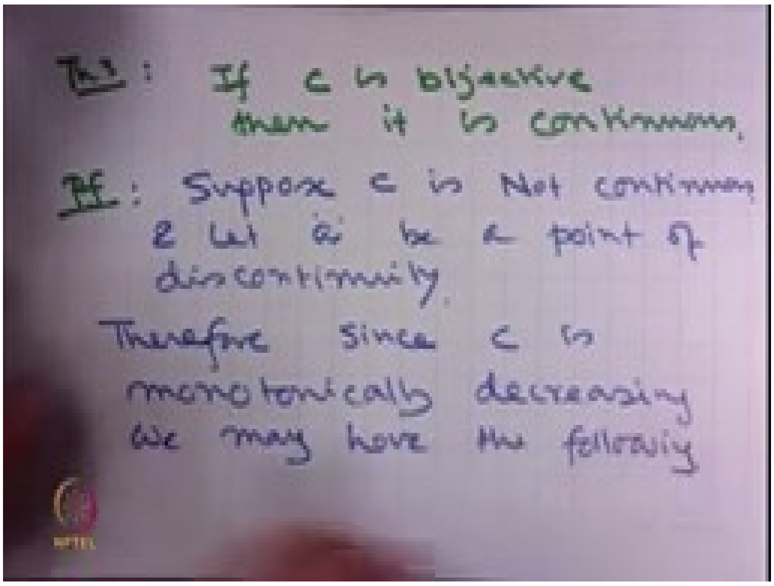

Another interesting result is:

Theorem: If is bijective then, it is continuous.

Let me give you a proof.

Suppose is not continuous and let be a point of discontinuity.

Therefore, since is monotonically decreasing we may have the following.

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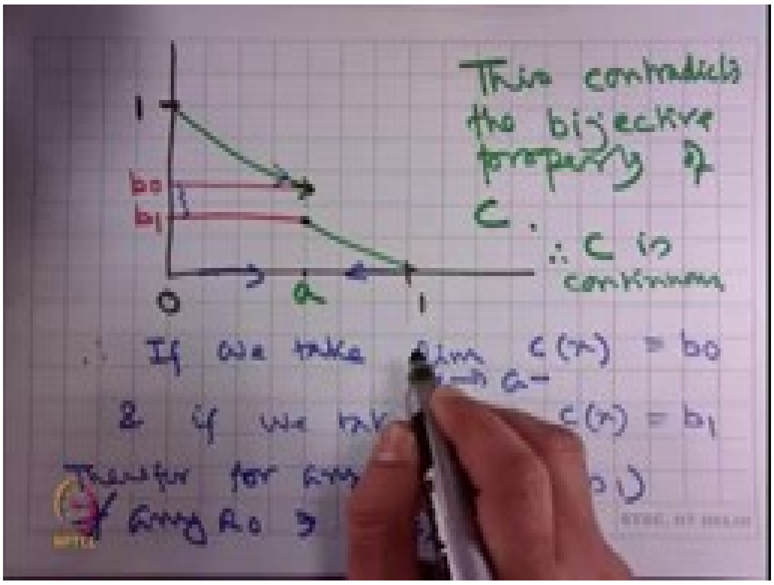

At it is , at it is and suppose ‘’ is a point of discontinuity.

Therefore, we shall get such a type of picture if ‘’ is the point of discontinuity.

Therefore, if we take that means we are moving towards ‘’ from the left side. Therefore, the limit is going to be .

And if we take that means we are moving towards from the right side then the limit will come to the point .

Therefore, for any there does not exist any such that .

Because for no values of in the interval we can achieve an intermediate point of and to be the value of the complement.

This contradicts the bijective property of

Therefore, is continuous.

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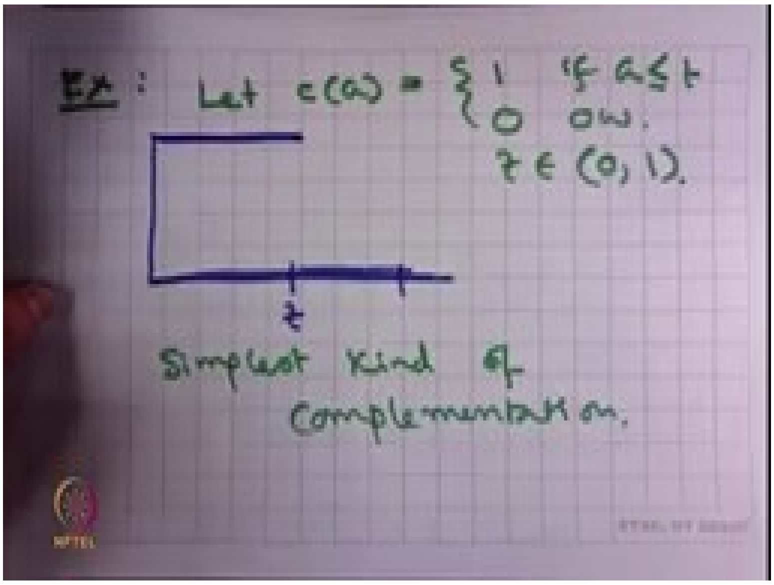

Example:

Let ,

Therefore, if this is we are looking at a function such that for these values it is and for these values it is

So, it is one of the simplest kind of complementation.

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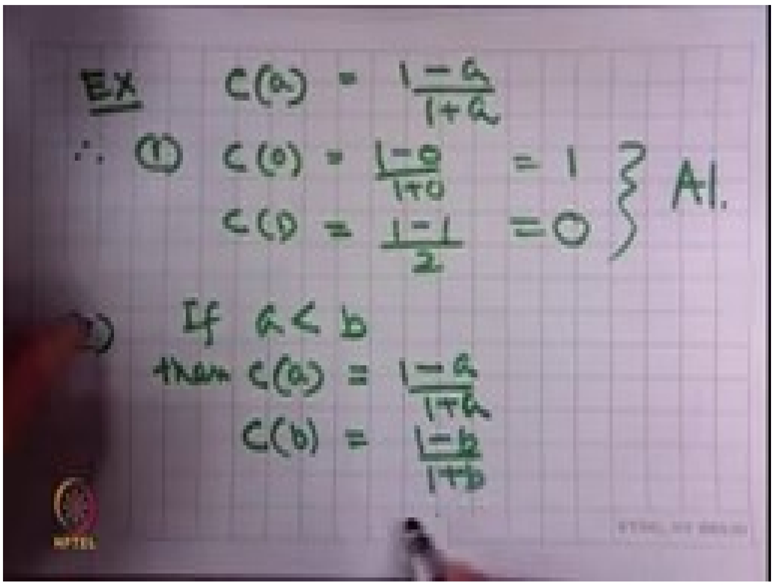

Another example:

Therefore,

1)

So, this is the Axiom 1.

2) If then, and

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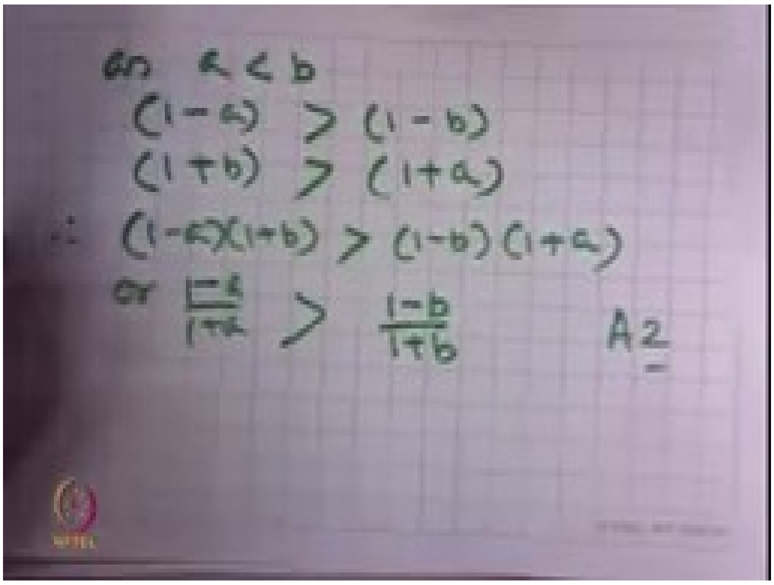

As ,

and

Or

This is Axiom 2.

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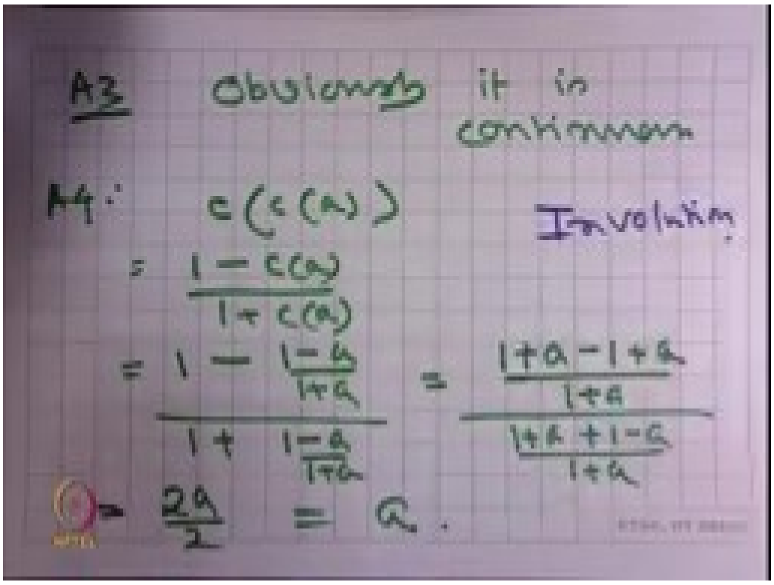

Axiom 3: Obviously it is continuous.

Axiom 4:

Therefore, it follows involution also.

In the tutorials I will give you many different problems involving different complementation functions.

With that I stop here today.

In the next class I shall investigate complementation slightly deeper and also look at generalizations of Fuzzy Union and Fuzzy Intersections, Thank you.