Introduction to Fuzzy Sets Arithmetic & Logic Prof. Niladri Chatterjee Department of Mathematics Indian Institute of Technology - Delhi

Lecture- 30 Fuzzy Sets Arithmetic & Logic

Welcome students, to the MOOC'S course on fuzzy sets, arithmetic and logic. This is lecture number 30. If you recall in the last class I started with, Evidence theory.

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Evidence Theory. Given a set X of alternative Decisions, are have given three definition: n: P(x) → [0, 1] Basic Probability arrighment to a subset of X Bel(A): $\sum m(B)$, A, B E P(X). BEA

In fact, given a set X of alternative decisions we have given 3 definitions:

- $m: P(X) \rightarrow [0, 1]$ this is the *Basic Probability Assignment* to a subset of X,
- $Bel(A) = \sum_{B \subseteq A} m(B)$ where $A, B \in P(X)$

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PI(A): <u>J</u>m(B) B|BAA = Ø And we have talked about Dempster's Rule for combination - of multiple evidences NPTEL

- $Pl(A) = \sum_{B \mid B \cap A \neq \phi} m(B)$
- And we have talked about *Dempster's Rule* for the combination of multiple evidences.

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We started working on a problem : X = ZA, B, C3 Two sources of Evidences. m, 2 m2 And we started working on computing their Bel 2 also Compining them.

We started working on a problem such that

$$X = \{A, B, C\}$$

Two sources of evidences m_1 and m_2 .

And we started working on computing, their *Belief* and also combining them.

In this respect, we have come to this table.

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45	et w	o fir	st Ta	bulat	e the	mi experts
for	m	Beli	m2	Bel2	MIL	Buliz
A	0.05	-	0.15		0-21+	
B	0		0		0.01	
C	0.05		20.0		0.15	
AUB	All		0.05		0-2	
Ave	01		0.05		0.06	
Bre	0.05	The	0.5		0.31	
Ve	0.6		1031		10 011	

Where the m_1 and m_2 are the evidences for different non empty subsets and using the formula we have calculated the m_{12} a combination of the evidences for different subsets of X.

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The formula for the combination was $m_{12}(P) = \sum_{Y \cap Z} m_1(Y) m_2(Z)$ Y \n Z = P

The formula for the combination was

$$m_{12}(P) = \frac{\sum_{Y \cap Z = P} m_1(Y) \times m_2(Z)}{1 - K}$$

Where $K = \sum_{B \cap C = \phi} m_1(B) \times m_2(C)$

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 $m_{12}(A)$

$$= \frac{\begin{pmatrix} m_1(A) \times m_2(A) + m_1(A) \times m_2(A \cup B) + m_1(A) \times m_2(A \cup C) \\ +m_1(A) \times m_2(A \cup B \cup C) + m_1(A \cup B) \times m_2(A) + m_1(A \cup C) \times m_2(A) \\ +m_1(A \cup B \cup C) \times m_2(A) + m_1(A \cup B) \times m_2(A \cup C) + m_1(A \cup C) \times m_2(A \cup B) \end{pmatrix}}{1 - K}$$

Thus we have 9 pairs of subsets whose intersection is $\{A\}$

In a similar way, you can compute these values and all other values in particular.

Let us now look at calculation of *Belief*, since, for each one of them, we can calculate the *Belief* for illustration, I am just showing for one of them.

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So, let us compute Bel_{12}

	m_{12}	Bel_{12}	<i>Pl</i> ₁₂
Α	0.21		
В	0.01		
С	0.09		
$A \cup B$	0.12		
$A \cup C$	0.2		
$B \cup C$	0.06		
$A \cup B \cup C$	0.31		

We are going to calculate Bel_{12} and Pl_{12} . How to compute them? .

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So far, you recall

• $Bel(P) = \sum_{Q \subseteq P} m(Q)$

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Let us compute Bel 12						
	MIZ	Beliz	Pliz			
A	0-21	0.51				
B	0.01	0.01				
C	0.09	60.0				
AUB	0.15.	0.34				
AVL	0.5	0.5				
BUC	0.00	0.12				
AUBIC	0.31	1.0				
mention						

Therefore, what we will have

	m_{12}	Bel_{12}	<i>Pl</i> ₁₂
Α	0.21	0.21	
В	0.01	0.01	
С	0.09	0.09	
$A \cup B$	0.12	0.34	
$A \cup C$	0.2	0.5	
$B \cup C$	0.06	0.16	
$A \cup B \cup C$	0.31	1.0	

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Let us now compute Plausibility. Can be computed in 2 different asy: a) $PI(P) = \sum m(Q)$ $P \cap Q \neq \emptyset$ b) $PI(P) = I - Bal(\overline{P})$

Let us now come to *Plausibility*, can be computed in two different ways:

- $Pl(P) = \sum_{Q|P \cap Q \neq \phi} m(Q)$
- $Pl(P) = 1 Bel(\overline{P})$

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L	et wa	o compu	compute Bel 1		
	112	Beliz	Pliz		
A	0-21	0.51	0.84	1- BU(BVS)	
B	0.01	0.01	0.5	1 - PallAug	
C	0.09	0.09	0.66		
AUB	0.15	0.34	0.91		
AVL	0.2	0.2	0.99		
BUC	0.00	0.12			
AUBK	0.31	1.0	1		
NATCE			1		

Therefore,

	m_{12}	Bel_{12}	<i>Pl</i> ₁₂	
Α	0.21	0.21	0.84	$1 - Bel_{12}(B \cup C)$
В	0.01	0.01	0.5	

С	0.09	0.09	0.66	$1 - Bel_{12}(A \cup C)$
$A \cup B$	0.12	0.34	0.91	
$A \cup C$	0.2	0.5	0.99	
$B \cup C$	0.06	0.16	0.79	
$A \cup B \cup C$	0.31	1.0	1.0	

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Let us now compute Plaumbility using the other formula: PI(P) = Zm(Q) $PnQ \neq D$ Me have $x = qA, B, c_{3}^{2}$ $PI(qA_{3}) = mi2(A) + mi2(AUB)$ t miR(AUC) + mi2(AUB) t MiR(AUC) + mi

So, let us now compute Plausibility using the other formula and that formula is

$$Pl(P) = \sum_{Q|P \cap Q \neq \phi} m(Q)$$

Therefore, we have $X = \{A, B, C\}$

Therefore,

$$Pl_{12}(\{A\}) = m_{12}(A) + m_{12}(A \cup B) + m_{12}(A \cup C) + m_{12}(A \cup B \cup C)$$
$$= 0.21 + 0.12 + 0.2 + 0.3 = 0.84$$

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$$T_{A} = similar & G_{AU} :$$

$$P(B) = m_{12}(B) + m_{12}(AUB) + m_{12}(AUC) + m_{12}(A$$

In a similar way,

$$Pl(B) = m_{12}(B) + m_{12}(A \cup B) + m_{12}(B \cup C) + m_{12}(A \cup B \cup C)$$
$$= 0.01 + 0.12 + 0.06 + 0.31 = 0.5$$

Let me calculate one more for you so

$$Pl(A \cup B) = m_{12}(A) + m_{12}(B) + m_{12}(A \cup B) + m_{12}(A \cup C) + m_{12}(B \cup C) + m_{12}(A \cup B \cup C) = 0.21 + 0.01 + 0.12 + 0.2 + 0.6 + 0.31 = 0.91$$

(Refer Slide Time: 20:50)

	m_{12}	Bel_{12}	Pl_{12}
Α	0.21	0.21	0.84
В	0.01	0.01	0.5
С	0.09	0.09	0.66
$A \cup B$	0.12	0.34	0.91
$A \cup C$	0.2	0.5	0.99
$B \cup C$	0.06	0.16	0.79
$A \cup B \cup C$	0.31	1.0	1.0

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For any set PEP(x) is called a "focal element m if m(P) because the non-zero Probabilition are focal elements values called

Now, let me give you some definitions

For any set $P \in P(X)$, P is called a *focal element* of m if m(P) > 0

This is because the nonzero basic probabilities are our concern.

Therefore we will focus only on those for which m(P) > 0

Another definition is this

The set of focal elements along with their m values together is called *Body of evidence*. And we denoted it as (\mathfrak{F}, m) . where \mathfrak{F} is the set of all *focal elements* and m is the *basic probability assignment*.

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with the above definition are go for a repectal case of Evidence Theory. Suppose the "body of Evidence" is s.t the \$ focal elements are nested them the corresponding theory called Possibility Theory"

Now with the above definition we go for some special case of evidence theory.

Suppose the *body of evidence* is such that the focal elements are nested then the corresponding theory is called *Possibility Theory*.

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So let me first give an illustration, suppose we have n elements in X and the focal elements are:

 $\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\} \dots \{x_1, x_2 \dots x_n\}$

It is not mandatory that all the elements will have to be there. This is a simplest way of looking at it.

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X1 = 717273 = 717273 ×r This is also Nested sets. That also around give us " possibility Theory: NPTEL

One could have x_1 , x_1 x_2 x_3 then x_1 x_2 x_3 x_4 x_5 . So like that if these are the focal elements, then also we could have got nested sets and that also would have given us possibility theory.

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The nested cases has one advantage: Bel (ANB) = min (Bel (A), Bel (B)) = max (Bel (A), Bel (B))

The nested cases has one advantage that:

$$Bel(A \cap B) = \min(Bel(A), Bel(B))$$
$$Pl(A \cup B) = \max(Bel(A), Bel(B))$$

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This is very early to visualize in the example I have given Let ACB $Bel(A \cap B) = Bel(A)$ Bel(A) < Bel(B) $Bel(A \cap B) = min(Bel(A), Bel(B))$

This is very easy to visualize in the example that I have given.

Let
$$A \subset B$$

 $\therefore Bel(A \cap B) = Bel(A)$
 $\therefore A \cap B = A$
 $\therefore Bel(A) < Bel(B)$
 $\therefore Bel(A \cap B) = \min(Bel(A), Bel(B))$
(Refer Slide Time: 31:32)

We note prove the record for
the general case:
Let
$$Je = \{A_1, A_2, \dots, A_n\}$$

 \Rightarrow Each Ai is a focal element.
Abo as they are merted, we
can order them linearly.
i.e. $A_1 \subseteq A_2 \subseteq A_3 \dots \subseteq A_n$
i.e. $A_2 \subseteq A_3 \dots \subseteq A_n$
i.e. $A_2 \subseteq A_3 \dots \subseteq A_n$

We now prove the result for general case

Let $\mathfrak{F} = \{A_1, A_2 \dots A_n\}$ such that each A_i is a focal element also as they are nested, we can order them linearly.

 $i.e.A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \subseteq A_n$ $i.e.A_i \subseteq A_j \text{ if } i < j$

(Refer Slide Time: 33:06)

Consider two arbitrary sets $A, B \leq X.$ Let X = 21,2,3, - - 103 Suppose se in an follown: $A_{1} = \{1, 2, 3\}$ $A_{2} = \{1, 2, 3\}$ $A_{3} = \{1, 2, 3, 6\}$ $A_{3} = \{1, 2, 3, 6\}$ $A_{4} = \{1, 2, 3, 6, 7\}$ $A_{5} = \{1, 2, 3, 6, 7, 8\}$ $A_{5} = \{1, 2, 3, 6, 7, 8\}$ if ici.

Consider two arbitrary sets $A, B \subseteq X$

Let $X = \{1, 2, 3, ..., 10\}$ Suppose \mathfrak{F} is as follows: $A_1 = \{1\}$ $A_2 = \{1, 2, 3\}$ $A_3 = \{1, 2, 3, 6\}$ $A_4 = \{1, 2, 3, 6, 7\}$ $A_5 = \{1, 2, 3, 6, 7, 8\}$ $A_6 = \{1, 2, 3, 6, 7, 8, 10\}$ You can easily verify that $A_i \subseteq A_j$ if i < j(Refer Slide Time: 35:10)

Let us take two subsets A 2 B of X. Let 2, be the largest index > Az, C A 110 Let 22 be the largest index + Az C B ex: A = E1,2,3,43 B= {1,2,3,5,63

Let us take two subsets A and B of X.

Let i_1 be the largest index such that $A_{i_i} \subset A$

Similarly let i_2 be the largest index such that $A_{i_2} \subset B$.

So example:

 $A = \{1, 2, 3, 4\}$

$$B = \{1, 2, 3, 5, 6\}$$

Then if we compare then, I can see A_2 is the largest index such that $A_2 \subset A$ and therefore $i_1 = 2$ And A_3 is the largest index such that $A_3 \subset B$. Therefore $i_2 = 3$

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Note that $\forall j \leq i_1 A_j \leq A$ $z \forall j \leq i_2 A_j \leq B$ $\therefore A_j \leq A \cap B$ if $j \leq \min(i_1, i_2)$ Hence Bel(A \cap B) = $\sum_{i=1}^{\min(i_1, i_2)} m(A_i)$ $= \min\left(\sum_{i=1}^{2i} m(A_i), \sum_{i=1}^{2i} m(A_i)\right)$ = min (Bel (A), Bel (B)) (*)

Note that:

 $\forall j \leq i_1 \quad A_j \subseteq A$ $\& \forall j \leq i_2 \quad A_j \subseteq B$ $\therefore A_j \subseteq A \cap B \text{ if } j \leq \min(i_1, i_2)$ Hence $Bel(A \cap B) = \sum_{i=1}^{\min(i_1, i_2)} m(A_i)$ $= \min(\sum_{i=1}^{i_1} m(A_i), \sum_{i=1}^{i_2} m(A_i)) = \min(Bel(A), Bel(B))$ (Refer Slide Time: 39:12)



One very interesting point with respect to nested focal elements is that *Belief* is increasing.

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For illotrakin consider two element case: A, = 2, 2, 3 A2 = 2, , 2, 3 · Bel (A1) = m(x1) Bul (A2) = Bel {x, xe} $= m(x_{1}) + m(x_{2})^{-} + m(x_{1}, x_{2}) = m(x_{1}) + 0 + m(x_{1}, x_{2})$ = m(x_{1}) + 0 + m(x_{1}, x_{2}) = Bel(A_{1}) + 2 (E>0)

For illustration, consider two element case

$$A_{1} = \{x_{1}\} \text{ and } A_{2} = \{x_{1}, x_{2}\}$$

$$\therefore Bel(A_{1}) = m(x_{1})$$

$$Bel(A_{2}) = Bel(\{x_{1}, x_{2}\}) = m(x_{1}) + m(x_{2}) + m(\{x_{1}, x_{2}\}) = m(x_{1}) + 0 + m(\{x_{1}, x_{2}\}) > 0$$

$$\therefore Bel(A_{2}) = Bel(A_{1}) + \epsilon \quad (\epsilon > 0)$$

(Refer Slide Time: 41:32)

Therefore, we can see that as one element is added the Belief is increased. This is true if are keep on adding more elements, such that focal elements are nested Note: The above in Not true for a general case.

Therefore, we can see that as one element is added the belief is increased.

This is true if we keep on adding more elements such that focal elements are nested.

Note: The above is NOT TRUE for a general case.

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PI(AUB) = 1 - Bel(AUB) = I- Bel (A OB) = 1 - min (Bul (A), Bel (B)) = max (1 - Bx1(A) 1- Bx1(B)) = marx (PI(A), PI(B))

In a similar way

$$Pl(A \cup B) = 1 - Bel(\overline{A \cup B})$$

= 1 - Bel($\overline{A} \cap \overline{B}$)
= 1 - min(Bel(\overline{A}), Bel(\overline{B}))
= max(1 - Bel(\overline{A}), 1 - Bel(\overline{B}))
= max(Pl(A), Pl(B))

(Refer Slide Time: 44:16)

In the nested case Bel -> Necessity (denoted as Pl -> Possibility Nec) (denoted as Pos) Therefore are can arrite: Nec (ANB) = min (Nec (A), Nec (B) Ron(AUB) = max (Pon (A), Pon (B))

Since in the nested case

 $Bel \rightarrow Necessity (denoted as Nec)$

 $Pl \rightarrow Possibility (denoted as Pos)$

Therefore we can write

$$Nec(A \cap B) = \min(Nec(A), Nec(B))$$
$$Pos(A \cup B) = \max(Pos(A), Pos(B))$$

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Since we have abready seen 1) Bel (A) + Bel (A) < 1 => Nec (A) + Nec (A) 51 2) PI(A) + PI(A) 21 =) Pon(A) + Pon(A) > Por(A) = 1- Nec(A)

Since we have already seen in the last class

- 1. $Bel(A) + Bel(\bar{A}) \le 1 \Rightarrow Nec(A) + Nec(\bar{A}) \le 1$
- 2. $Pl(A) + Pl(\bar{A}) \ge 1 \Rightarrow Pos(A) + Pos(\bar{A}) \ge 1$
- 3. $Pos(A) = 1 Nec(\bar{A})$

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We may further observe () Min (Nec (A), Nec (A)) = O This is : min (Hec (A), Nec (A)) = Nec(ANA) = Nec(Q) (3) max (Pos(A), Pos(A)) This is because = 1 man(pon(A), pon(A)) = pon(AVA) Pon (x) = 1.

We may further observe

1. $Min(Nec(A), Nec(\bar{A})) = 0$

This is $: Min(Nec(A), Nec(\overline{A})) = Nec(A \cap \overline{A}) = Nec(\phi) = 0$

2. $Max(Pos(A), Pos(\bar{A})) = 1$

This is $: Max(Pos(A), Pos(\bar{A})) = Pos(A \cup \bar{A}) = Pos(X) = 1$

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Thus poppibility values in a neoted case of "Body of Evidence" E [0, 1] In that sense it close to Probability Theory. But one major difference is that : In probability we arright watter to each element in "possibility Theory"

So, thus possibility values in a nested case of *Body Of Evidence* \in [0, 1]

In that sense it is close to probability theory.

But one major difference is that: In probability we assign values to each element but here, in possibility theory.

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The values are arrighed to different subsets of the universe - i.e the set x. and that too they are nented.

The values are assigned to different subsets of the universe that is the set X and that too they are nested.

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Let me now conclude the talk with the following the name. (Nec (A) > O => Por (A) = 1 3 POR (A) <1 => Nec (A) =0 PS: 1) Since Min (Nec (A), Nec(A)) = 0 :: IF Nec(A) 70= 0 Nec(A) = 0= 1 - Nec(A) = 1 - 0 = 1

Let me now conclude the talk with the following theorems

- 1. $Nec(A) > 0 \Rightarrow Pos(A) = 1$
- 2. $Pos(A) < 1 \Rightarrow Nec(A) = 0$

Proof:

1. Since $Min(Nec(A), Nec(\bar{A})) = 0$ If $Nec(A) > 0 \Rightarrow Nec(\bar{A}) = 0$

$$\therefore Pos(A) = 1 - Nec(\overline{A}) = 1 - 0 = 1$$

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Pf 2: PORCA) < 1 => Mec(A)=0 We KNOW MOX (POR(A), POR(A)) = 1 : POR(A) < 1 => POR(A)=1 : Nec(A) = 1- POD(A)

2. $Pos(A) < 1 \Rightarrow Nec(A) = 0$ We know $Max(Pos(A), Pos(\bar{A})) = 1$ $\therefore Pos(A) < 1 \Rightarrow Pos(\bar{A}) = 1$ $\therefore Nec(A) = 1 - Pos(\bar{A}) = 1 - 1 = 0$

Ok friends with that I conclude this talk, so if we summarize what we have done. We have started with,

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Basic Definitions of FU338 sets 2 FUBBY memberalips. Barric Set operations on FUSIO sets, and examiner different properties of different Fuzzy set operators,

Basic definitions of fuzzy sets and fuzzy memberships.

Then we have looked at basic set operations on fuzzy sets, and examined different properties of different fuzzy set operators.

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FU33y Numburs Types of FUB3y mumbur. - How to carry out Arithmetic operation on FUBBY Numburg How to order Engs & sumpus How to approximate remits of Fuggy arith-metic with TFND.

Then we have seen fuzzy numbers

Types of fuzzy numbers.

How to carry out arithmetic operations on fuzzy numbers.

How to order fuzzy numbers.

How to approximate results of fuzzy arithmetic with triangular fuzzy numbers etcetera.

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We have also examined - Decomposition Theorems Extension Principle. FU33y-Logic Starks with propositional Logic - Multivalued logic - Logical operators Inforence up for FU33 y logical

We have also examined

different decompositions theorems and

the very important concept of extension principle.

Then we have studied fuzzy logic starting with propositional logic,

we studied multivalued logic,

different logical operators and

inferencing from fuzzy logical statements.

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Inforence - Different types statement modifiona un conditiona d also fer frm nen leas to them. practical Approally schem

In inference our focus has been on different types of logical statements that is conditional,

unconditional,

qualified,

unqualified

and also seen how to infer from them.

Then we have given a very practical approach namely Mamdani scheme.

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We have also shalled FU33y Relation Evidence Theory (Possibility And Finally

We have also studied Fuzzy's relations and finally the evidence theory or possibility theory.

It was a long journey of 30 lectures. But I hope these 30 lectures have given you a very strong foundation of how to deal with uncertainty using fuzzy mathematics fuzzy logic etc.

As I said in the very beginning that the concept of fuzzy came around the Year 1965. So over the last 50 years or so the theory has been extended and many new concepts have come. That I could not touch in the very introductory class namely say in,

Mathematics FU338 Car FU33Y Automata More advanced types of FU334 sels: FUSSY set of Type II, Intuitionistic FUSS y Iteritant FUSSY set

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Mathematics, people are talking about fuzzy complex numbers, fuzzy geometry. People also talk about inferencing fuzzy systems. There are many other methods of inferencing other than Mamdani in computer science people talk about fuzzy automata, more advanced type of fuzzy sets such as fuzzy set of type II, Intuitionistic fuzzy set, hesitant fuzzy set.

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Abo, in different Learning or in other applications are can find use of Fussy logic Fuzzy Control FUSSY Data Analysis Fuzzy chrotering FUBBY database, 2 generies

Also, in different learning or in other applications we can find use of fuzzy logic, such as fuzzy control, fuzzy data analysis, fuzzy clustering, fuzzy databases and queries etcetera.

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Abo . people talk about FUSSY OPENISAHA

Also people talk about fuzzy optimization, where constraints or resources are expressed in a fuzzy manner.

Like that there are plethora of developments in the fuzzy set theory it was not possible to complete all of them in a 30 lecture series. But again say that if you follow the thirty lectures you will have

a very strong foundation of how to deal with the uncertainty using fuzzy mathematics, fuzzy logic etc.

With that I conclude my series of lectures I wish you all the success in life.

Thank you very much.