Introduction to Fuzzy Sets Arithmetic & Logic
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Lecture- 30
Fuzzy Sets Arithmetic & Logic

Welcome students, to the MOOC'S course on fuzzy sets, arithmetic and logic. This is lecture number 30. If you recall in the last class I started with, Evidence theory.

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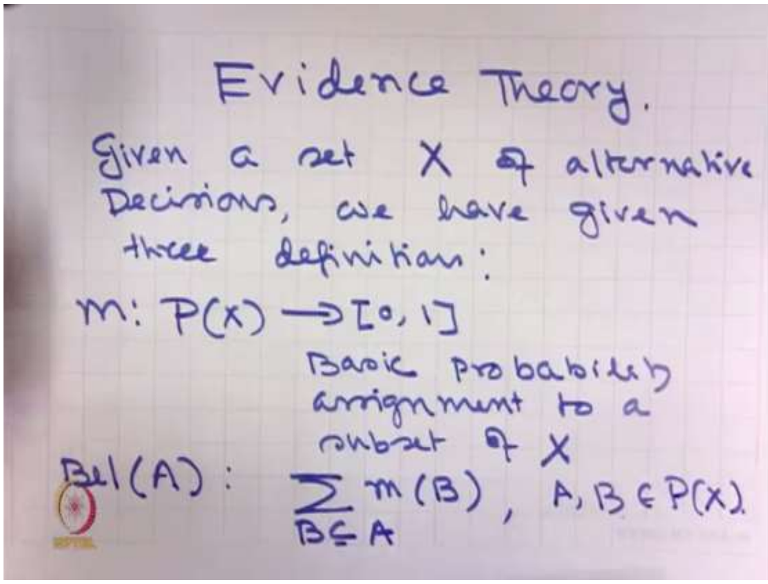

In fact, given a set X of alternative decisions we have given 3 definitions:

- $m: P(X) \rightarrow [0, 1]$ this is the $Basic\ Probability\ Assignment$ to a subset of X,
- $Bel(A) = \sum_{B \subseteq A} m(B)$ where $A, B \in P(X)$

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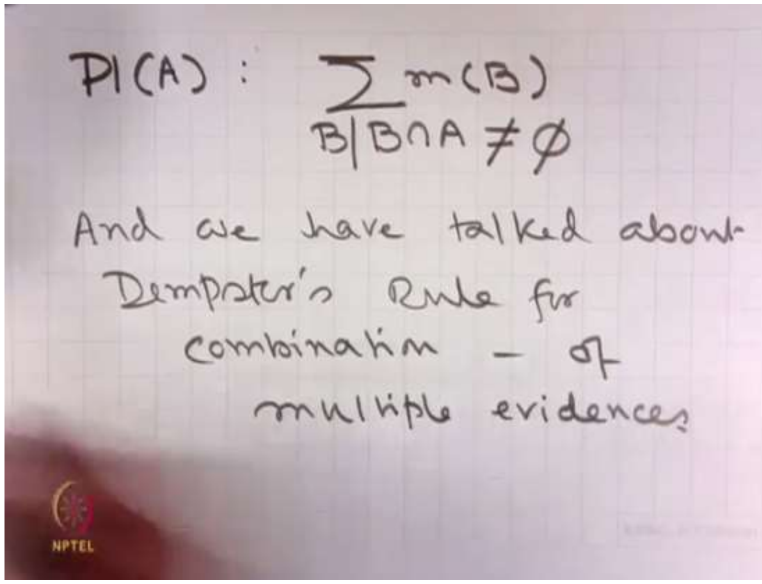

- $Pl(A) = \sum_{B|B\cap A\neq\phi} m(B)$
- And we have talked about *Dempster's Rule* for the combination of multiple evidences.

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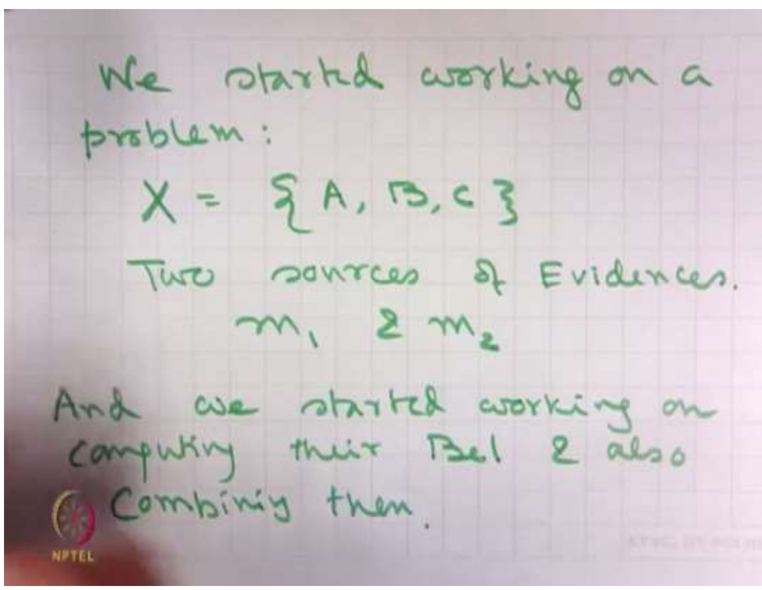

We started working on a problem such that

$$X = \{A, B, C\}$$

Two sources of evidences m_1 and m_2.

And we started working on computing, their *Belief* and also combining them.

In this respect, we have come to this table.

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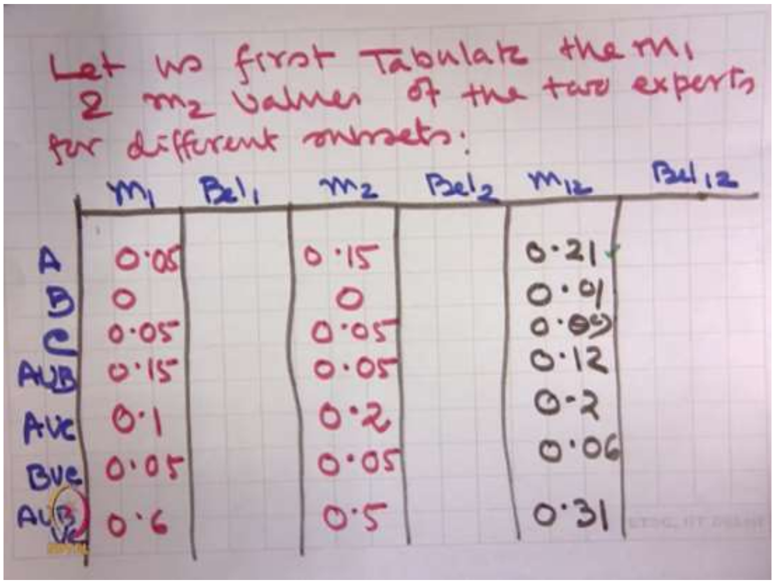

Where the m_1 and $m_2$are the evidences for different non empty subsets and using the formula we have calculated the m_{12} a combination of the evidences for different subsets of X.

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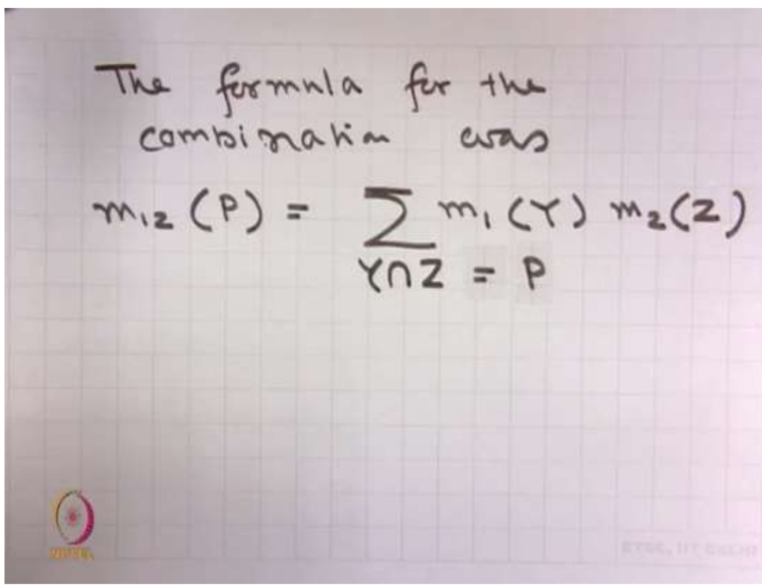

The formula for the combination was

$$m_{12}(P) = \frac{\sum_{Y\cap Z=P} m_1(Y) \times m_2(Z)}{1-K}$$

Where $K = \sum_{B\cap C=\phi} m_1(B) \times m_2(C)$

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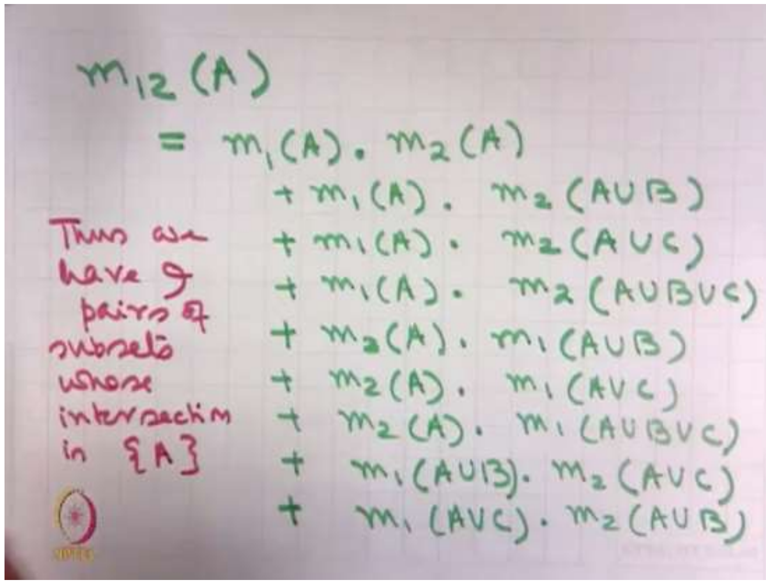

$$m_{12}(A) = \frac{\begin{pmatrix} m_1(A) \times m_2(A) + m_1(A) \times m_2(A \cup B) + m_1(A) \times m_2(A \cup C) \\ + m_1(A) \times m_2(A \cup B \cup C) + m_1(A \cup B) \times m_2(A) + m_1(A \cup C) \times m_2(A) \\ + m_1(A \cup B \cup C) \times m_2(A) + m_1(A \cup B) \times m_2(A \cup C) + m_1(A \cup C) \times m_2(A \cup B) \end{pmatrix}}{1 - K}$$

Thus we have 9 pairs of subsets whose intersection is $\{A\}$

In a similar way, you can compute these values and all other values in particular.

Let us now look at calculation of $Belief$, since, for each one of them, we can calculate the $Belief$ for illustration, I am just showing for one of them.

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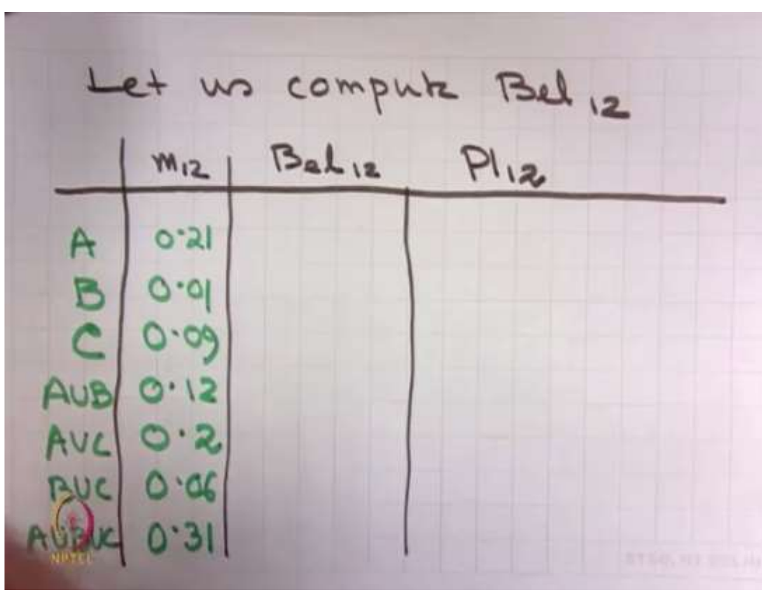

So, let us compute Bel_{12}

	m_{12}	Bel_{12}	Pl_{12}
A	0.21		
B	0.01		
C	0.09		
$A \cup B$	0.12		
$A \cup C$	0.2		
$B \cup C$	0.06		
$A \cup B \cup C$	0.31		

We are going to calculate Bel_{12} and Pl_{12}. How to compute them? .

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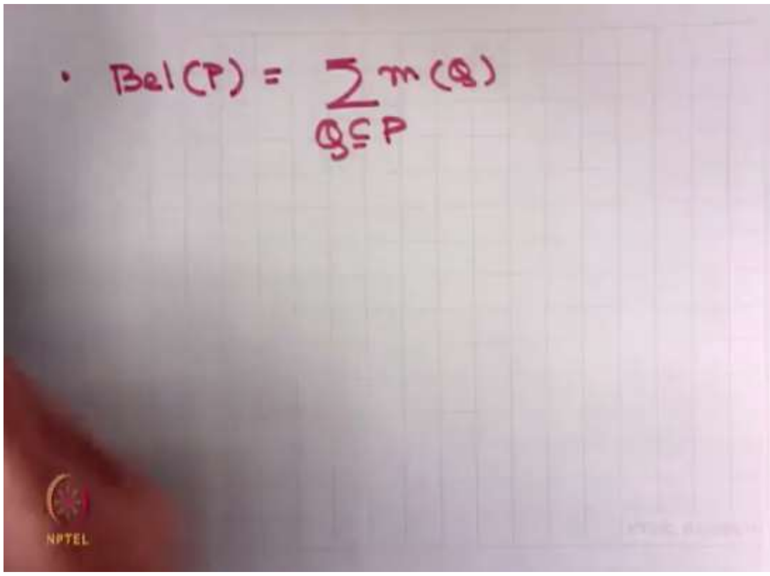

So far, you recall

- $Bel(P) = \sum_{Q \subseteq P} m(Q)$

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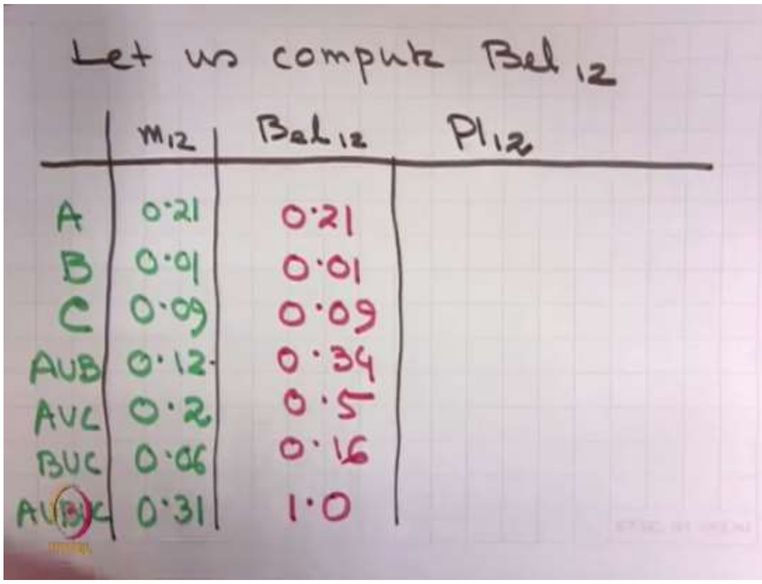

Therefore, what we will have

	m_{12}	Bel_{12}	Pl_{12}
A	0.21	0.21	
B	0.01	0.01	
C	0.09	0.09	
$A \cup B$	0.12	0.34	
$A \cup C$	0.2	0.5	
$B \cup C$	0.06	0.16	
$A \cup B \cup C$	0.31	1.0	

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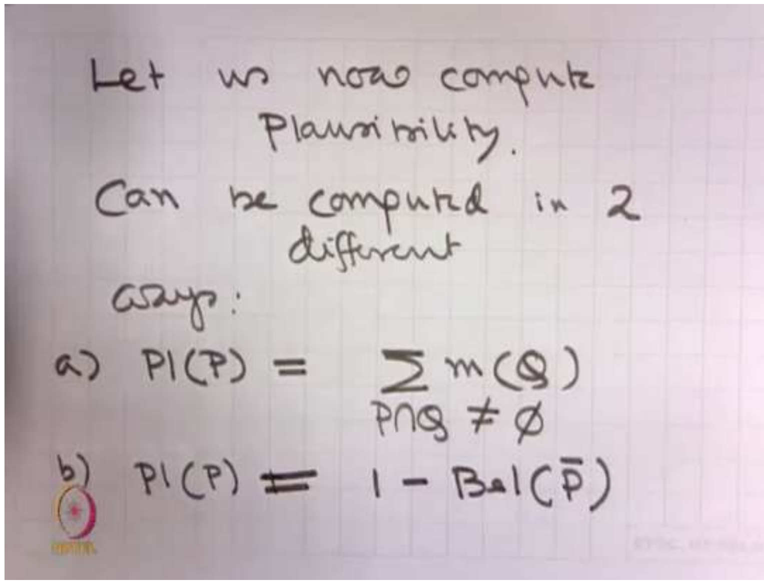

Let us now come to $Plausibility$, can be computed in two different ways:

- $Pl(P) = \sum_{Q|P \cap Q \neq \phi} m(Q)$
- $Pl(P) = 1 - Bel(\bar{P})$

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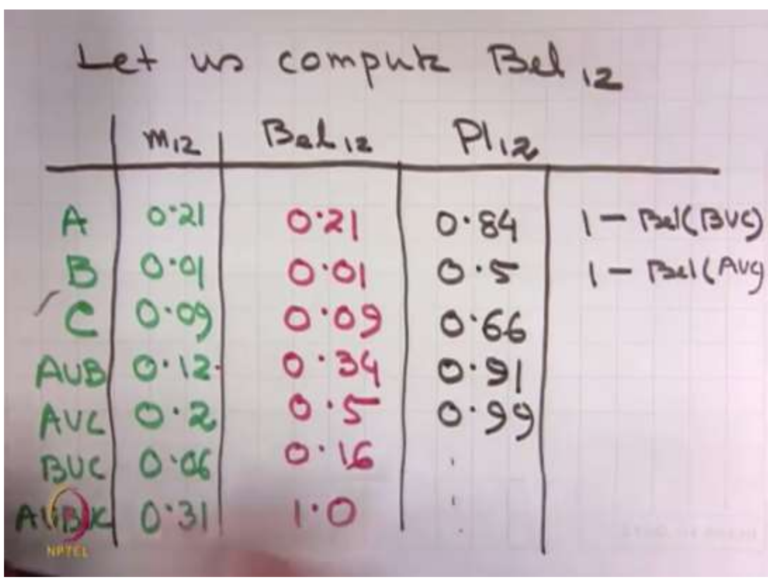

Therefore,

	m_{12}	Bel_{12}	Pl_{12}	
A	0.21	0.21	0.84	$1 - Bel_{12}(B \cup C)$
B	0.01	0.01	0.5	

C	0.09	0.09	0.66	$1 - Bel_{12}(A \cup C)$
$A \cup B$	0.12	0.34	0.91	
$A \cup C$	0.2	0.5	0.99	
$B \cup C$	0.06	0.16	0.79	
$A \cup B \cup C$	0.31	1.0	1.0	

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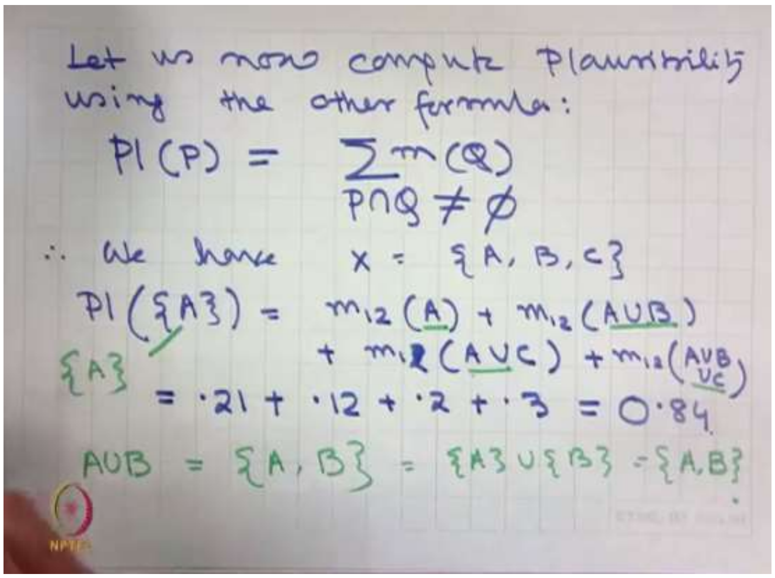

So, let us now compute $Plausibility$ using the other formula and that formula is

$$Pl(P) = \sum_{Q|P \cap Q \neq \phi} m(Q)$$

Therefore, we have $X = \{A, B, C\}$

Therefore,

$$Pl_{12}(\{A\}) = m_{12}(A) + m_{12}(A \cup B) + m_{12}(A \cup C) + m_{12}(A \cup B \cup \ C)$$
$$= 0.21 + 0.12 + 0.2 + 0.3 = 0.84$$

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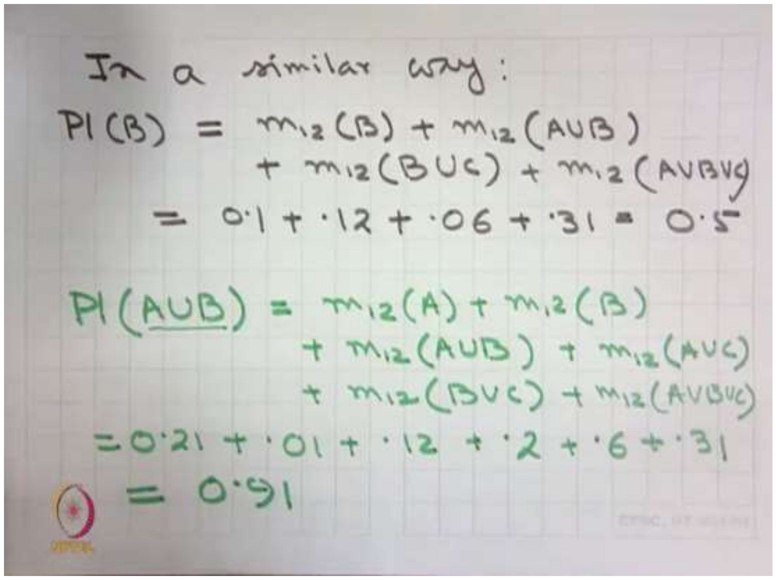

In a similar way,

$$Pl(B) = m_{12}(B) + m_{12}(A \cup B) + m_{12}(B \cup C) + m_{12}(A \cup B \cup C)$$
$$= 0.01 + 0.12 + 0.06 + 0.31 = 0.5$$

Let me calculate one more for you so

$$Pl(A \cup B) = m_{12}(A) + m_{12}(B) + m_{12}(A \cup B) + m_{12}(A \cup C) + m_{12}(B \cup C)$$
$$+ m_{12}(A \cup B \cup C) = 0.21 + 0.01 + 0.12 + 0.2 + 0.6 + 0.31 = 0.91$$

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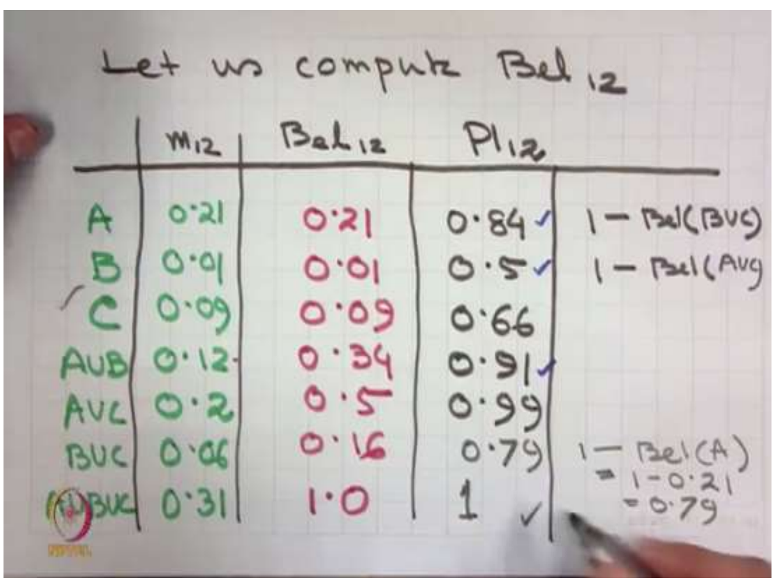

	m_{12}	Bel_{12}	Pl_{12}
A	0.21	0.21	0.84
B	0.01	0.01	0.5
C	0.09	0.09	0.66
$A \cup B$	0.12	0.34	0.91
$A \cup C$	0.2	0.5	0.99
$B \cup C$	0.06	0.16	0.79
$A \cup B \cup C$	0.31	1.0	1.0

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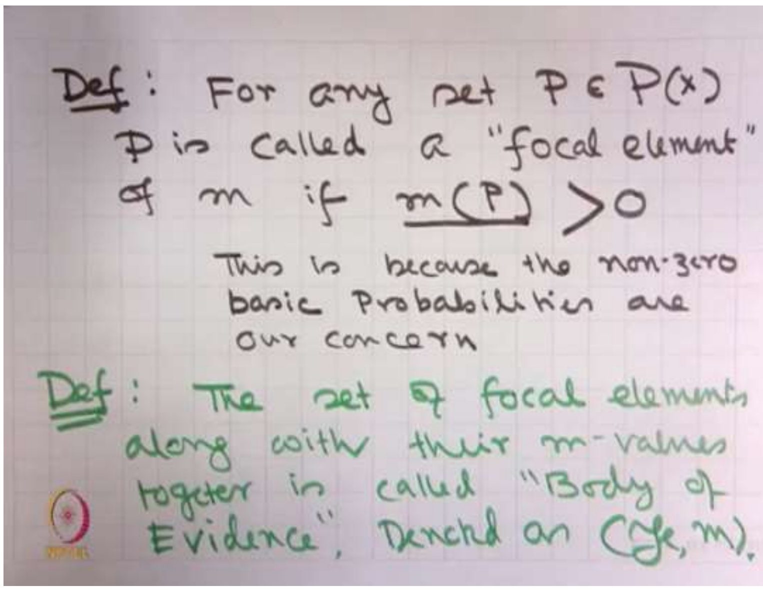

Now, let me give you some definitions

For any set $P \in P(X)$, P is called a $focal\ element$ of m if $m(P) > 0$

This is because the nonzero basic probabilities are our concern.

Therefore we will focus only on those for which $m(P) > 0$

Another definition is this

The set of focal elements along with their m values together is called $Body\ of\ evidence$. And we denoted it as $(\mathfrak{F}, m)$. where $\mathfrak{F}$ is the set of all $focal\,elements$ and m is the $basic\,probability\,assignment$.

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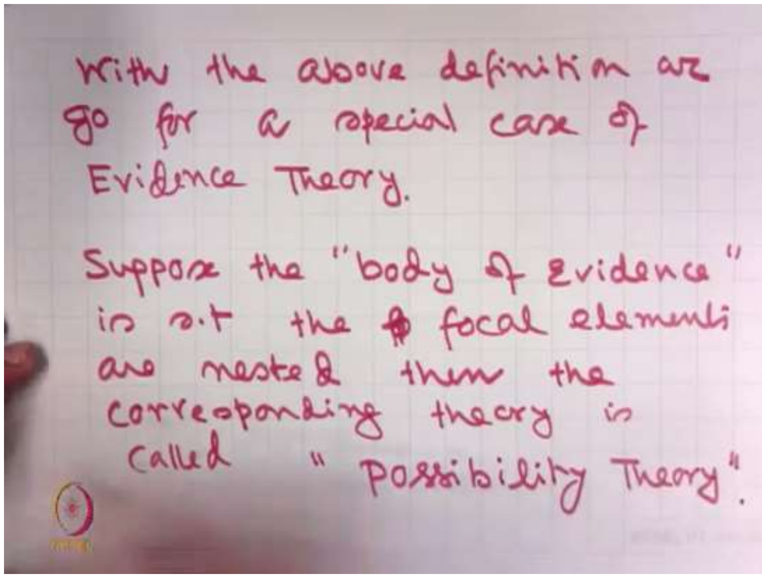

Now with the above definition we go for some special case of evidence theory.

Suppose the *body of evidence* is such that the focal elements are nested then the corresponding theory is called *Possibility Theory*.

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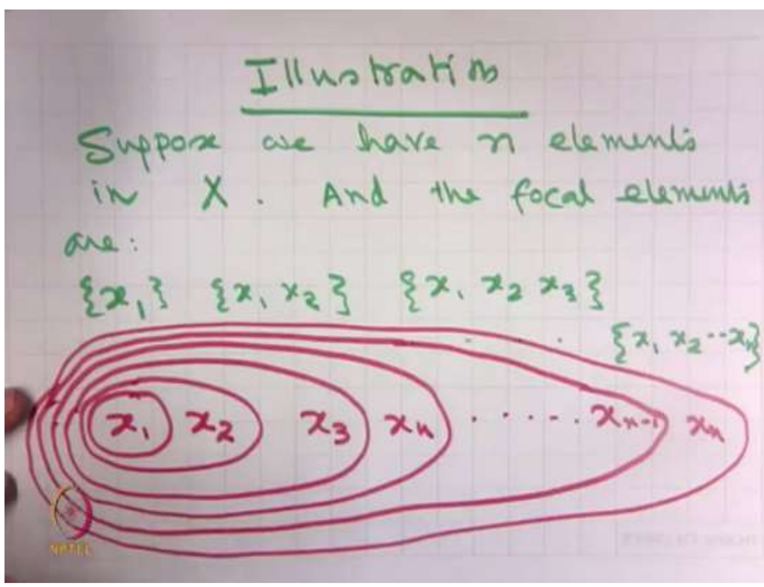

So let me first give an illustration, suppose we have n elements in X and the focal elements are:

$$\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\} \dots \{x_1, x_2 \dots x_n\}$$

It is not mandatory that all the elements will have to be there. This is a simplest way of looking at it.

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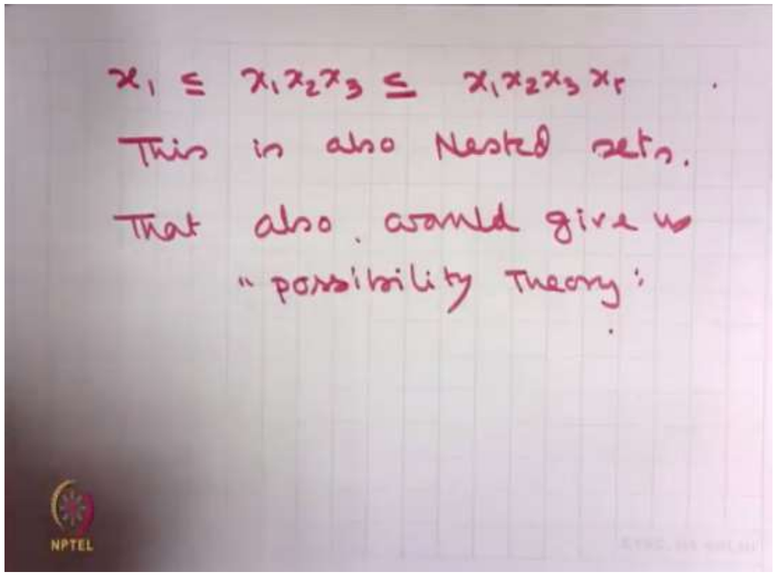

One could have x1, x1 x2 x3 then x1 x2 x3 x4 x5. So like that if these are the focal elements, then also we could have got nested sets and that also would have given us possibility theory.

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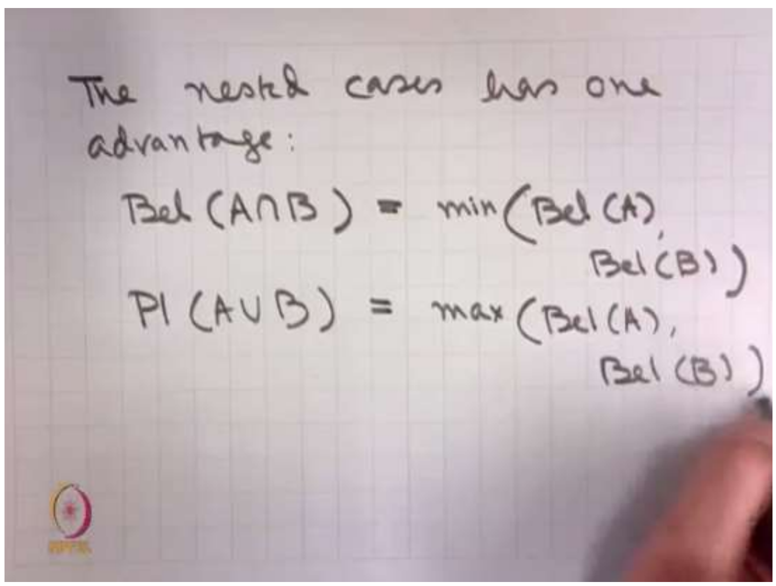

The nested cases has one advantage that:

$$Bel(A \cap B) = \min\big(Bel(A), Bel(B)\big)$$

$$Pl(A \cup B) = \max\big(Bel(A), Bel(B)\big)$$

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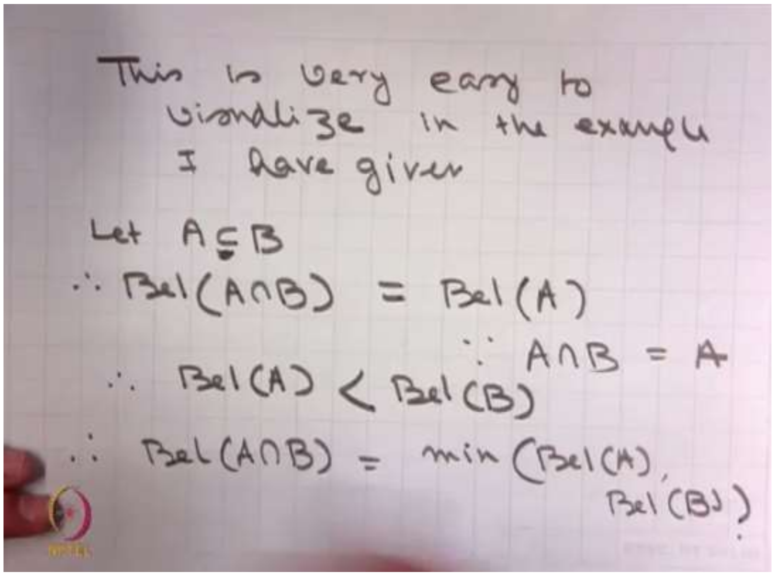

This is very easy to visualize in the example that I have given.

Let $A \subset B$

$\therefore Bel(A \cap B) = Bel(A)$

$\because A \cap B = A$

$\therefore Bel(A) < Bel(B)$

$\therefore Bel(A \cap B) = \min\big(Bel(A), Bel(B)\big)$

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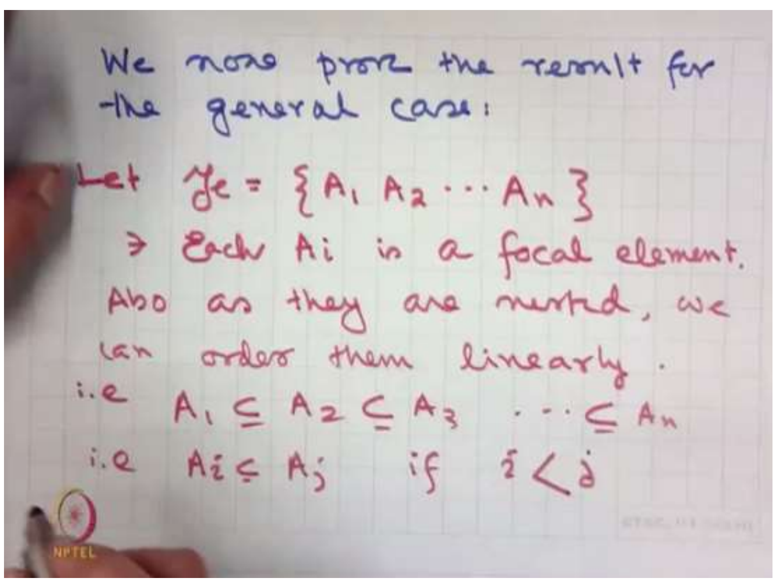

We now prove the result for general case

Let $\mathfrak{F} = \{A_1, A_2 \dots A_n\}$ such that each A_i is a focal element also as they are nested, we can order them linearly.

$i.e. A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n$

$i.e. A_i \subseteq A_j$ if $i < j$

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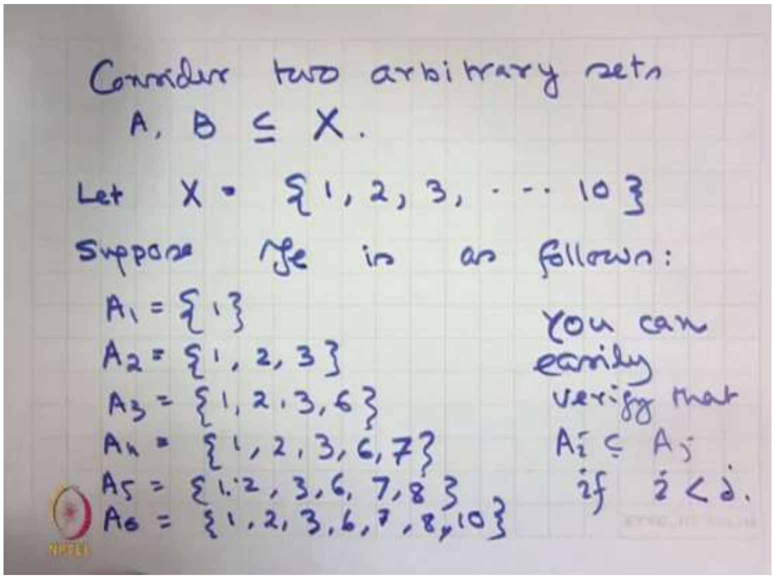

Consider two arbitrary sets $A, B \subseteq X$

Let $X = \{1, 2, 3, \ldots, 10\}$

Suppose $\mathfrak{F}$ is as follows:

$A_1 = \{1\}$

$A_2 = \{1, 2, 3\}$

$A_3 = \{1, 2, 3, 6\}$

$A_4 = \{1, 2, 3, 6, 7\}$

$A_5 = \{1, 2, 3, 6, 7, 8\}$

$A_6 = \{1, 2, 3, 6, 7, 8, 10\}$

You can easily verify that $A_i \subseteq A_j$ if $i < j$

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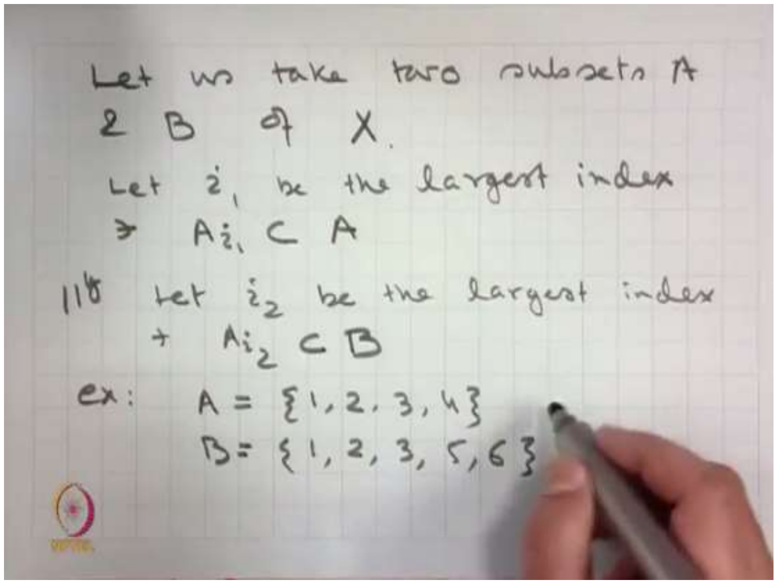

Let us take two subsets A and B of X.

Let i_1 be the largest index such that $A_{i_i} \subset A$

Similarly let i_2 be the largest index such that $A_{i_2} \subset B$.

So example:

$A = \{1, 2, 3, 4\}$

$B = \{1, 2, 3, 5, 6\}$

Then if we compare then, I can see A_2 is the largest index such that $A_2 \subset A$ and therefore $i_1 = 2$

And A_3 is the largest index such that $A_3 \subset B$. Therefore $i_2 = 3$

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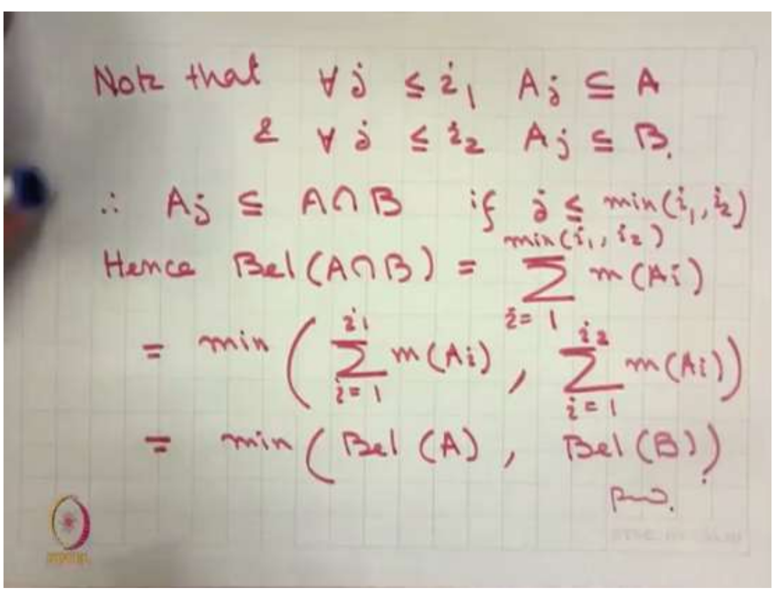

Note that:

$\forall\ j \le i_1\ \ A_j \subseteq A$

$\&\ \forall\ j \le i_2\ \ A_j \subseteq B$

$\therefore A_j \subseteq A \cap B$ if $j \le \min(i_1, i_2)$

Hence $Bel(A \cap B) = \sum_{i=1}^{\min(i_1,i_2)} m(A_i)$

$= \min\left(\sum_{i=1}^{i_1} m(A_i), \sum_{i=1}^{i_2} m(A_i)\right) = \min\left(Bel(A), Bel(B)\right)$

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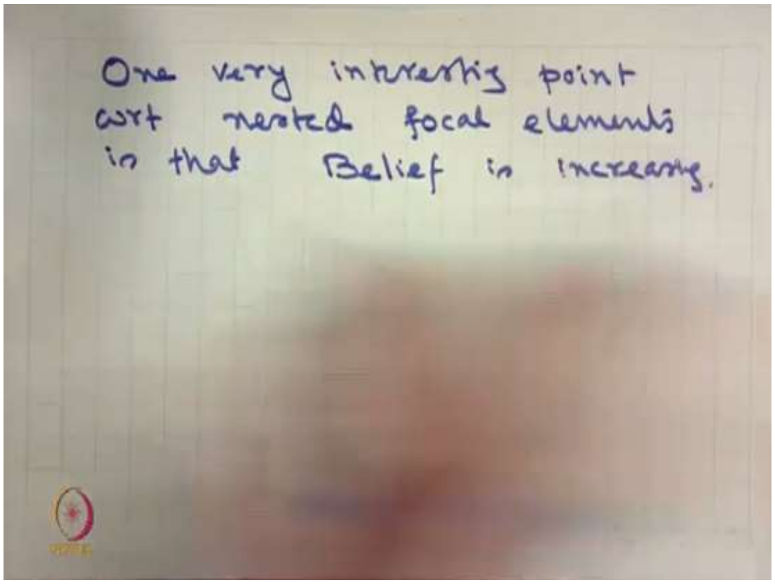

One very interesting point with respect to nested focal elements is that $Belief$ is increasing.

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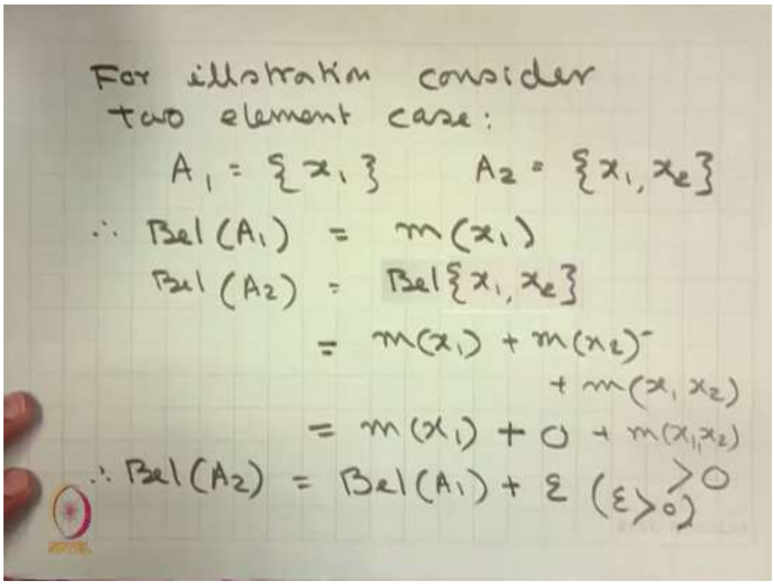

For illustration, consider two element case

$A_1 = \{x_1\}$ and $A_2 = \{x_1, x_2\}$

$\therefore Bel(A_1) = m(x_1)$

$Bel(A_2) = Bel(\{x_1, x_2\}) = m(x_1) + m(x_2) + m(\{x_1, x_2\}) = m(x_1) + 0 + m(\{x_1, x_2\}) > 0$

$\therefore Bel(A_2) = Bel(A_1) + \epsilon \;\; (\epsilon > 0)$

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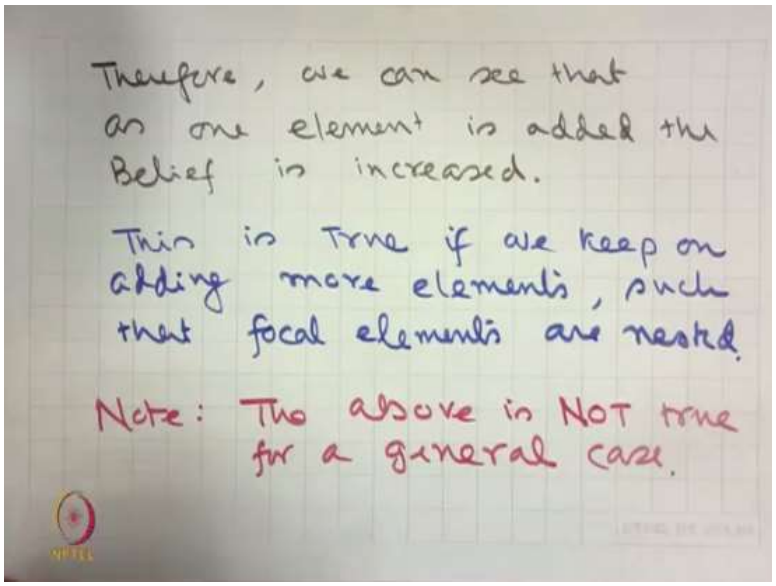

Therefore, we can see that as one element is added the belief is increased.

This is true if we keep on adding more elements such that focal elements are nested.

Note: The above is NOT TRUE for a general case.

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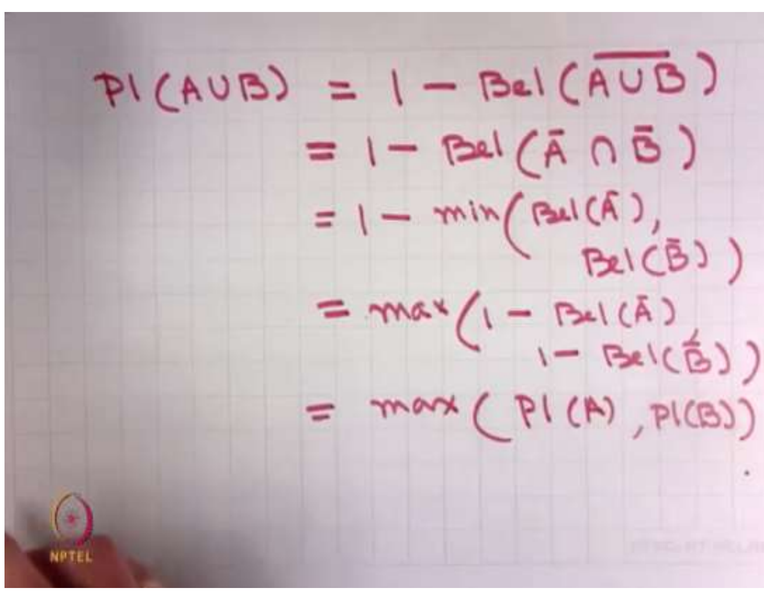

In a similar way

$$Pl(A \cup B) = 1 - Bel(\overline{A \cup B})$$
$$= 1 - Bel(\bar{A} \cap \bar{B})$$
$$= 1 - \min(Bel(\bar{A}), Bel(\bar{B}))$$
$$= \max(1 - Bel(\bar{A}), 1 - Bel(\bar{B}))$$
$$= \max\big(Pl(A), Pl(B)\big)$$

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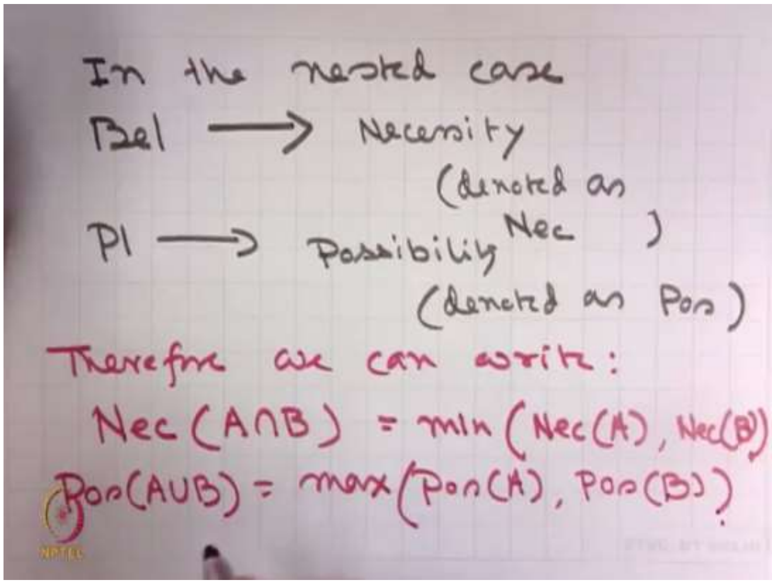

Since in the nested case

$Bel \rightarrow Necessity$ (denoted as Nec)

$Pl \rightarrow Posssibility$ (denoted as Pos)

Therefore we can write

$$Nec(A \cap B) = \min\big(Nec(A), Nec(B)\big)$$
$$Pos(A \cup B) = \max\big(Pos(A), Pos(B)\big)$$

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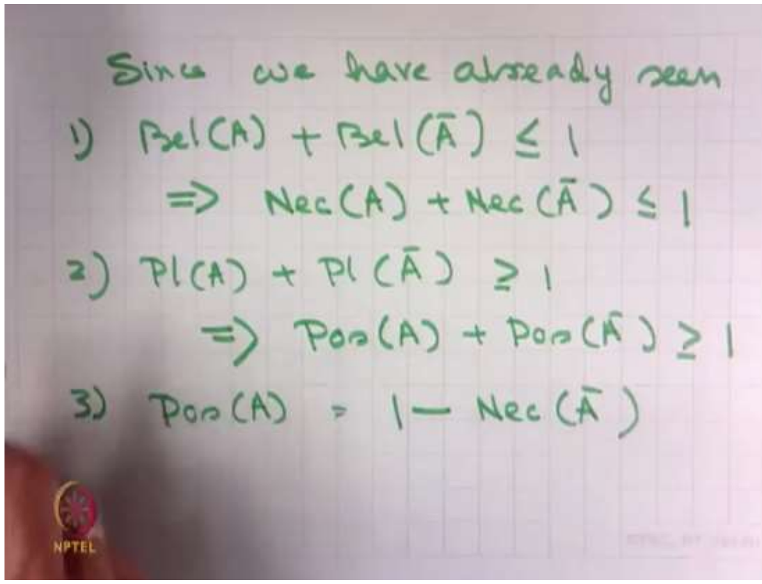

Since we have already seen in the last class

1. $Bel(A) + Bel(\bar{A}) \leq 1 \Rightarrow Nec(A) + Nec(\bar{A}) \leq 1$
2. $Pl(A) + Pl(\bar{A}) \geq 1 \Rightarrow Pos(A) + Pos(\bar{A}) \geq 1$
3. $Pos(A) = 1 - Nec(\bar{A})$

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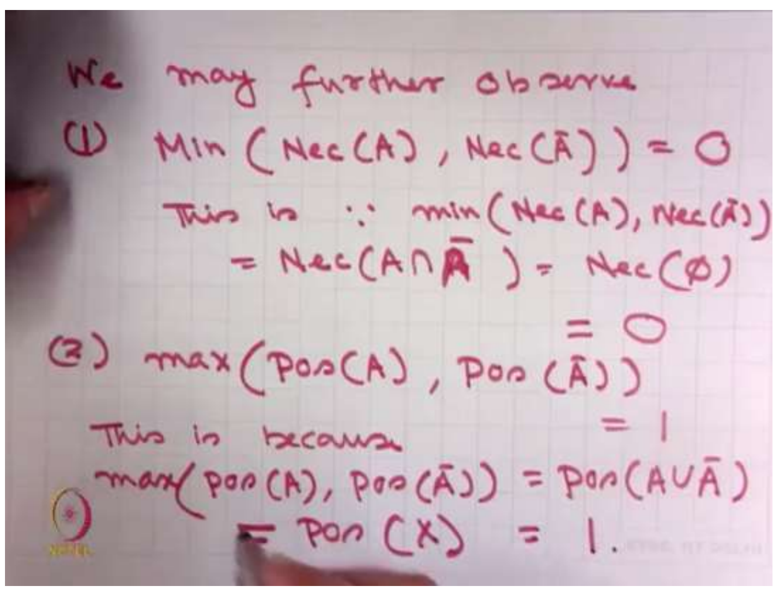

We may further observe

1. $Min\big(Nec(A), Nec(\bar{A})\big) = 0$

   This is $\because Min\big(Nec(A), Nec(\bar{A})\big) = Nec(A \cap \bar{A}) = Nec(\phi) = 0$
2. $Max\big(Pos(A), Pos(\bar{A})\big) = 1$

This is $\because Max(Pos(A), Pos(\bar{A})) = Pos(A \cup \bar{A}) = Pos(X) = 1$

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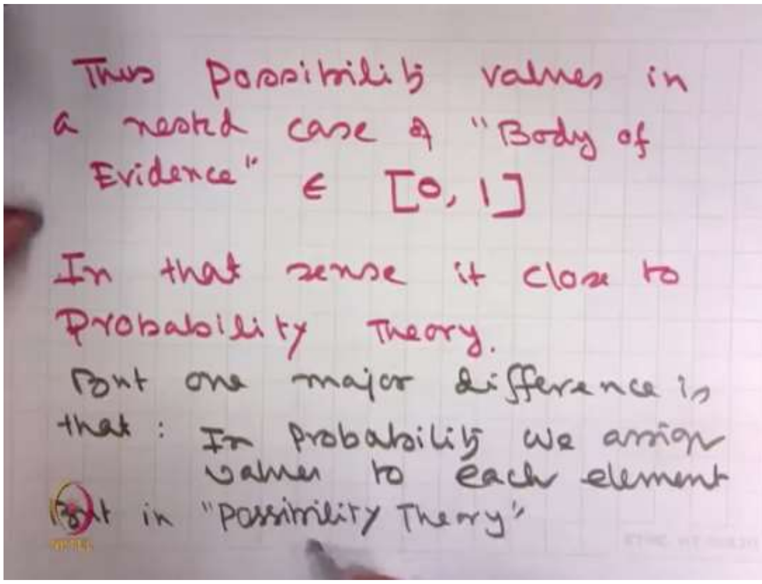

So, thus possibility values in a nested case of $Body\ Of\ Evidence \in [0,\ 1]$

In that sense it is close to probability theory.

But one major difference is that: In probability we assign values to each element but here, in possibility theory.

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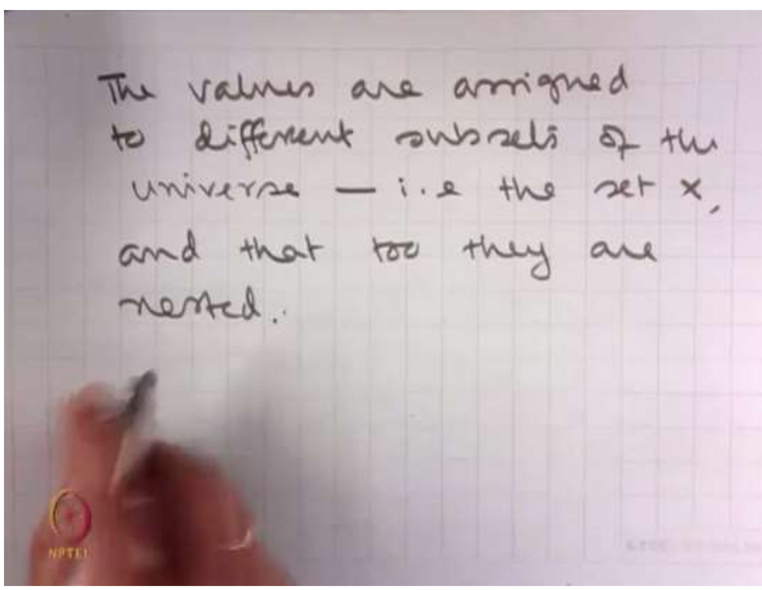

The values are assigned to different subsets of the universe that is the set X and that too they are nested.

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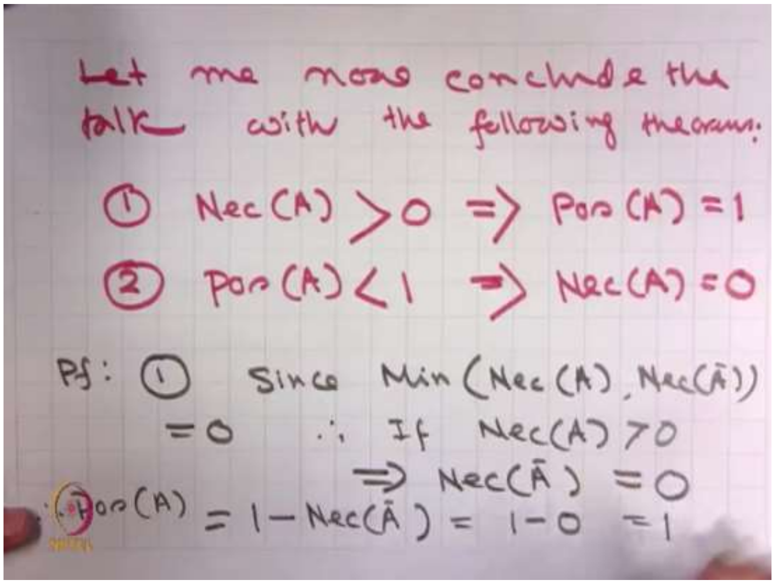

Let me now conclude the talk with the following theorems

1. $Nec(A) > 0 \Rightarrow Pos(A) = 1$
2. $Pos(A) < 1 \Rightarrow Nec(A) = 0$

Proof :

1. Since $Min\big(Nec(A), Nec(\bar{A})\big) = 0$

   If $Nec(A) > 0 \Rightarrow Nec(\bar{A}) = 0$

   $\therefore Pos(A) = 1 - Nec(\bar{A}) = 1 - 0 = 1$

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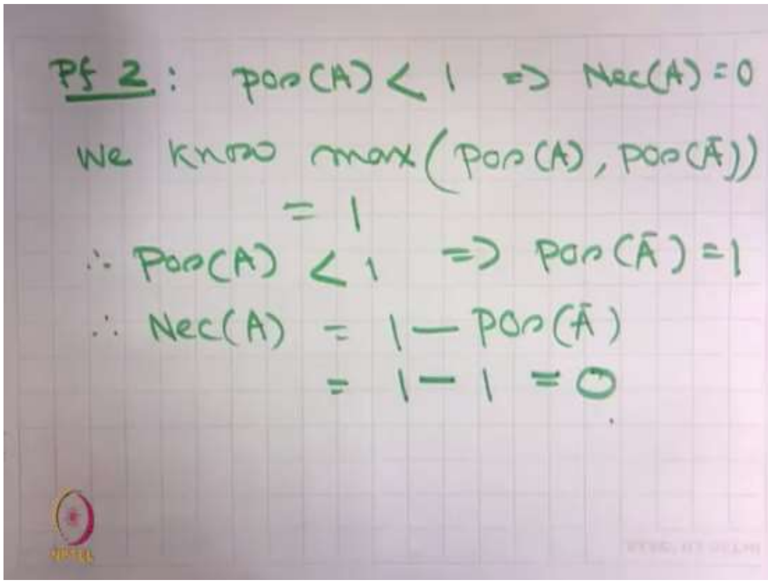

2. $Pos(A) < 1 \Rightarrow Nec(A) = 0$

We know $Max\big(Pos(A), Pos(\bar{A})\big) = 1$

$\therefore Pos(A) < 1 \Rightarrow Pos(\bar{A}) = 1$

$\therefore Nec(A) = 1 - Pos(\bar{A}) = 1 - 1 = 0$

Ok friends with that I conclude this talk, so if we summarize what we have done.

We have started with,

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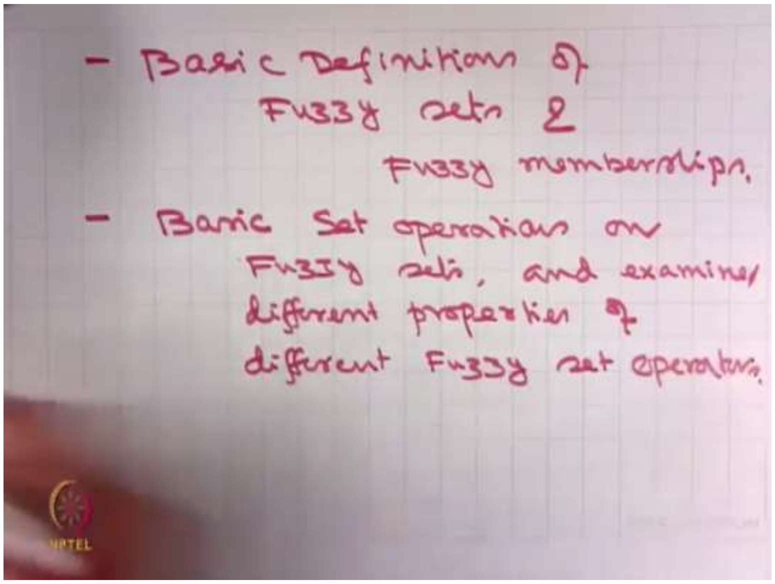

Basic definitions of fuzzy sets and fuzzy memberships.

Then we have looked at basic set operations on fuzzy sets, and examined different properties of different fuzzy set operators.

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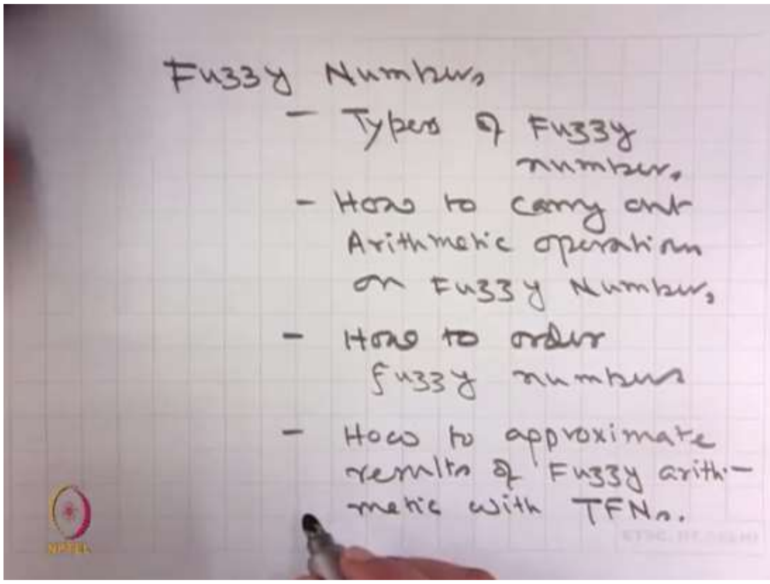

Then we have seen fuzzy numbers

Types of fuzzy numbers.

How to carry out arithmetic operations on fuzzy numbers.

How to order fuzzy numbers.

How to approximate results of fuzzy arithmetic with triangular fuzzy numbers etcetera.

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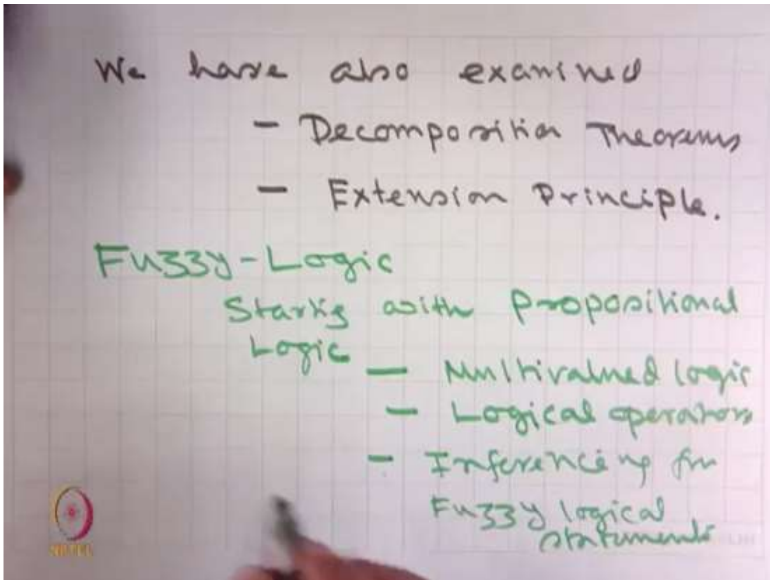

We have also examined

different decompositions theorems and

the very important concept of extension principle.

Then we have studied fuzzy logic starting with propositional logic,

we studied multivalued logic,

different logical operators and

inferencing from fuzzy logical statements.

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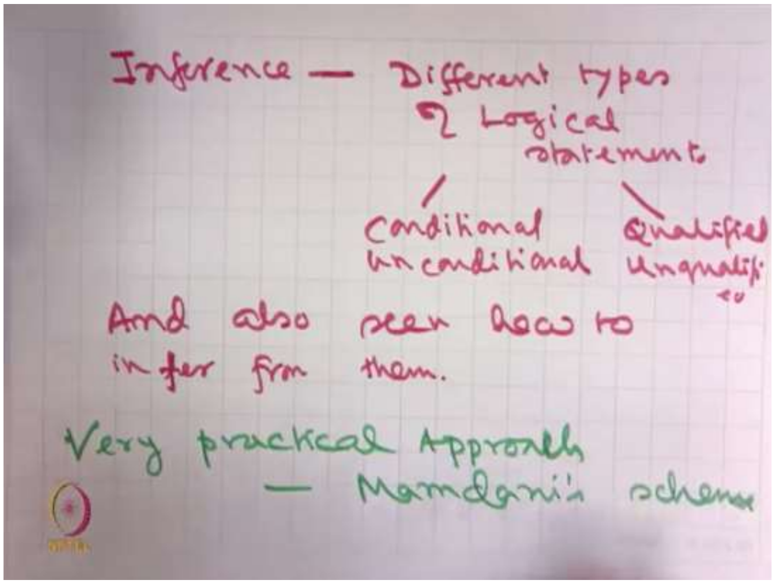

In inference our focus has been on different types of logical statements that is

conditional,

unconditional,

qualified,

unqualified

and also seen how to infer from them.

Then we have given a very practical approach namely Mamdani scheme.

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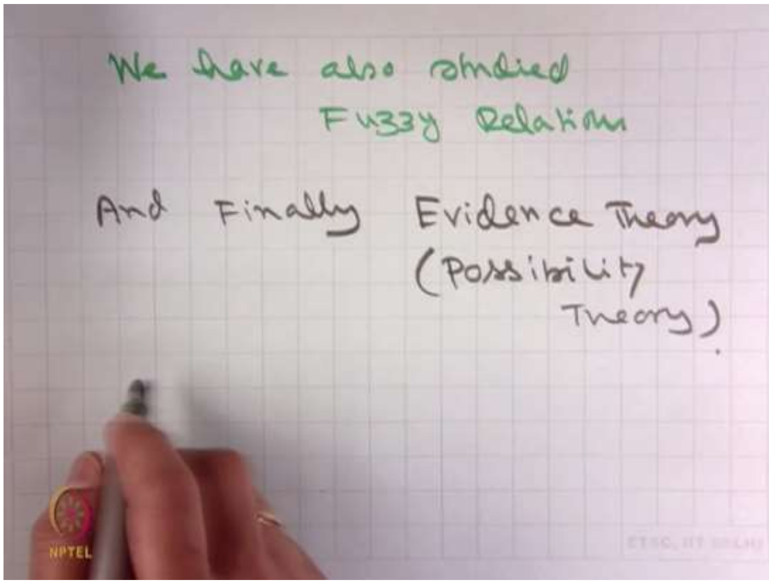

We have also studied Fuzzy's relations and finally the evidence theory or possibility theory.

It was a long journey of 30 lectures. But I hope these 30 lectures have given you a very strong foundation of how to deal with uncertainty using fuzzy mathematics fuzzy logic etc.
As I said in the very beginning that the concept of fuzzy came around the Year 1965. So over the last 50 years or so the theory has been extended and many new concepts have come. That I could not touch in the very introductory class namely say in,

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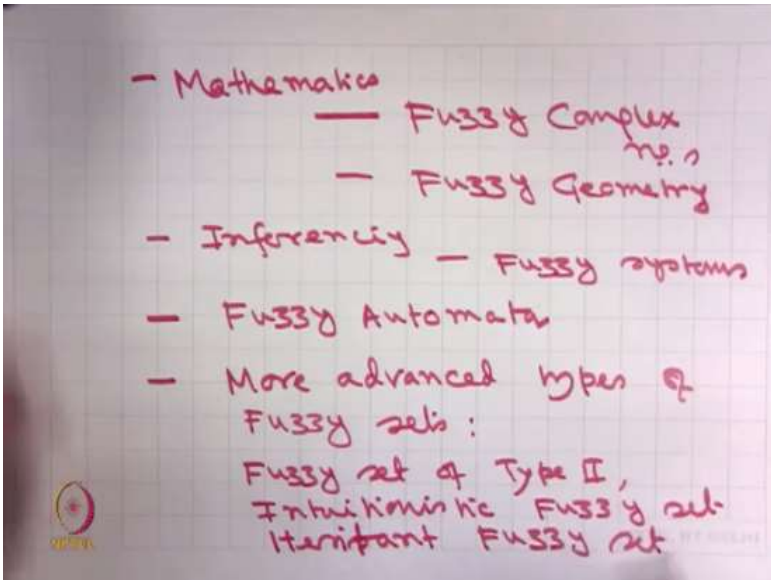

Mathematics, people are talking about fuzzy complex numbers, fuzzy geometry. People also talk about inferencing fuzzy systems. There are many other methods of inferencing other than

Mamdani in computer science people talk about fuzzy automata, more advanced type of fuzzy sets such as fuzzy set of type II, Intuitionistic fuzzy set, hesitant fuzzy set.

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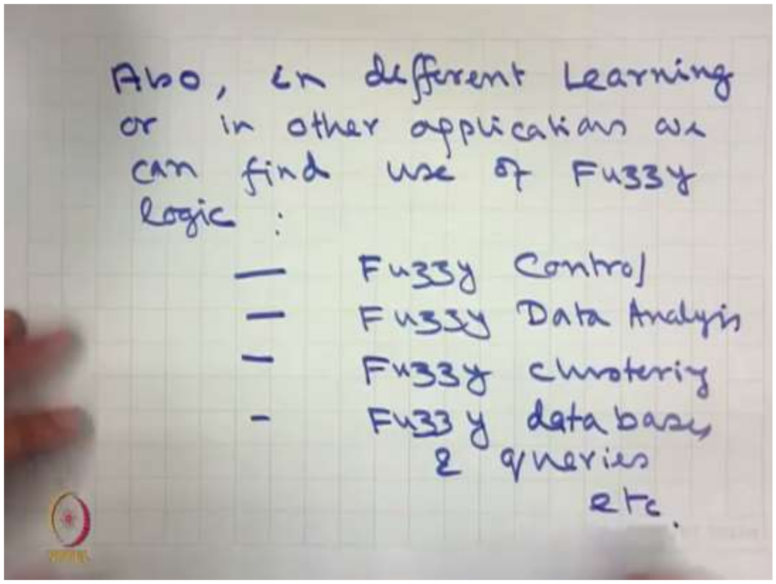

Also, in different learning or in other applications we can find use of fuzzy logic, such as fuzzy control, fuzzy data analysis, fuzzy clustering, fuzzy databases and queries etcetera.

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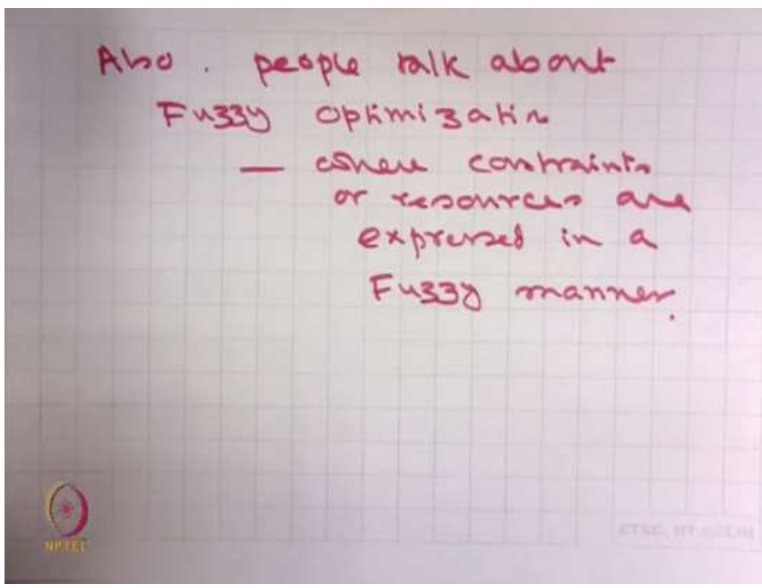

Also people talk about fuzzy optimization, where constraints or resources are expresssed in a fuzzy manner.

Like that there are plethora of developments in the fuzzy set theory it was not possible to complete all of them in a 30 lecture series. But again say that if you follow the thirty lectures you will have a very strong foundation of how to deal with the uncertainty using fuzzy mathematics, fuzzy logic etc.

With that I conclude my series of lectures I wish you all the success in life.

Thank you very much.