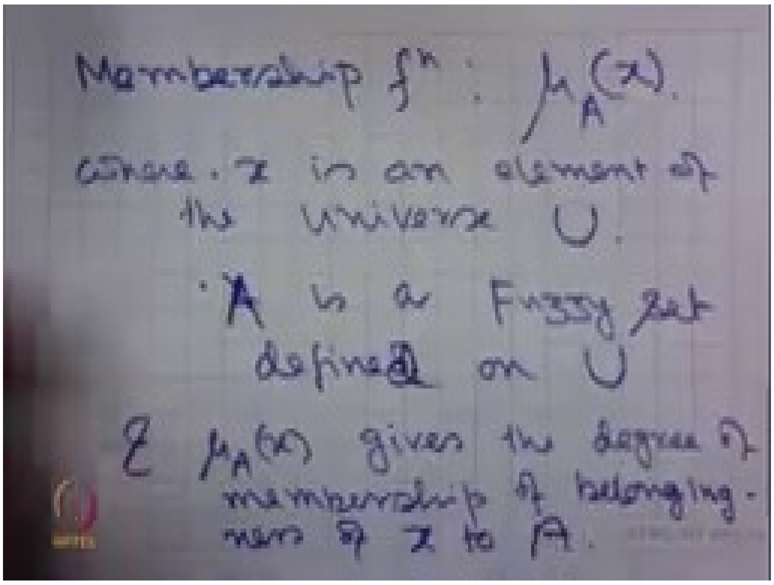
Introduction to Fuzzy Set Theory, Arithmetic and Logic

Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology- Delhi

Lecture 03
Fuzzy Sets Arithmetic and Logic

Welcome students to the third lecture on introduction to Fuzzy sets, Arithmetic and Logic. In the last class we have introduced that concept of membership functions.

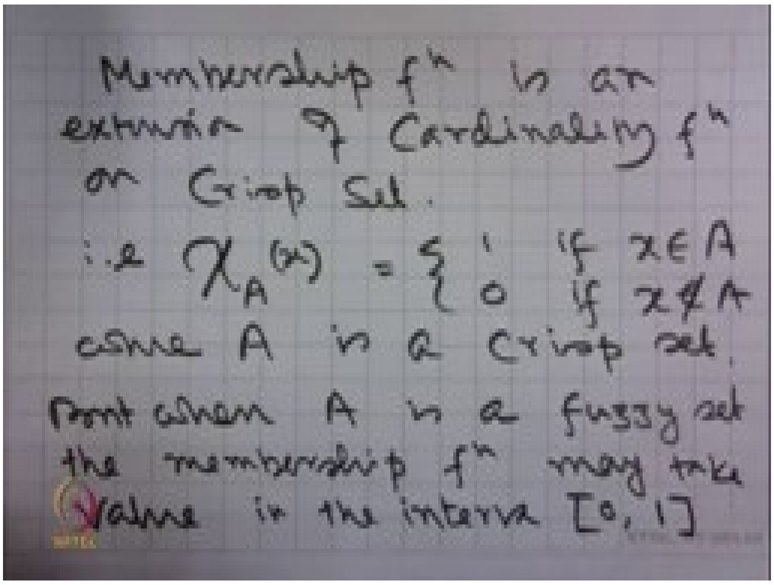
(Refer Slide Time: 00:34)



So, which we have denoted like this: where is an element of the universe .

is a fuzzy set defined on and gives the degree of membership of belongingness of to .

(Refer Slide Time: 01:55)

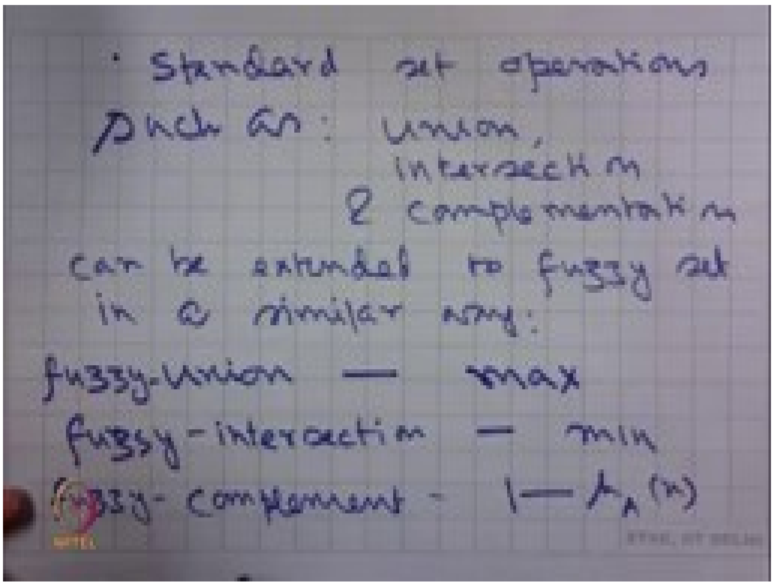


We have seen that this membership function is an extension of cardinality function on crisp set. That is, cardinality of in a set ,

where is a crisp set.

But when is a fuzzy set the degree of belongingness or the membership function may take value in the interval [0,1]. That is why, I say it is an extension from the binary set (it has now) the range of the membership has gone to the interval

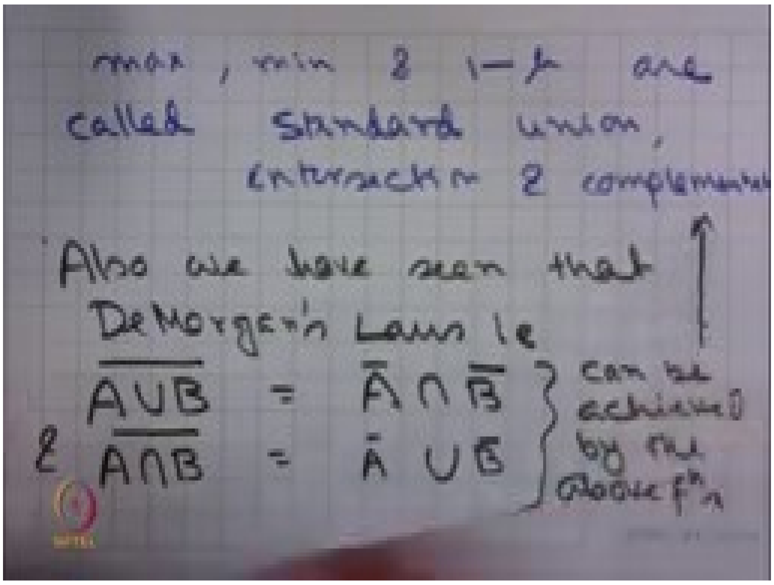
(Refer Slide Time: 03:41)



We have also seen that standards set operations such as Union, Intersection and Complementation can be extended to fuzzy set in a similar way. Union or Fuzzy Union can be implemented with a function, fuzzy intersection can be done using function and fuzzy complement using function.

Of course, these are not the only way of achieving Union, intersection or complementation. Over the years, people have developed different Union, Intersection and complement function which we shall study in some of my subsequent lectures.

(Refer Slide Time: 05:40)



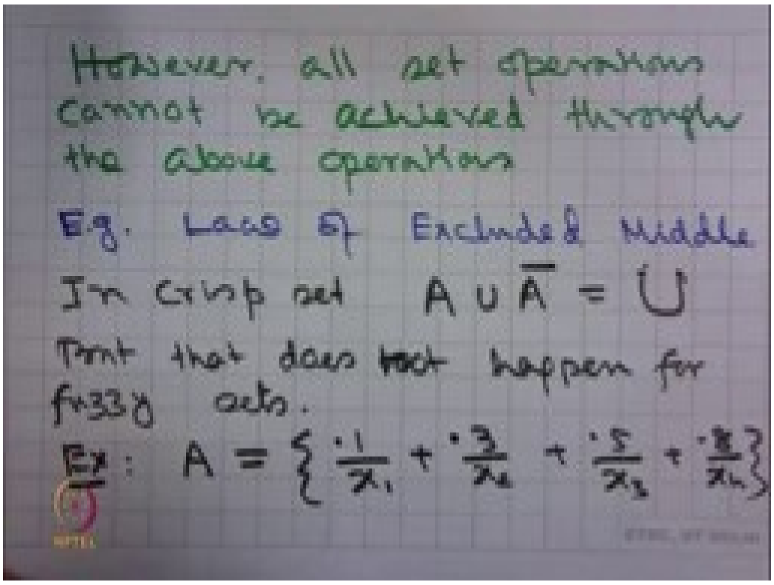
So, these and complementation and say are called standard Union, Intersection and complementation.

Also, we have seen that De Morgan's law, that is

- , and
-

can be achieved by the above functions. That is the Standard Union, Standard Intersection and Standard Complementation.

(Refer Slide Time: 07:19)



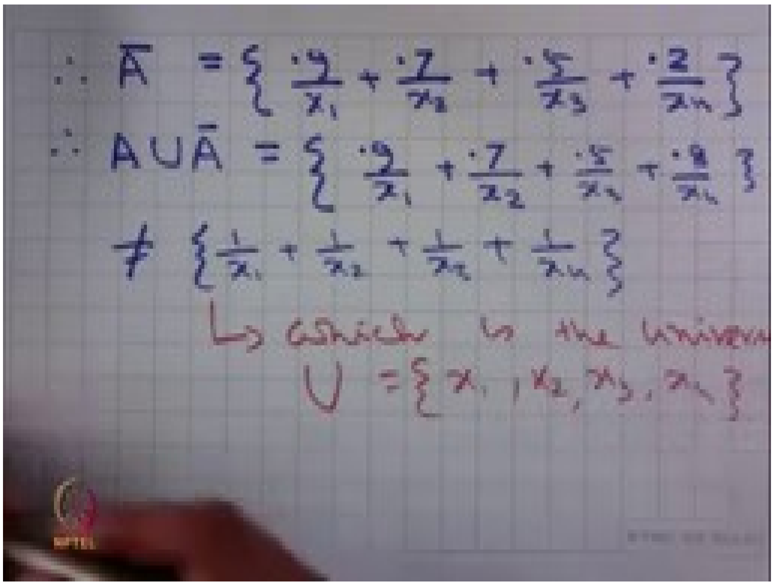
However, all set operations cannot be achieved through the above operations.

For example, Law of excluded middle:

In crisp set , where is the universe. But that does not happen for fuzzy sets.

For example, let be the set .

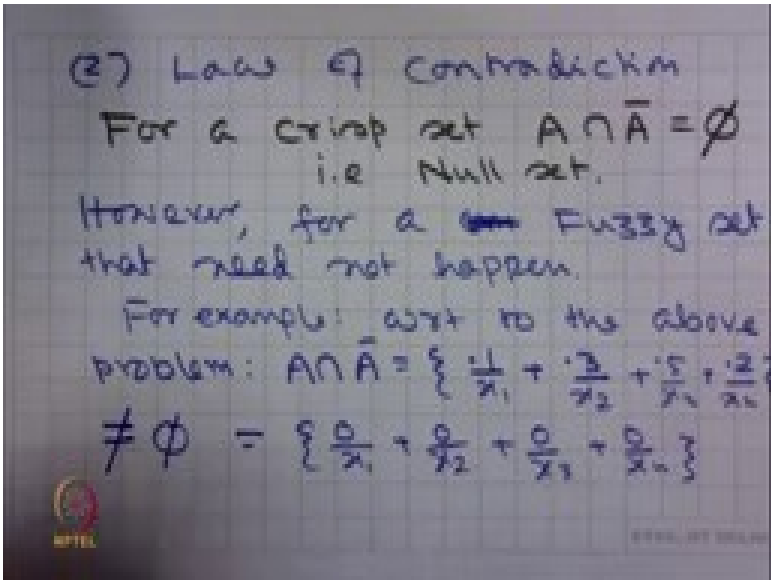
(Refer Slide Time: 09:13)



Therefore,   which is a fuzzy set is equal to

Therefore, is equal to, since you are using the operation for union for each element we will check its membership in both and and I will take the maximum of that. Therefore,   which is equivalent to universe .

(Refer Slide Time: 10:54)



Another example is the Law of contradiction.

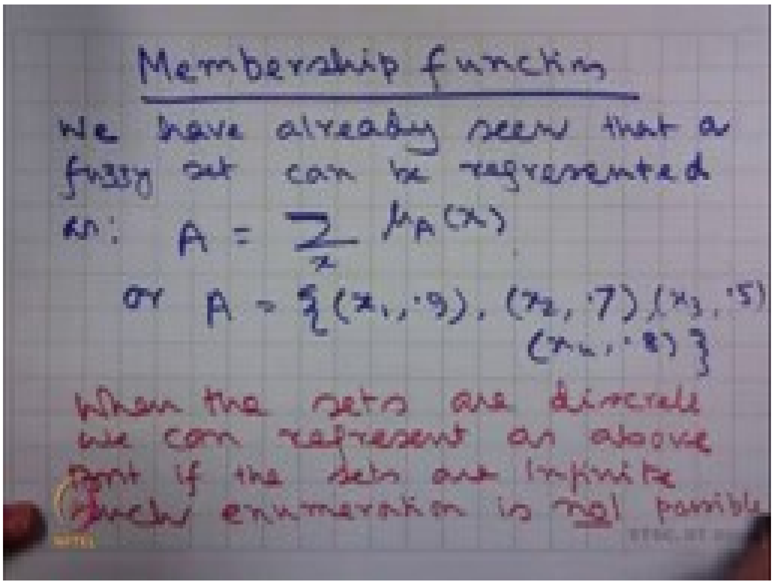
For a crisp set, , that is null set. However, for a fuzzy set that need not happen.

For example, with respect to the above problem.

That is, for each element we are looking at the minimum of the memberships to and .

And this is surely not equal to , which is . Thus, the Law of contradiction does not hold with respect to the fuzzy set.

(Refer Slide Time: 13:02)



With that background let us study membership function in some more detail. We have already seen that a fuzzy set can be represented as .

The examples that we have done just now with or we have actually represented a fuzzy set in this way. Or it can also be represented as the set of tuples. Say, something like

So, when the sets are discrete, we can represent as above. But if the sets are infinite such enumeration is not possible.

(Refer Slide Time: 15:16)

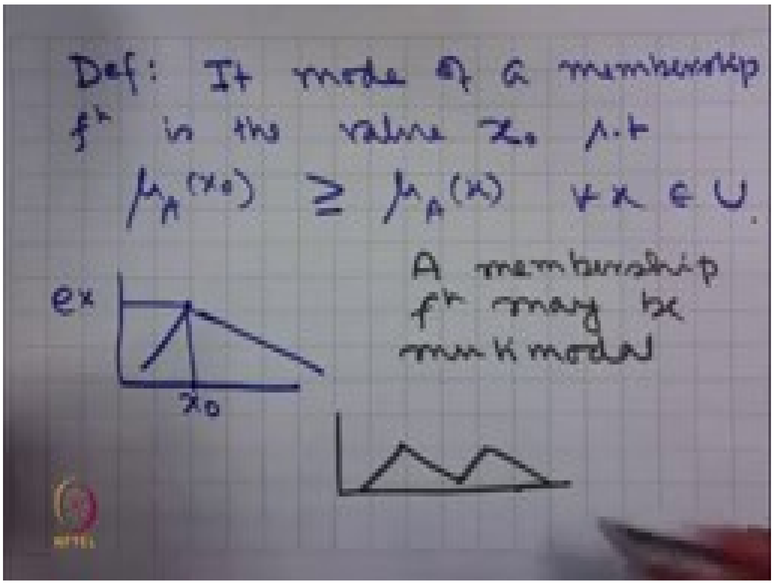
In such cases, we use mathematical functions to denote the membership of each element to the fuzzy set.

Since we use mathematical functions, it makes sense that the fuzzy set is defined on because on real numbers we can define the function.

Note that, it can be defined on that is or in general.

We can always define a function on -dimensional space. But in this class we shall stick to membership functions defined on real line.

(Refer Slide Time: 17:27)



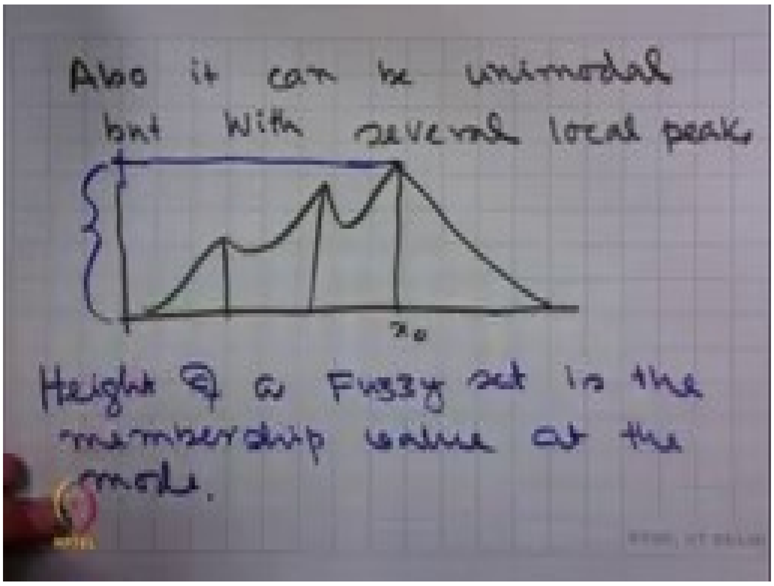
Before going into that, I give you a few definitions.

The mode of a membership function is the value such that   .

That means, the mode of the fuzzy set, the membership function to that, is that point where the membership value is maximum.

For example, if this is the membership function then this point is the . There can be multi modes. A membership function may be multimodal. Say something like this.

(Refer Slide Time: 19:10)

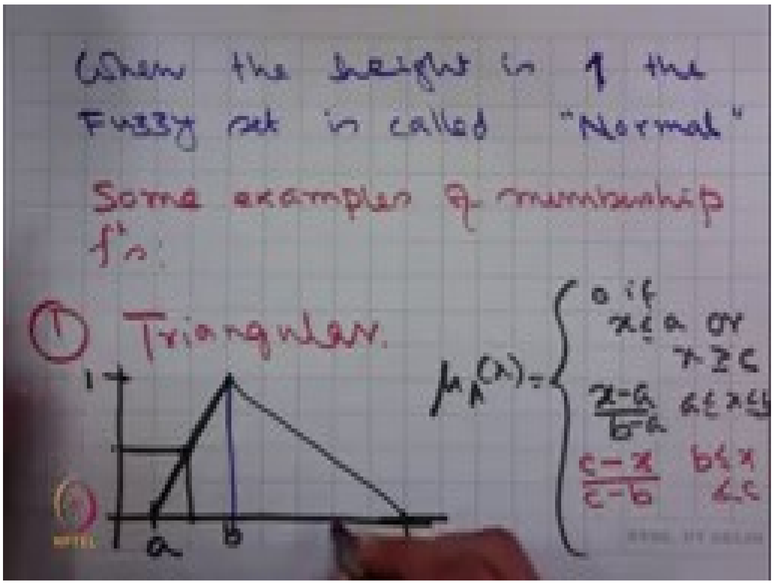


Also, it can be unimodal. That means there is only one mode to the membership function but with several local peaks.

For example: if a membership function is defined like this. It may have only one mode , but these are Peaks which appear like mode although, they do not have the maximum membership value. The height of a fuzzy set is the membership value at the mode.

For example, this is the height of this fuzzy set or its membership function.

(Refer Slide Time: 20:40)



When the height is the fuzzy set is called 'Normal'. We know that the membership function belongs to the interval so if the maximum possible value is achieved by at least one member then we call it Normal.

So, some examples of membership functions.

1) Triangular.

Suppose we have a fuzzy set whose support is the interval with a mode at and the membership function looks like a triangle. When this is one it is a Normal triangular fuzzy set.

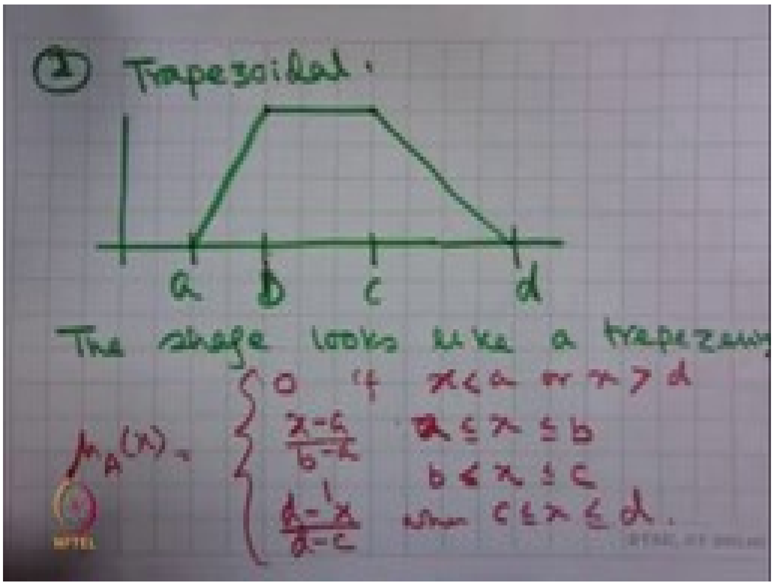
How do you define the membership function?

It is very simple.

That means on this side it is , on this side it is . It is defined over the entire real line.

But, actual support of the fuzzy set is the interval . This is understandable from simple geometry.

(Refer Slide Time: 23:51)



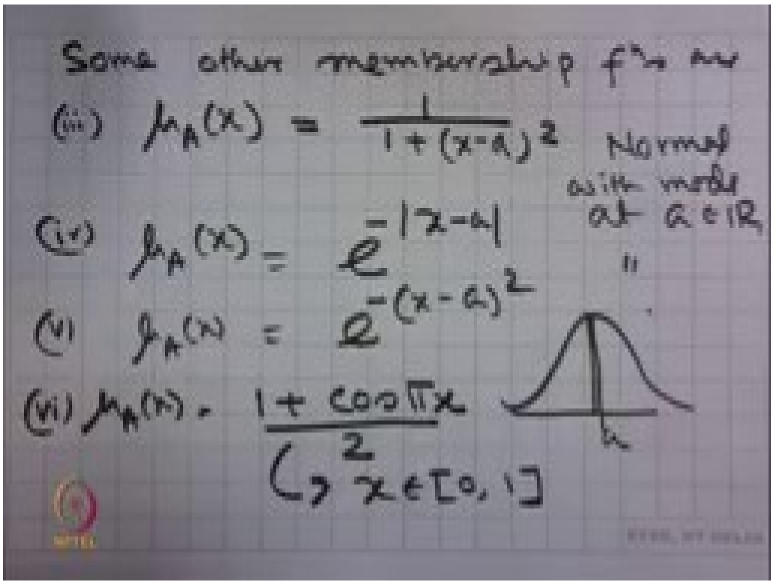
2) Trapezoidal

The shape is something like this.

It is and suppose. So, the support is from to . And it goes like a straight line. Up to point it remains at and from there it goes down up to . So, the shape looks like a trapezium. Hence, it is called a trapezoidal membership function and the membership values are given as follows:

Like that we can specify the membership function for each element in the interval .

(Refer Slide Time: 26:00)



Some other membership functions are:

3)

So, when is equal to this is going to be but as it deviates from then membership value decreases. So, it is a normal fuzzy set with mode at belonging to .

Some other functions I am writing I hope you can understand them very easily.

4) , so at it is , as deviates from it decreases and therefore, it is also a Normal Fuzzy set with mode at is equal to .

Another possibility may be this.

5) So, it will have a shape like this, where the mode is equal to .

And one trigonometric function that can be used is

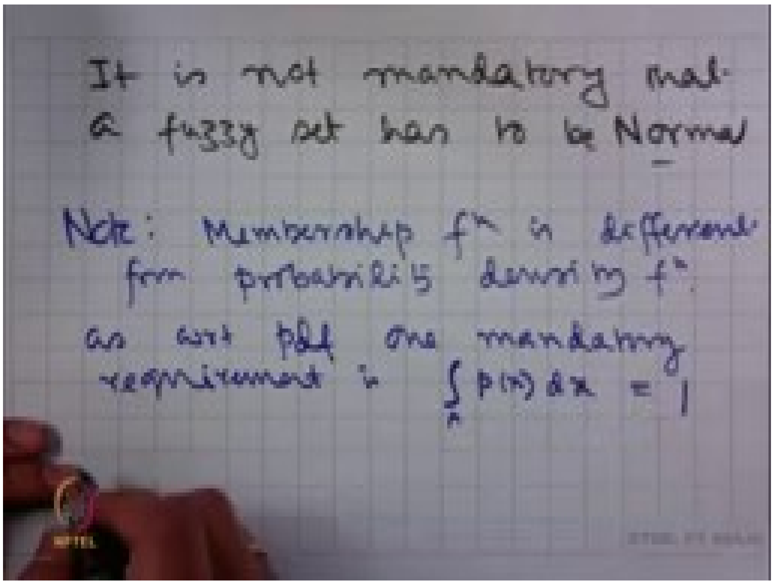
6)

So, what happens here?

When , it is , therefore it gives the value .

When it is and therefore, membership value will get and when , it will be . Therefore, membership value is going to be . So, like that one can define many mathematical functions which will give you membership values.

(Refer Slide Time: 29:17)

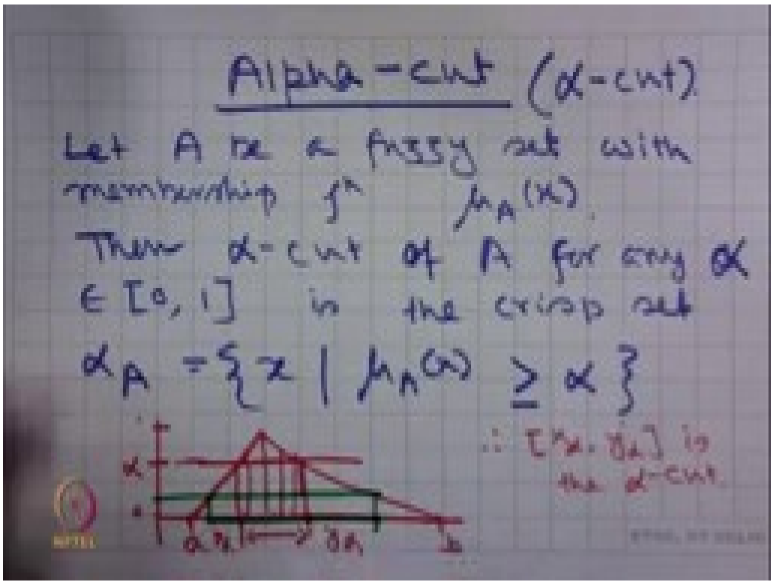


And it is not mandatory that a fuzzy set has to be Normal. One can actually play with these functions and one can define a membership which is sub-normal or whose range is different from what I have given.

One note is that the membership function is different from probability density function as with respect to one mandatory requirement is that . But such restrictions are not needed for a membership function.

We need to be careful that the values are between to .

(Refer Slide Time: 31:04)



With that background, I introduce you to an interesting concept of fuzzy set which is called \alpha-cut or often we write -cut.

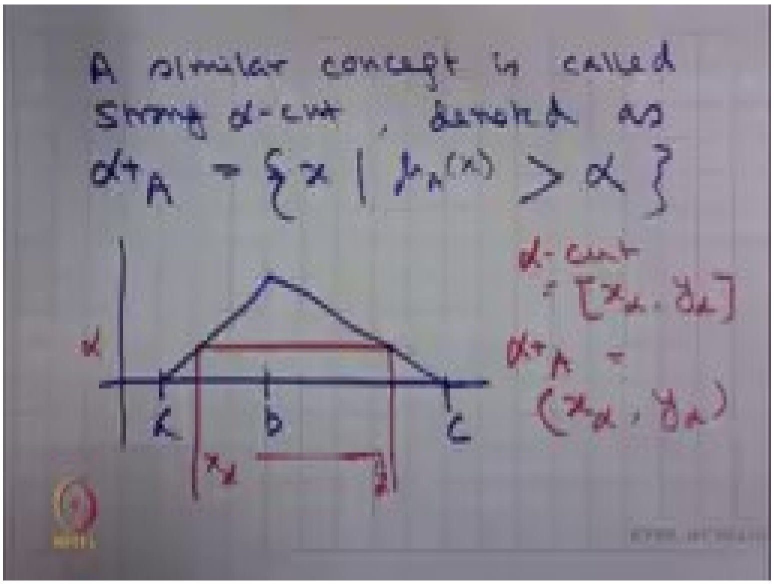
Let be a fuzzy set with membership function Then the -cut of for any is the crisp set,

So, we have a set of elements, each one of them has an associated membership value.

For a given we will look at the set of values for which the membership value is greater than equal to

For example, suppose this is a triangular fuzzy set and this is in the range and this is my that I have taken in the range [0,1]. Then if I look at this line then consider this interval on the real line. Let us call this interval to be then, for all these values the membership value is greater than . Therefore, this interval is the -cut and of course we can define for all between 0 and 1. For example if this is my then this is going to be my corresponding -cut.

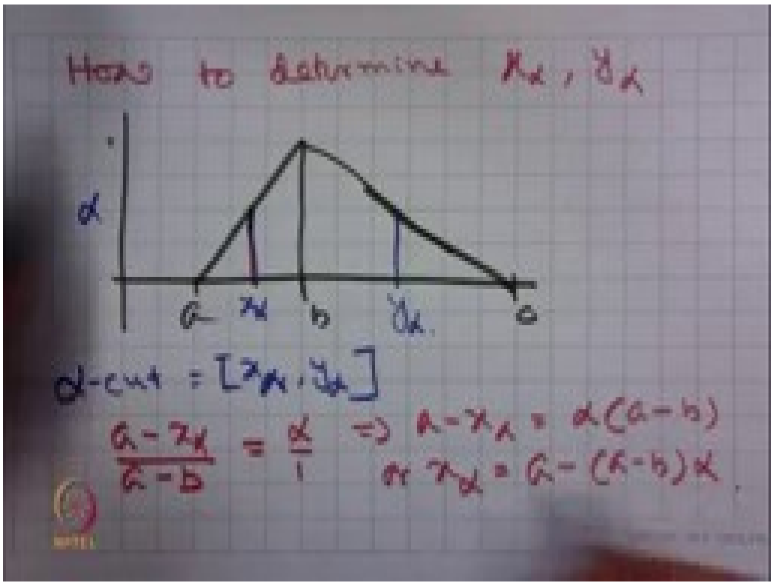
(Refer Slide Time: 34:04)



A similar concept is called strong \alpha-cut denoted as which consists of elements such that
.

Therefore, with respect to the previous example, if this is my , this is my and this is my. Then the -cut is as we have already seen is the interval but because if the value of is then it touches the membership value

(Refer Slide Time: 35:43)

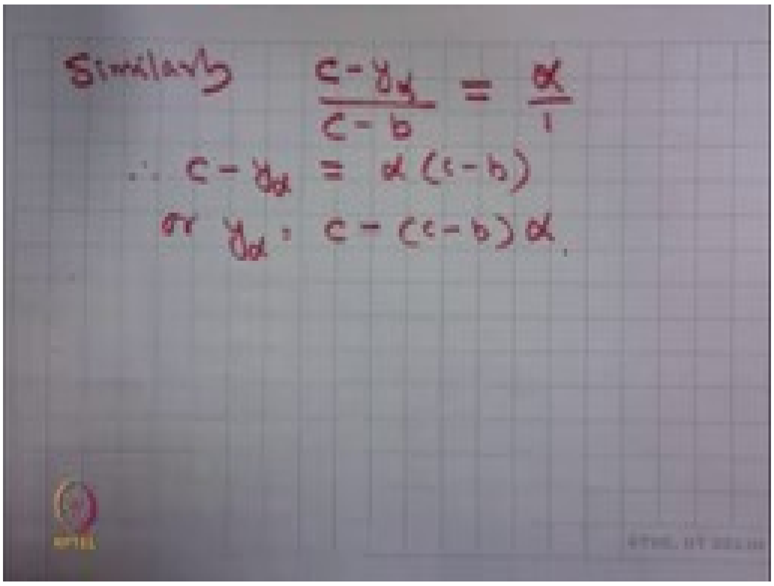


Question is how to determine and?

So with respect to triangular membership we can see that the membership value is at . Therefore, -cut is equal to . The values are determined as follows. Suppose this is my , so this is my and this is my .

From geometry we can see that,

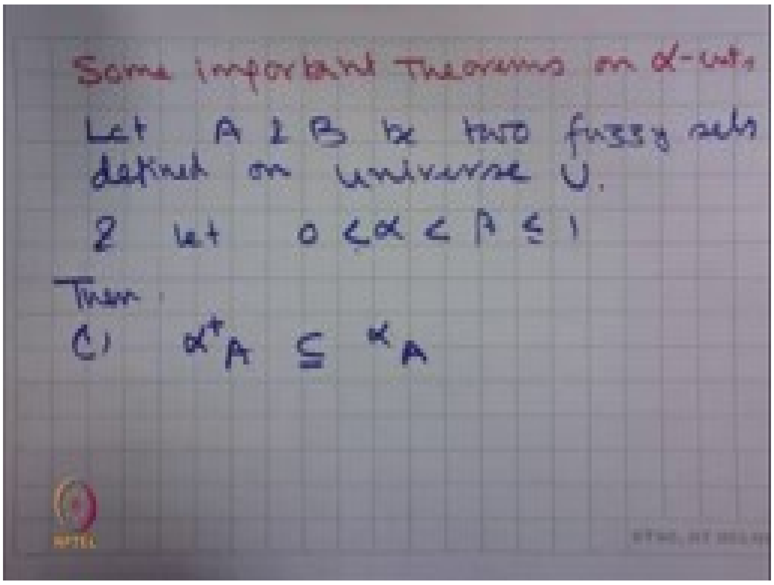
(Refer Slide Time: 37:35)



Similarly,

Now given one can therefore, easily calculate the corresponding -cuts.

(Refer Slide Time: 38:24)



Some important theorems on -cuts.

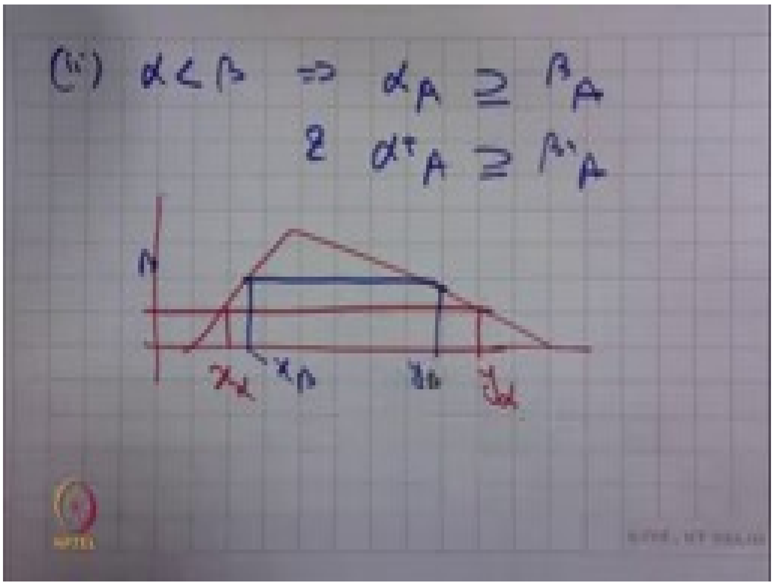
Let and be two fuzzy sets define on universe and let

Then

1)

This is obvious.

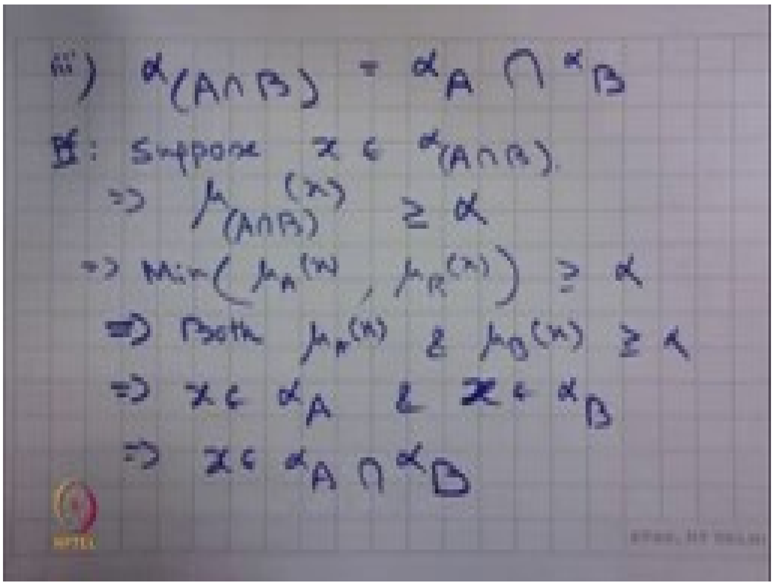
(Refer Slide Time: 39:31)



2) and

This is also obvious because if this is the membership functions and this is -cut and this is -cut we can easily see that the interval is contained in the interval

(Refer Slide Time: 40:30)



3)

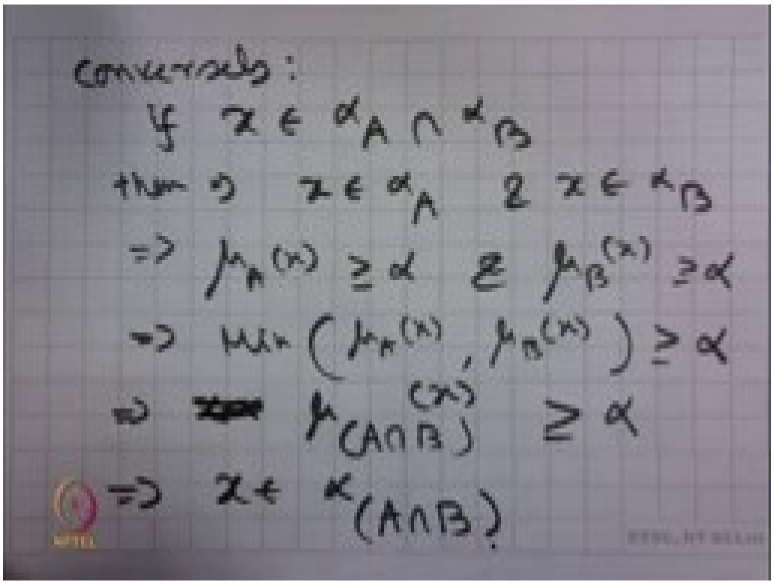
Proof:

Suppose

Both and

 and

(Refer Slide Time: 41:46)



Conversely,

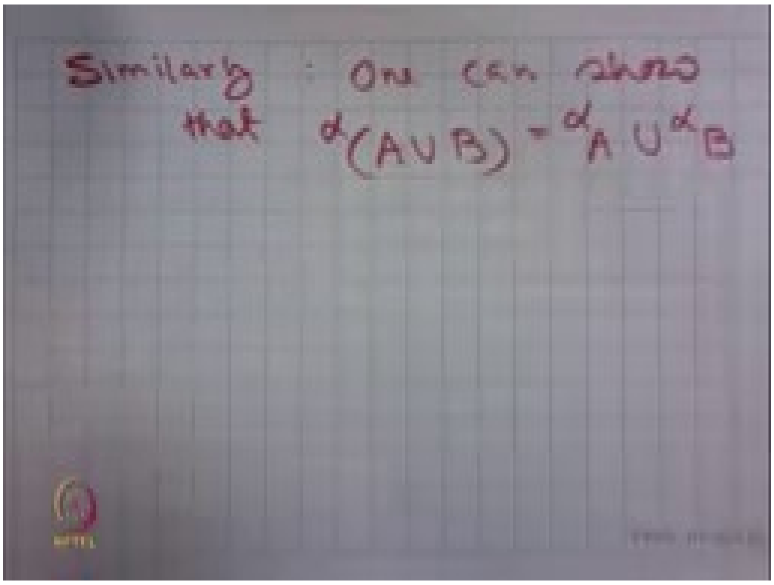
If

Then and

and

This shows that equivalence of the is same as the.

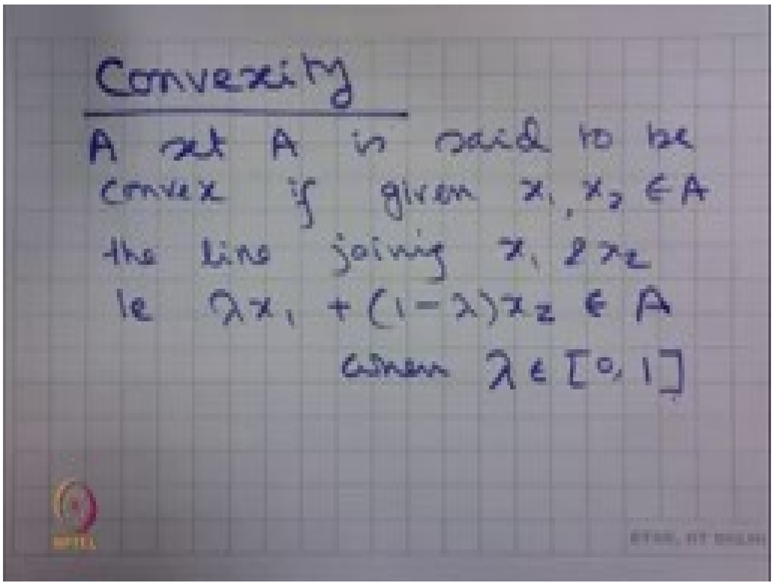
(Refer Slide Time: 43:07)



Similarly, one can show that is same as the

I leave it as an exercise. The proof is very similar; the way I did that in the case of intersection, in a very similar way you can prove for union.

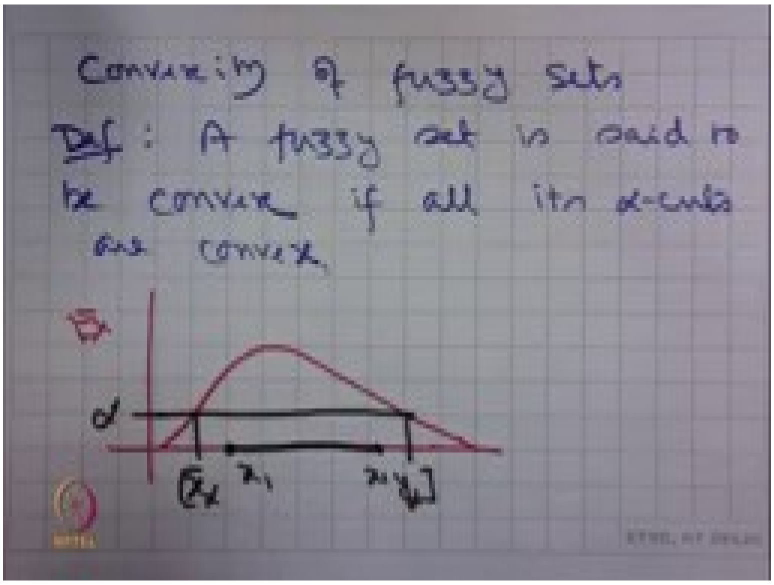
(Refer Slide Time: 43:50)



With that let me give you an important concept with respect to membership function which is called convexity.

We know that a set is said to be convex if given , the line joining and that is when

(Refer Slide Time: 44:52)



This concept of convexity is extended to fuzzy set.

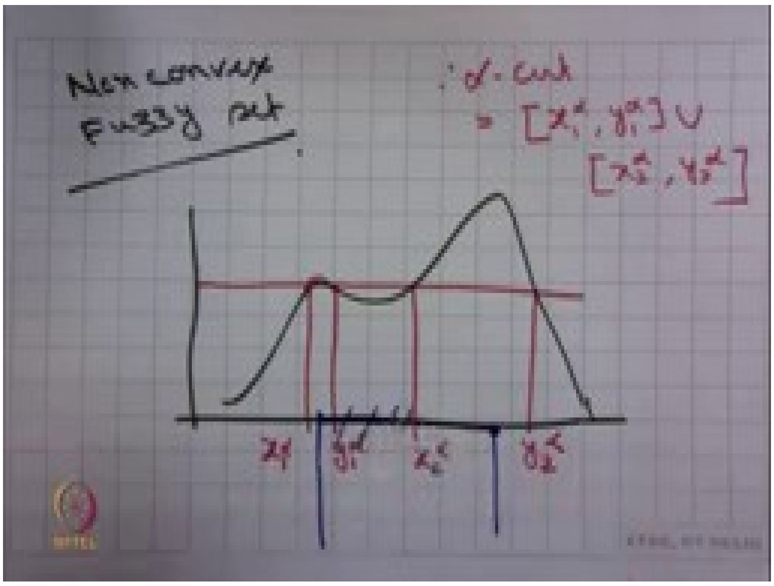
Definition:

A fuzzy set is said to be convex, if all its -cuts are convex.

That means for any between to if we construct the corresponding -cut then, that -cut has to be a convex set. We know that all -cuts are crisps set so it makes sense.

Example: Consider a fuzzy set membership function like this and suppose I consider to be like that then, the interval , we can see that this is a convex set because we take any arbitrary point and , the line joining them is actually contained in the set.

(Refer Slide Time: 46:41)



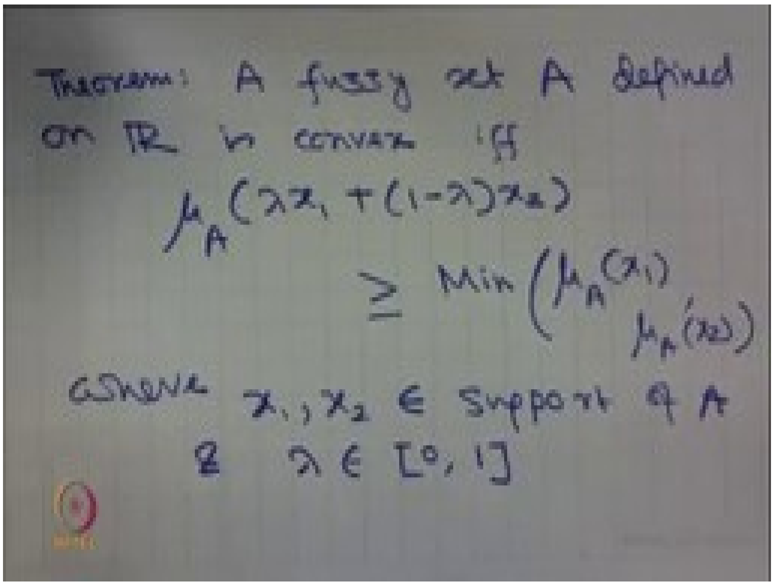
Non convex fuzzy set

Let me give you an example.

Considered a fuzzy membership function like this and consider like that then, the -cut is let me call them and then, the -cut is union of these two intervals and we can see that, this is not a convex set, because if I take a point here and another point here the line joining these two points is not contained, because this part is not part of the -cut.

Any such property like the convexity which one can extend from that domain of crisp sets to the domain of fuzzy sets with the help of -cuts is called a cut worthy property. So, I will stop the lecture today with one theorem.

(Refer Slide Time: 48:32)

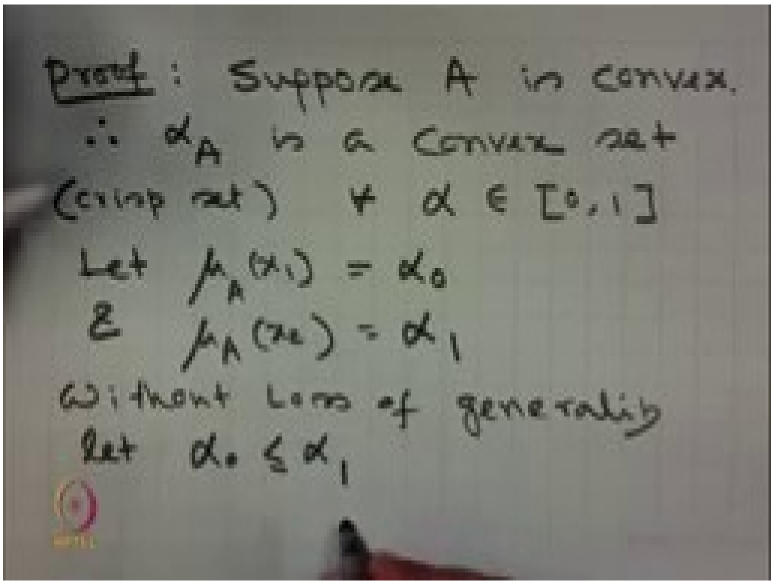


So, here is the theorem

A fuzzy set defined on is convex if and only if , where , support of and

So, effectively it is saying that if you take two points and in the support of then, each point on the line joining and will have a membership that is greater than or equal to the minimum of the memberships of and to the fuzzy set

(Refer Slide Time: 50:20)

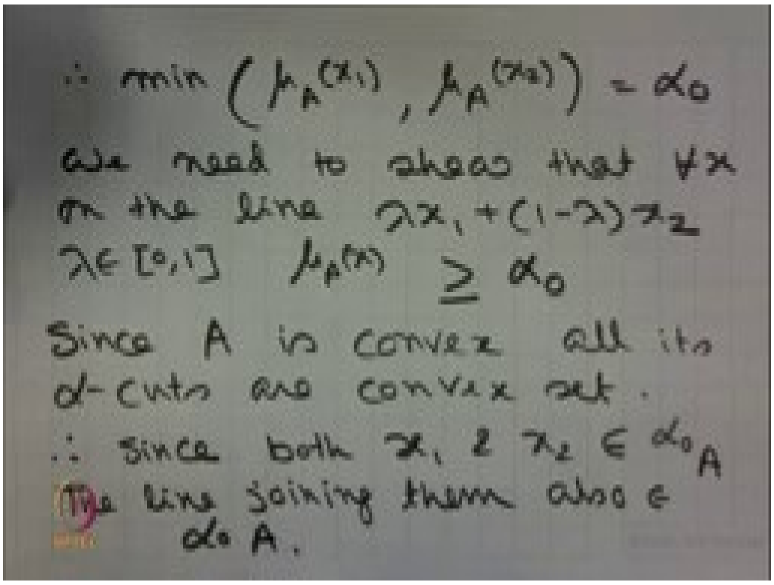


Proof: Suppose is convex. Means that is a convex set (crisp set) for all

Let and .

Without loss of generality, let

(Refer Slide Time: 51:51)



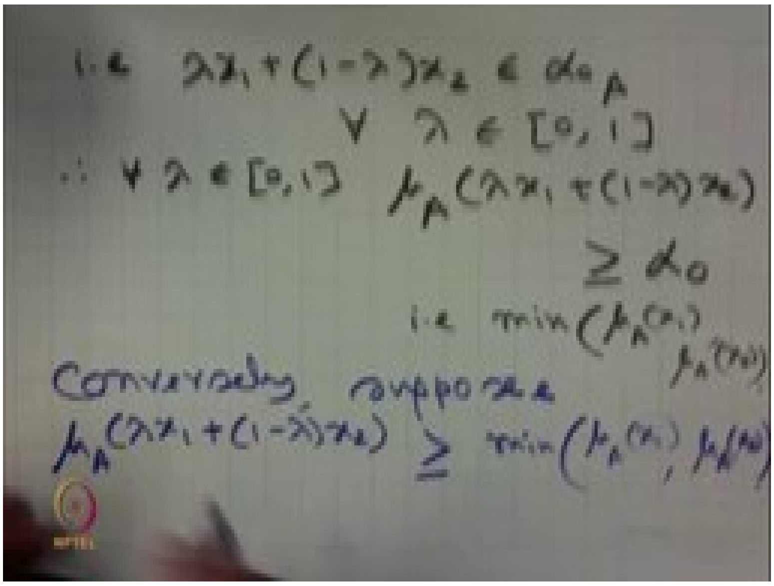
Therefore,

We need to show that, for all on the line ,

Since is convex, all its -cuts are convex set.

Therefore, since both and belong to the line joining them also belongs to .

(Refer Slide Time: 53:42)

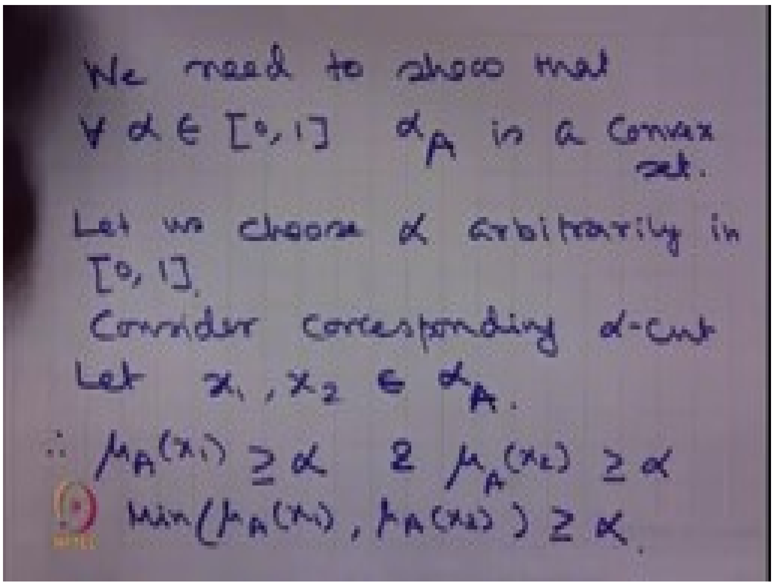


That is, , .

Therefore, , which is

Conversely suppose this is true. That is,

(Refer Slide Time: 55:13)



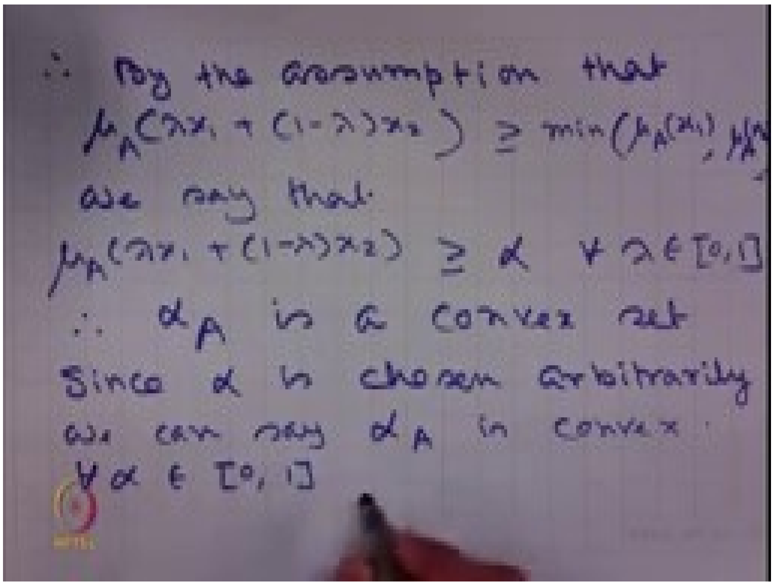
We need to show that for all , -cut of A is a convex set.

Let us choose arbitrarily in the interval Consider corresponding -cut.

Let belonging to that -cut of A.

Therefore, and . Therefore,

(Refer Slide Time: 56:55)



Therefore, by the assumption that .

We say that for all . Therefore, -cut of A is a convex set. Since is chosen arbitrarily, we can say -cut of is convex for all

(Refer Slide Time: 58:28)

is a fuzzy convex set.

That concludes the proof of the theorem

Okay students, I stop here today.

In the next class I shall explore more on membership functions, thank you.