Introduction of Fuzzy Sets Arithmetic & Logic

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Lecture-29
Fuzzy Sets Arithmetic & Logic

Welcome students, To the MOOCs series of lectures on Fuzzy Sets Arithmetic and Logic. This is lecture number 29.

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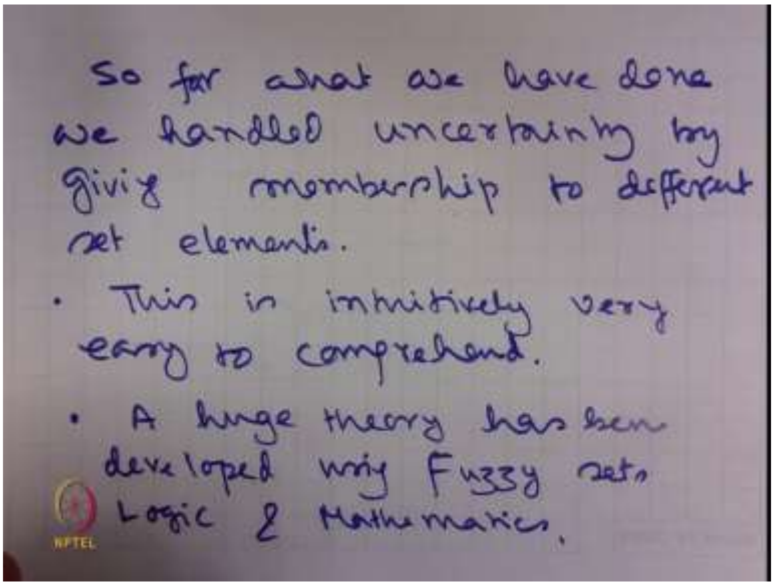

So, far what we have done, we handled uncertainty by giving membership to different set elements.

- This is intuitively very easy to comprehend.
- And we have seen that a huge theory has been developed using Fuzzy Sets, Logic and Mathematics.

Over the last 28 lectures, we have discussed these things in detail.

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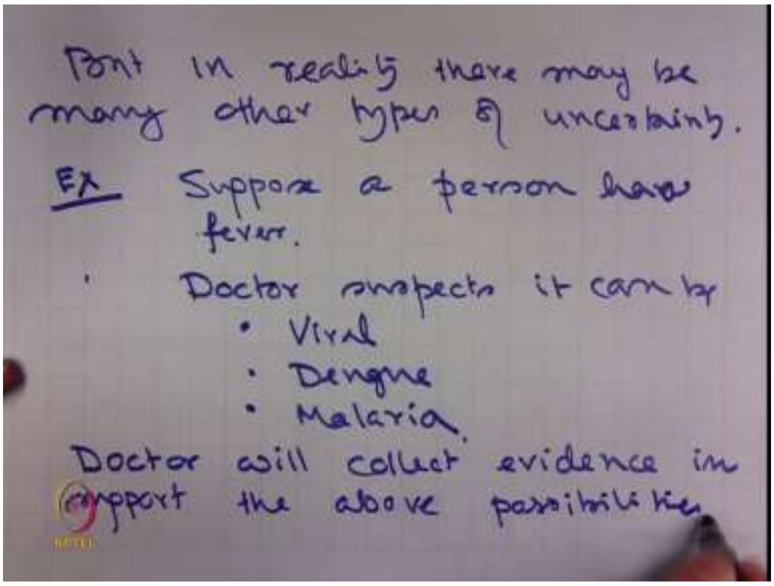

But in reality there may be many other types of uncertainty.

For example: Suppose a person has fever.

And Doctor suspects:

- it can be Viral,
- it can be Dengue
- it can be Malaria.

Now, what the doctor will do, doctor will collect evidence in support of the above possibilities.

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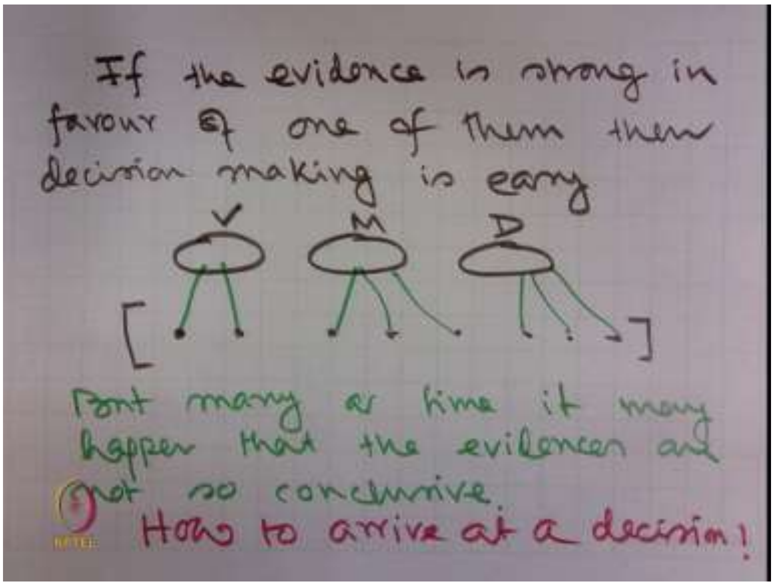

If the evidence is very strong in favour of one of them then decision making is easy.

Say for example, if this is viral, this is malaria, and this is a dengue, and these are the different set of tests that he has conducted and if presence of this is means that disease is viral presence of this means that disease is malaria and presence of this means it is that dengue then there is no problem.

But, many a time it may happen that the evidences are not so conclusive.

Question is how to arrive at a decision?

I hope you understand the point.

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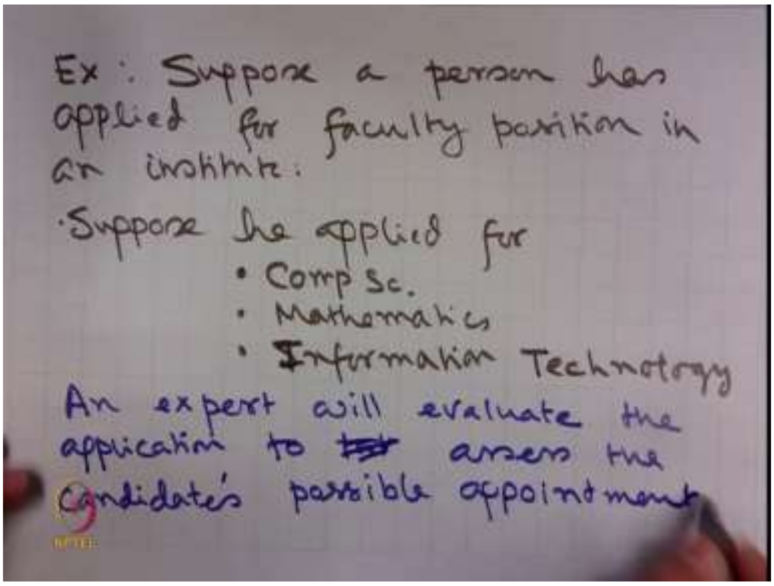

Let me give one more example

Suppose a person has applied for faculty position in an institute.

And suppose he applied for

- computer science,
- mathematics and
- information technology

An expert will evaluate the application to take to assess the candidate's possible appointment.

Now, depending upon the application.

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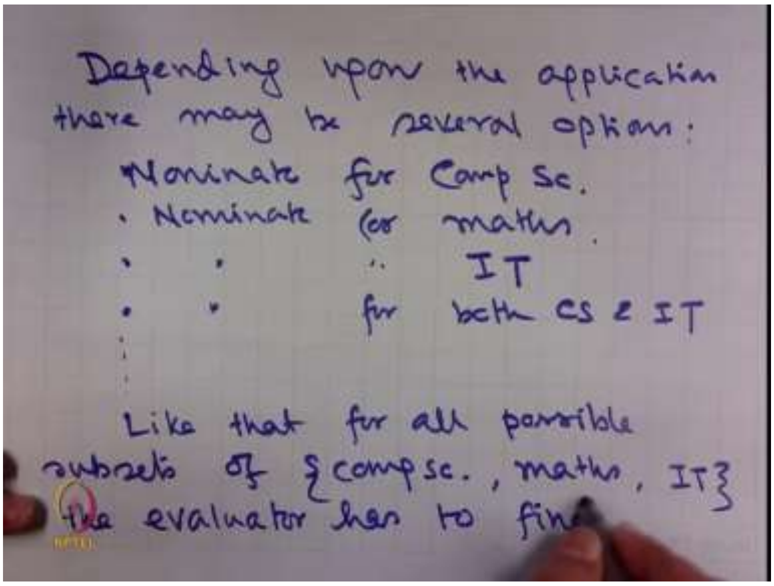

There may be several options:

- nominate for computer science
- nominate for maths
- similarly, nomination for IT
- he may nominate for both CS and IT

Like that for all possible subsets of in this case the universal set is three items,

$$\{Computers\ Science, Maths, IT\}$$

For each such subset of disparate set the evaluator has to find the possibility of nominating for one of them

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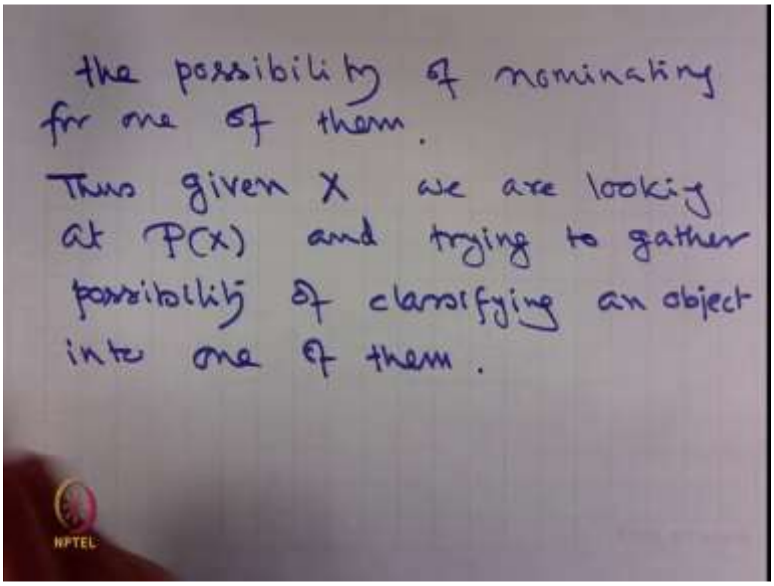

Thus given X the universal set we are looking at $P(X)$ and trying to gather possibility of classifying an object which in our case is a person into one of them.

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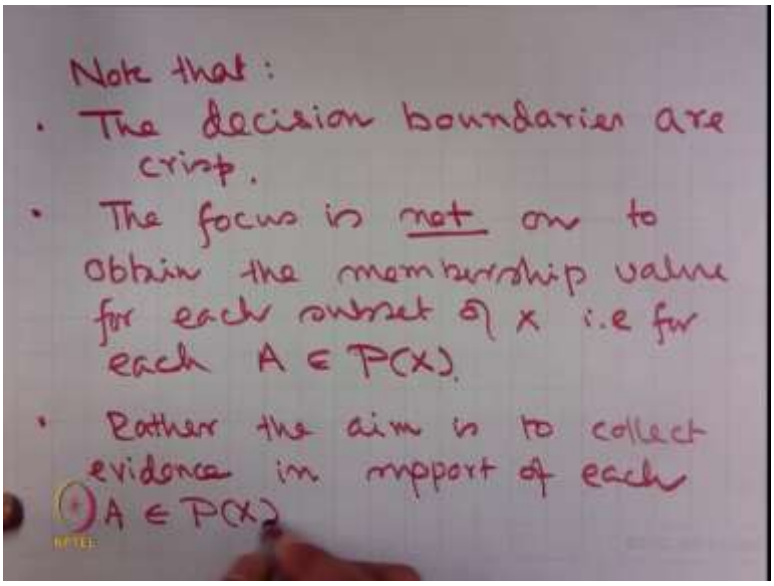

Note that

- The decision boundaries are crisp. It is one of these possible subsets that has to be nominated.
- The focus is not on to obtain the membership value for each subset of X that is for each $A \in P(X)$
- Rather the aim is to collect evidence in support of each $A \in P(X)$

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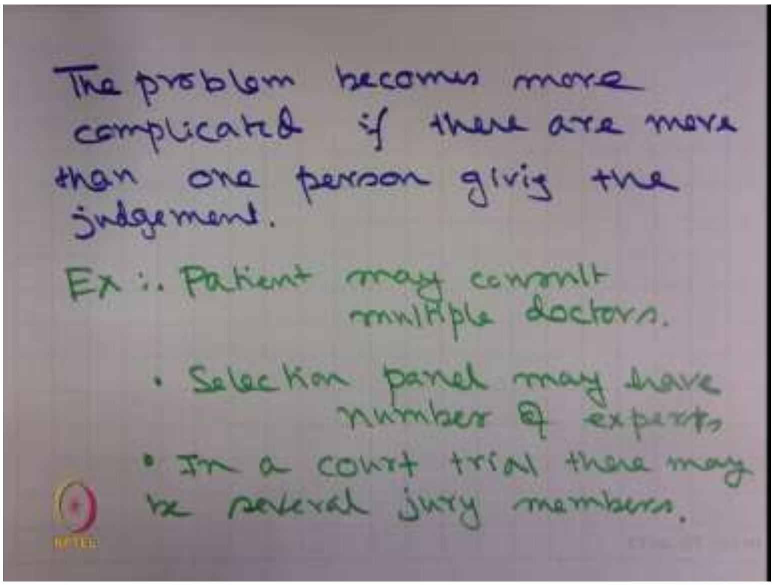

The problem becomes more complicated if there are more than one person giving the judgment. Example:

- Patient may consult multiple doctors
- Selection panel may have number of experts
- In a court trial there may be several jury members.

In each of this case, each concern person will have some opinion and they may differ among themselves. And therefore, we need to somehow combine the opinions or evidence is collected by the different experts to come into a conclusion.

So, that is the problem that we are taking up in this lecture. And, I will continue with that in the next class as well.

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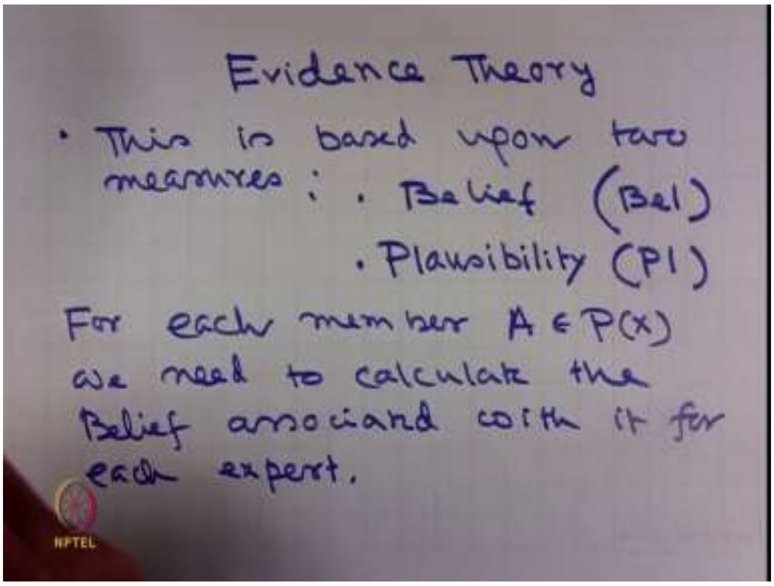

The theory that I will be discussing today is called Evidence Theory

- This is based upon two measures.
    - $Belief$ which I will write as, Bel
    - $Plausibility$ which I will write as Pl.

For each member $A \in P(X)$. We need to calculate the belief associated with it for each expert.

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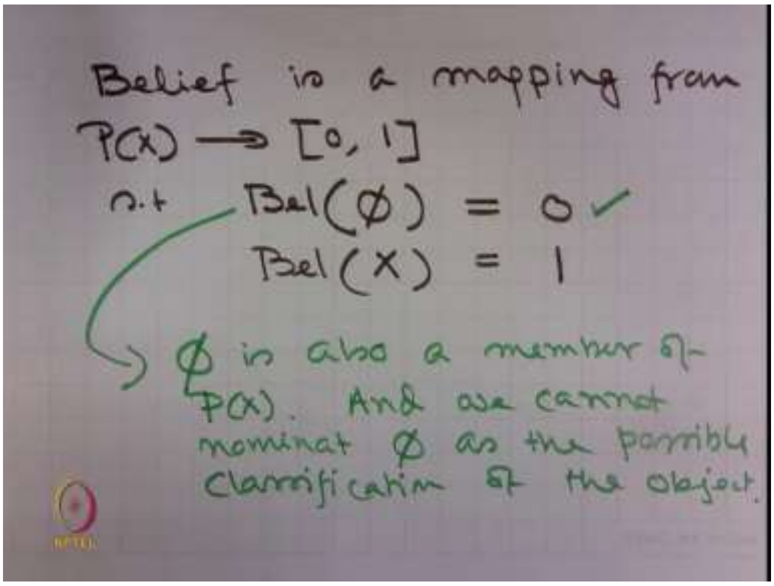

So, let us talk about $Belief$

$Belief$ is a mapping from $P(X) \to [0, 1]$ such that

$$Bel(\phi) = 0$$

$$Bel(X) = 1$$

This is because ϕ is also a member of $P(X)$ and we cannot nominate ϕ as the possible classification of the object.

For example:

- If the patient has a disease the doctor cannot say, No he does not have any disease.
- If a person is on trial, then the judge will have to take decision whether he is guilty or not guilty. That is the corresponding set here is only two members guilty and not guilty. Therefore, the $P(X)$ will have 4 elements ϕ, guilty, not guilty and both guilty and not guilty.

  Now, the judge will have to take a decision, it can be one of guilty not guilty and that can certainly not be ϕ That is the judge cannot say that I won't take any decision.

  Just like a doctor cannot say no, this patient does not have any ailment.

  Therefore, corresponding to ϕ the $Belief$ associated is 0.

In a very similar line, since, the decision has to be one of the possible members of X or some subset of X, the belief of the entire set it has to be one.

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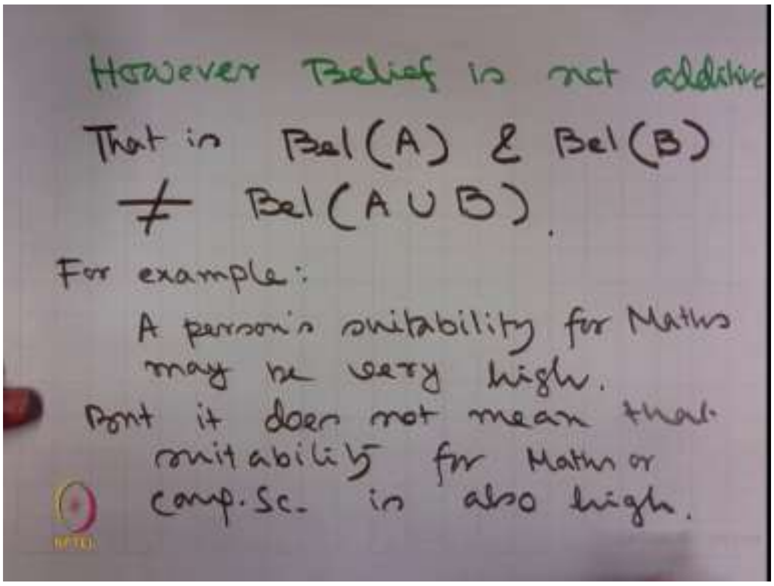

However, $Belief$ s not additive.

That is,

$$Bel(A \cup B) \neq Bel(A) + Bel(B)$$

For example,

A person's suitability for maths maybe very high.

But, it does not mean that suitability for maths or computing or computer science is also high, which we can easily understand.

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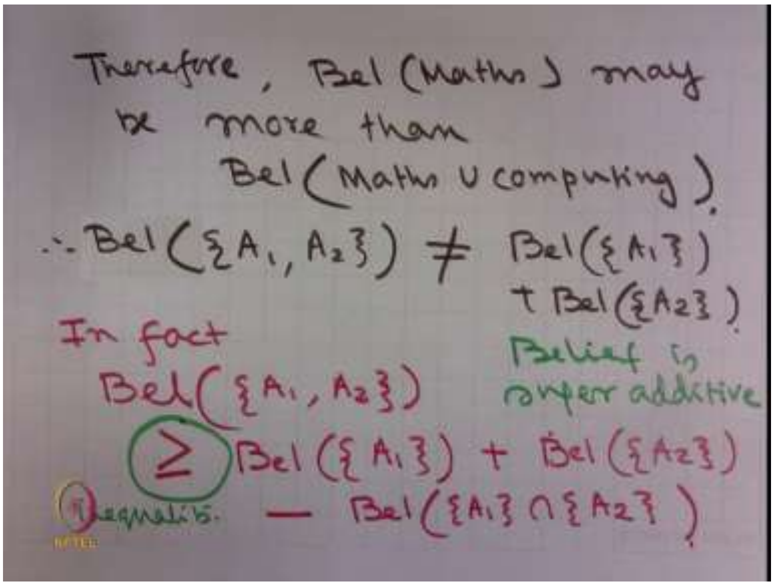

And therefore $Bel(Maths)$ maybe more than $Bel(Maths \cup Computing)$

I hope the concept is clear and

Therefore, $Bel(\{A_1, A_2\}) \neq Bel(\{A_1\}) + Bel(\{A_2\})$

In fact, $Bel(\{A_1, A_2\}) \geq Bel(\{A_1\}) + Bel(\{A_2\}) - Bel(\{A_1\} \cap \{A_2\})$

You have seen similar formula with respect to probability, but in that case, it was an equality. But, in case of evidence, this is inequality. And because of that, belief is super additive, not just additive, it is super additive, because it exceeds that value.

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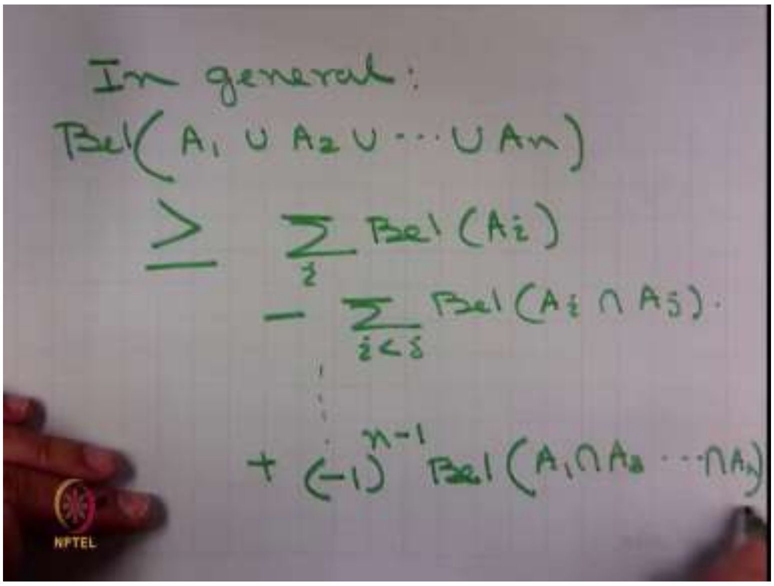

In general,

$$Bel(A_1 \cup A_2 \cup \ldots . \cup A_n)$$
$$\geq \sum_i Bel(A_i) - \sum_{i<j} Bel\left(A_i \cap A_j\right) \ldots . + (-1)^{n-1} Bel(A_1 \cap A_2 \cap \ldots \cap A_n)$$

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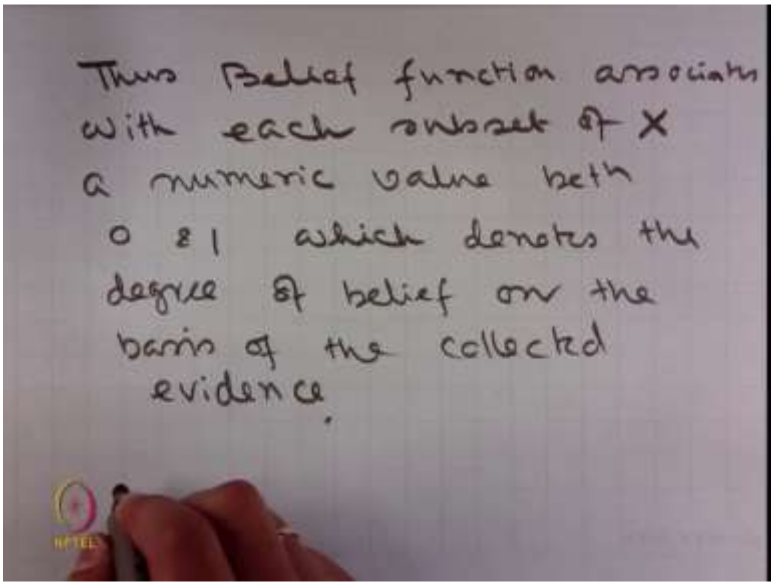

Thus, belief function associates with each subset of X, a numeric value between 0 and 1 which denotes the degree of belief on the basis of the collected evidence.

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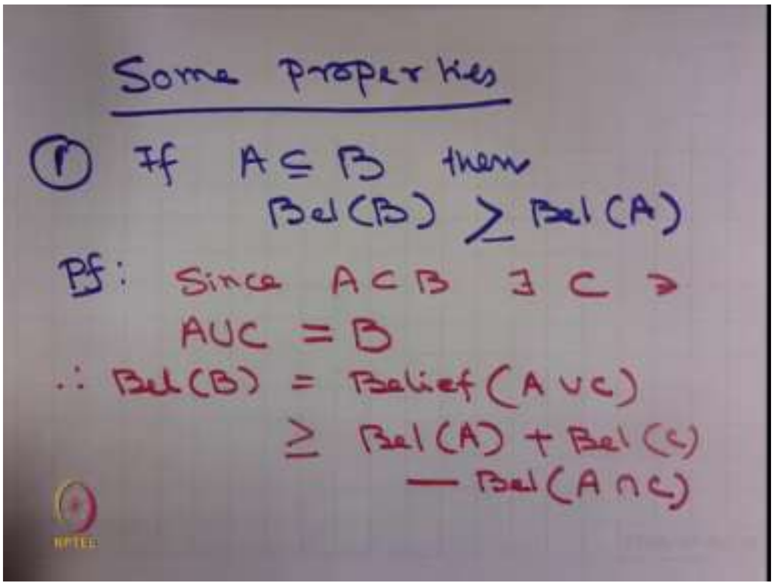

Some properties,

1. If $A \subseteq B$ then, $Bel(B) \geq Bel(A)$

   Proof:

   Since $A \subset B \quad \exists\, C$ such that $A \cup C = B$

$$\therefore Bel(B) = Bel(A \cup C) \geq Bel(A) + Bel(C) - Bel(A \cap C)$$

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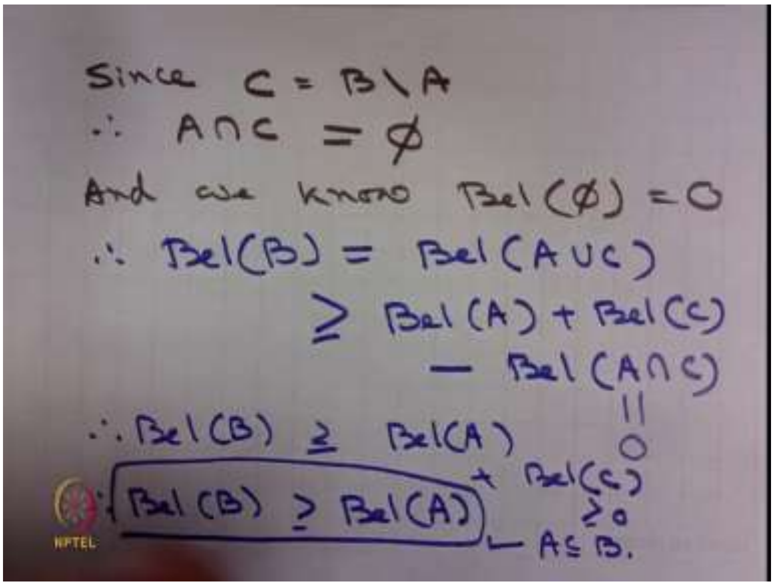

Since $C = B \backslash A$

Therefore, $A \cap C = \phi$

And we know that $Bel(\phi) = 0$

Therefore, $Bel(B) = Bel(A \cup C) \geq Bel(A) + Bel(C) - Bel(A \cap C)$

Since, $Bel(A \cap C) = 0$

Therefore, $Bel(B) \geq Bel(A) + Bel(C)$

Now, $Bel(C) \geq 0$

Therefore, $Bel(B) \geq Bel(A)$

Note, we started with $A \subseteq B$

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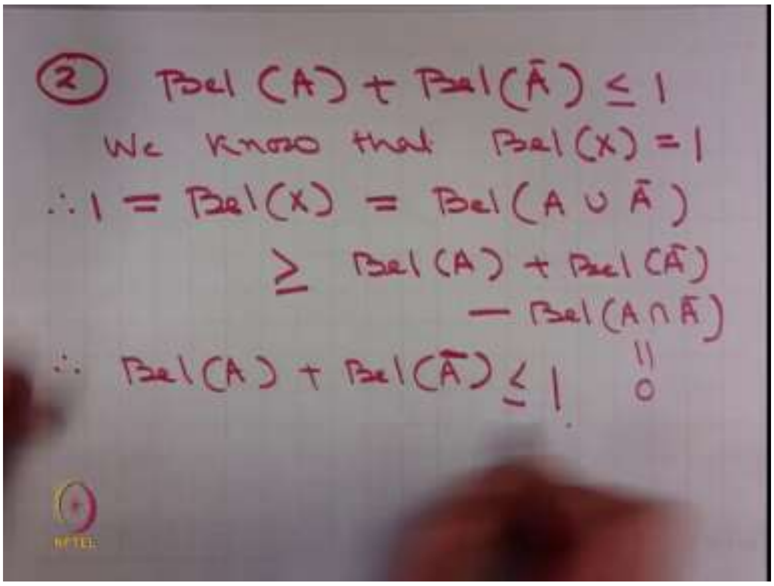

2. $Bel(A) + Bel(\bar{A}) \leq 1$

   We know that $Bel(X) = 1$

   Therefore, $1 = Bel(X) = Bel(A \cup \bar{A}) \geq Bel(A) + Bel(\bar{A}) - Bel(A \cap \bar{A})$

   Since, $Bel(A \cap \bar{A}) = Bel(\phi) = 0$

   Therefore, $Bel(A) + Bel(\bar{A}) \leq 1$

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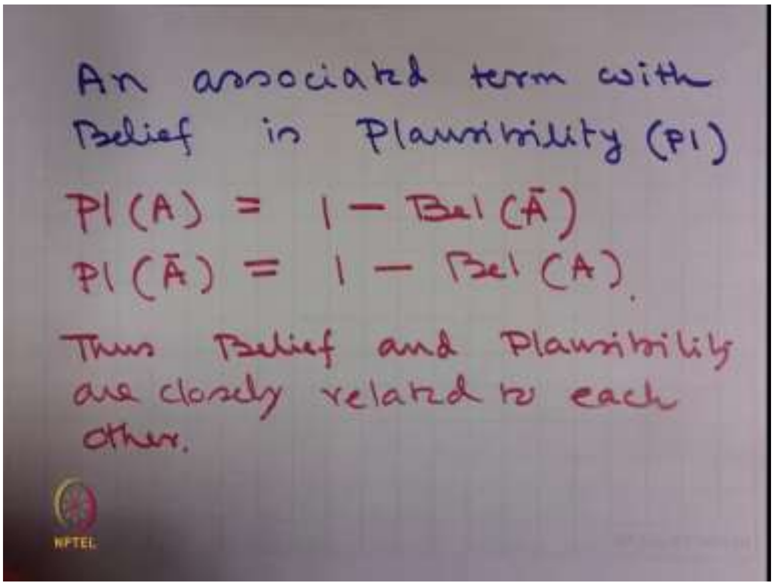

An associated, term with $Belief$ is $Plausibility$. As I indicated earlier we will denote it by Pl.

By definition:

$Pl(A) = 1 - Bel(\bar{A})$ and $Pl(\bar{A}) = 1 - Bel(A)$

Thus, $Belief$ and $Plausibility$ are closely related to each other.

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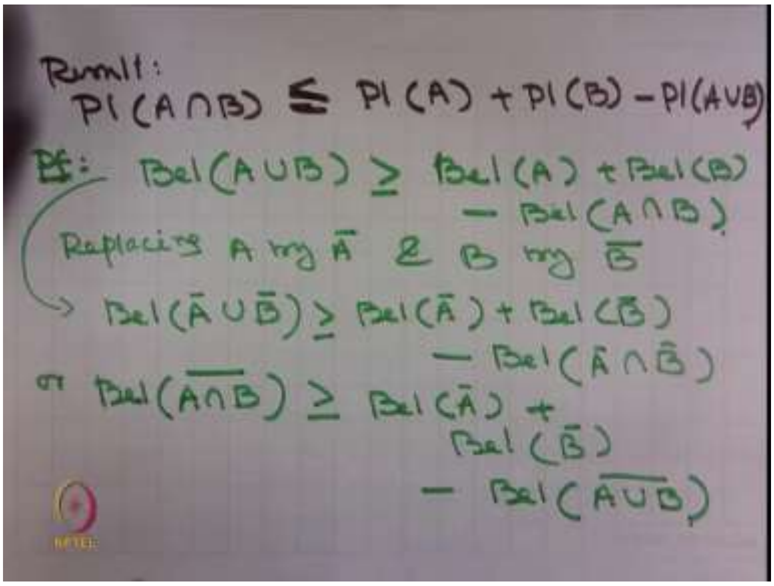

Result:

$Pl(A \cap B) \leq Pl(A) + Pl(B) - Pl(A \cup B)$

Proof:

We have $Bel(A \cup B) \geq Bel(A) + Bel(B) - Bel(A \cap B)$

Replacing A by $\bar{A}$ and B by $\bar{B}$

$Bel(\bar{A} \cup \bar{B}) \geq Bel(\bar{A}) + Bel(\bar{B}) - Bel(\bar{A} \cap \bar{B})$

Using De Morgan's Law

$$Bel(\overline{A \cap B}) \geq Bel(\bar{A}) + Bel(\bar{B}) - Bel(\overline{A \cup B})$$

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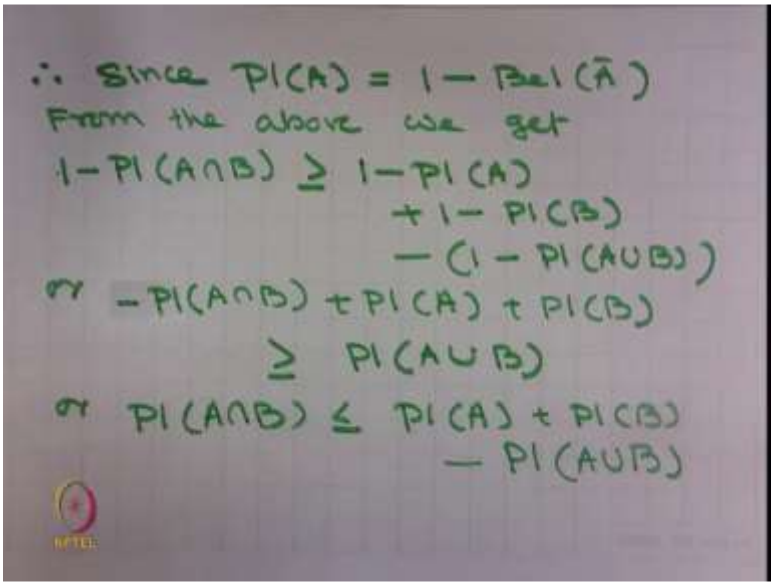

Therefore, Since $Pl(A) = 1 - Bel(\bar{A})$

From the above, we get

$$1 - Pl(A \cap B) \geq 1 - Pl(A) + 1 - Pl(B) - \big(1 - Pl(A \cup B)\big)$$

Or $-Pl(A \cap B) + Pl(A) + Pl(B) \geq Pl(A \cup B)$

Or $Pl(A \cap B) \leq Pl(A) + Pl(B) - Pl(A \cup B)$

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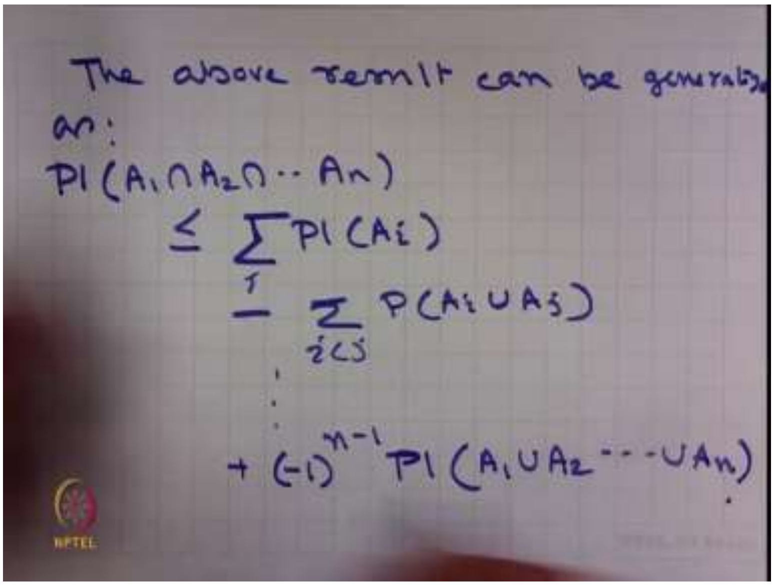

The above result can be generalized as

$$Pl(A_1 \cap A_2 \cap \ldots \cap A_n)$$
$$\leq \sum_{i} Pl(A_i) - \sum_{i<j} Pl\big(A_i \cup A_j\big) + \cdots + (-1)^{n-1} Pl(A_1 \cup A_2 \cup \ldots \cup A_n)$$

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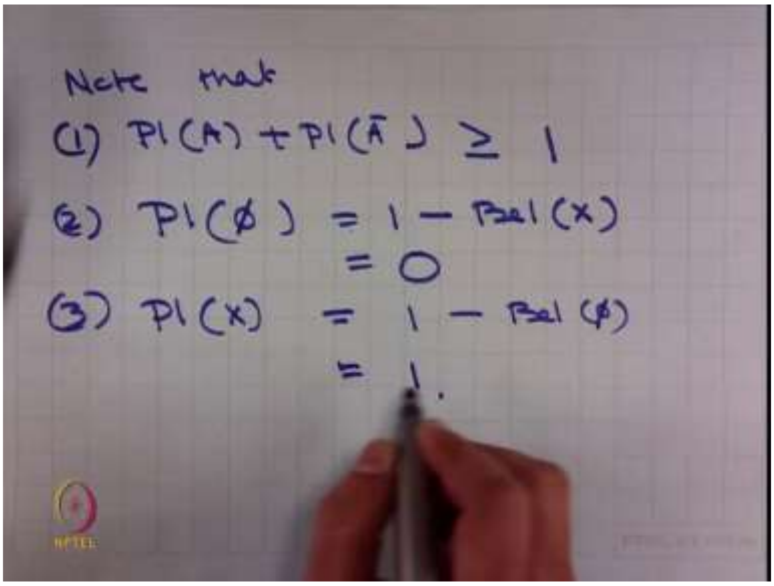

Note that

1. $Pl(A) + Pl(\bar{A}) \geq 1$
2. $Pl(\phi) = 1 - Bel(X) = 0$
3. $Pl(X) = 1 - Bel(\phi) = 1$

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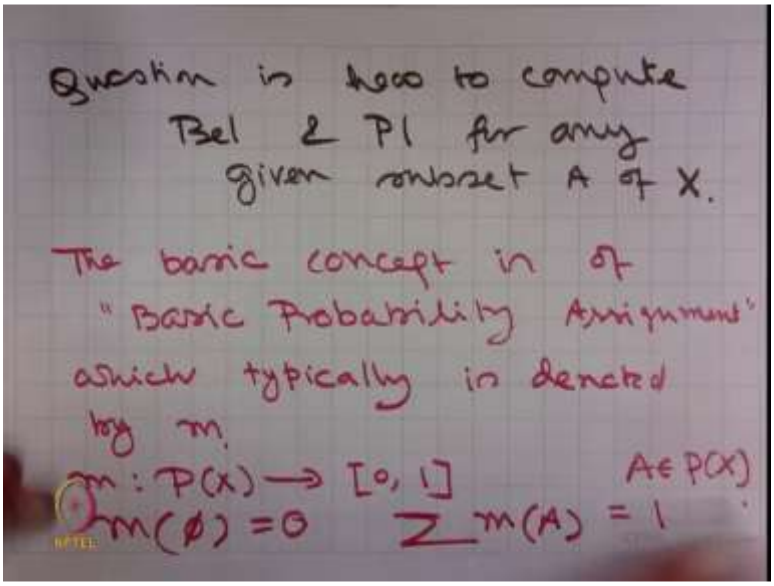

Question is How to compute $Belief$ and $Plausibility$ for any given subset A of X.

The basic concept here is of $Basic\ Probability\ Assignment$ which typically is denoted by m such that:

$m: P(X) \rightarrow [0, 1]$

$m(\phi) = 0$ and $\sum_{A \in P(X)} m(A) = 1$

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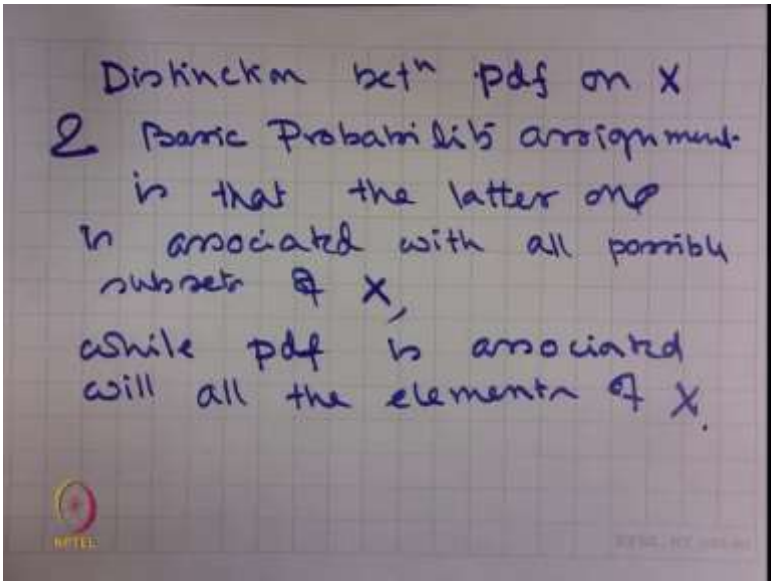

The distinction should be very clear any probability distribution function on X and basic probability assignment is that the latter one is associated with all possible subsets of X. While a probability distribution function is associated with all the elements of X.

So, this has to be remembered very carefully.

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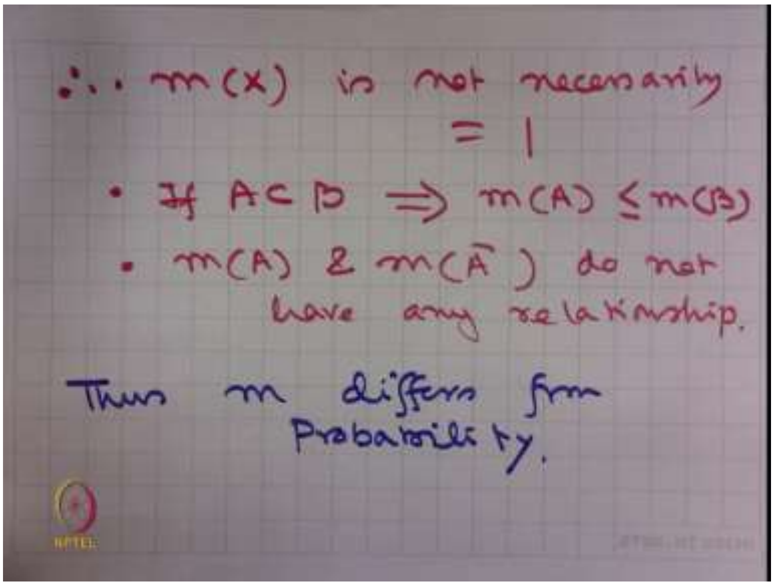

And therefore, $m(X)$ is not necessarily equal to 1.

- $m(X) \neq 1$
- If $A \subset B \Rightarrow m(A) \leq m(B)$
- $m(A)$ and $m(\bar{A})$ do not have any relationship.

Although with respect to probability, we know that:

- $P(X) = 1$
- If $A \subset B \Rightarrow P(A) \leq P(B)$
- $P(A) + P(\bar{A}) = 1$

So, m differs from probability, we have to remember this thing because this is conceptually different from what you have learned in your probability class.

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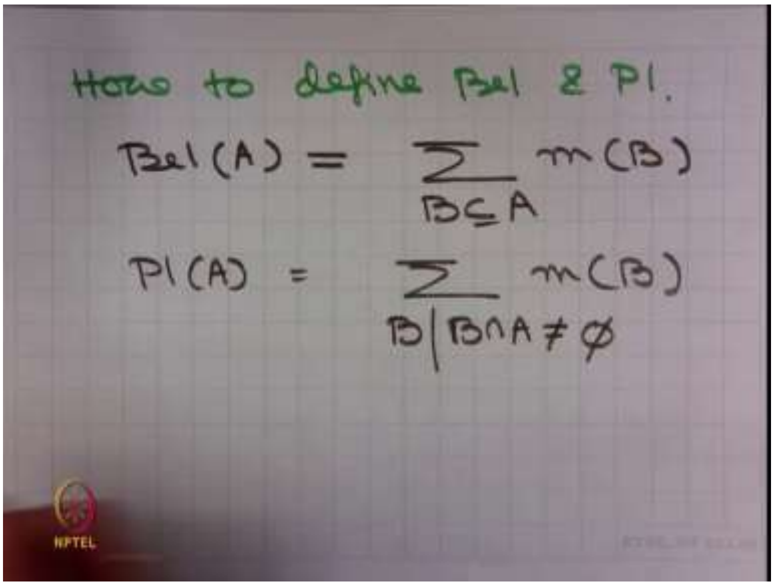

So, how to define $Belief$ and $Plausibility$ that definition is as follows:

$$Bel(A) = \sum_{B \subseteq A} m(B)$$

$$Pl(A) = \sum_{B | B \cap A \neq \phi} m(B)$$

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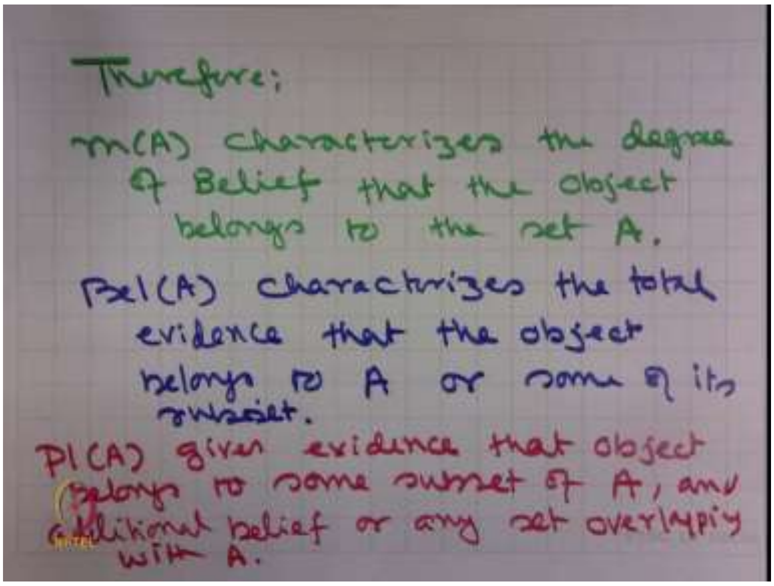

Therefore,

- $m(A)$ characterizes the degree of $Belief$ that the object to be classified belongs to the set A
- $Bel(A)$ characterizes the total evidence that the object belongs to A or some of its subset
- $Pl(A)$ gives evidence that object belongs to some subset of A and additional $Belief$ or any set overlapping with A.

So, that is the basic characterization of the 3 measures:

- $Plausibility$
- $Belief$
- m

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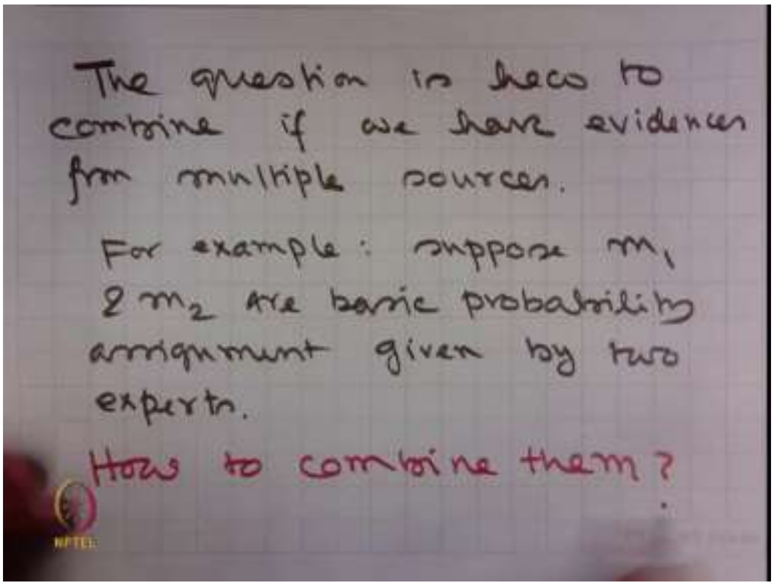

The question is how to combine if we have evidences from multiple sources?

For example:

Suppose m_1 and m_2 are the $Basic\ Probability\ Assignment$ given by two experts.

Our problem is how to combine them?

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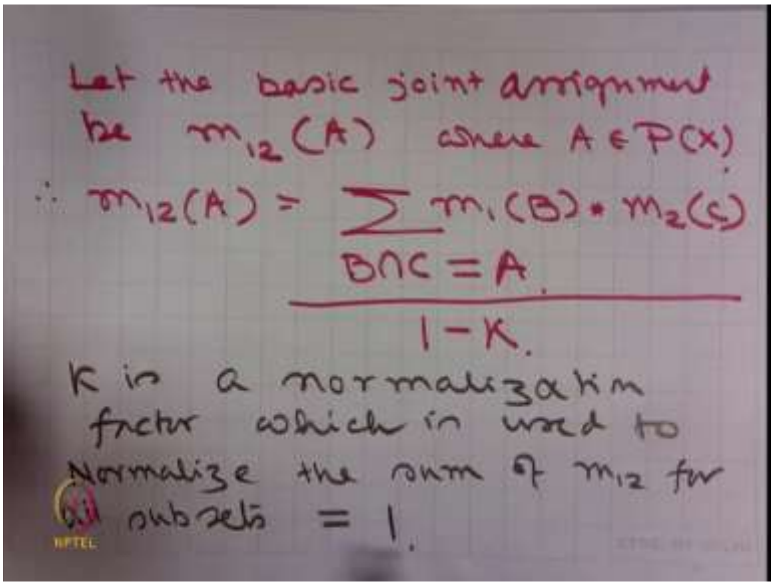

So, let the basic joint assignment be $m_{12}(A)$ where, $A \in P(X)$

$$\therefore m_{12}(A) = \frac{\sum_{B \cap C = A} m_1(B) \times m_2(C)}{1 - K}$$

What is K?

K is a normalization factor which is used to normalize that the sum of m_{12} for all subsets is equal to 1.

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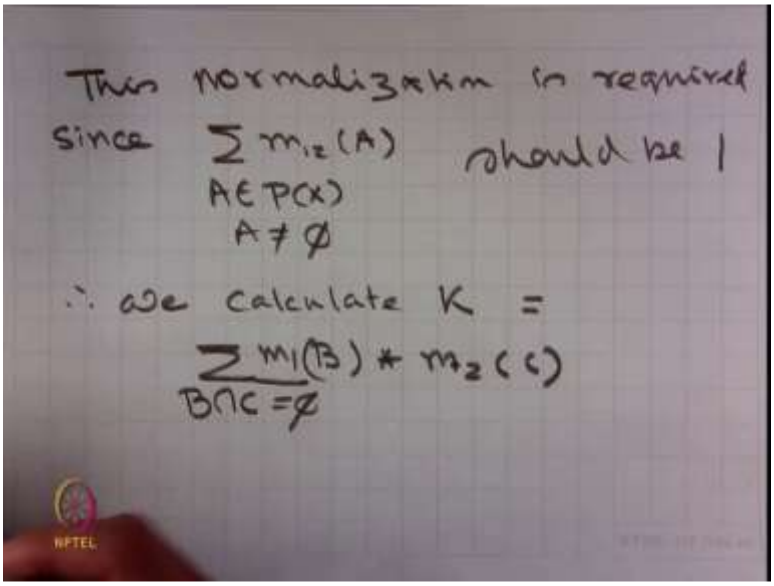

We know that this normalization is required.

$$\because \sum_{A \in P(X) | A \neq \phi} m_{12}(A) = 1$$

Therefore, we calculate

$$K = \sum_{B \cap C = \phi} m_1(B) \times m_2(C)$$

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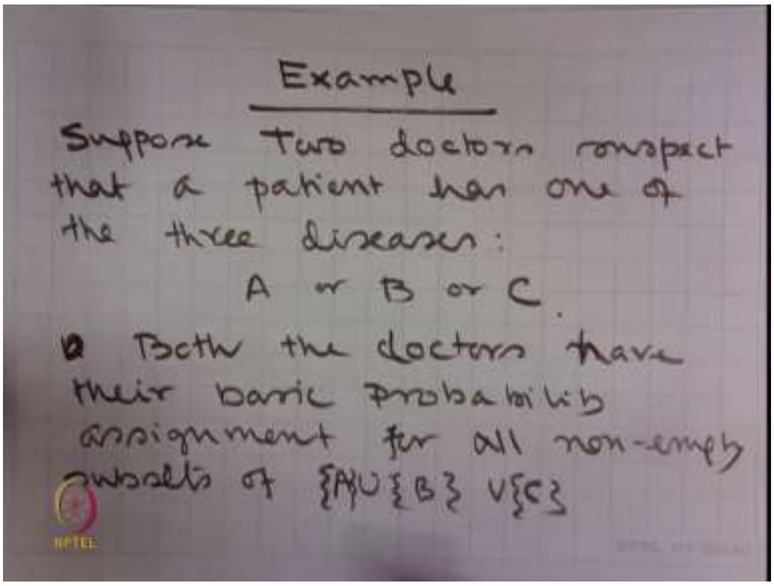

Example:

Suppose two doctors suspect that a patient has one of the three diseases

$$A \text{ or } B \text{ or } C$$

Both the doctors have their basic probability assignment for all non-empty subsets of $X = \{A, B, C\}$

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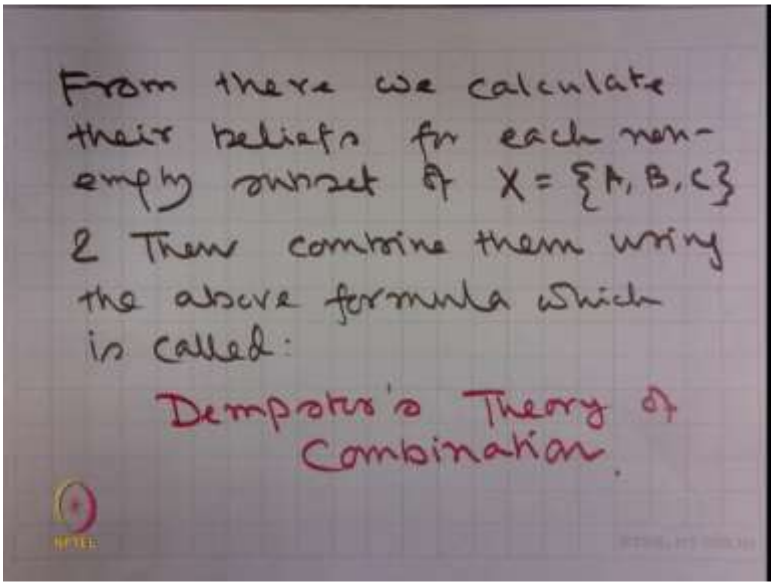

From there we calculate their $Beliefs$ for each non empty subset of the whole set $X = \{A, B, C\}$.

And then combine them using the about formula which is called

$$Dempster's\ Theory\ Of\ Combination.$$

There are other ways of combining also, but Dempster's Theory is one of the most popular one.

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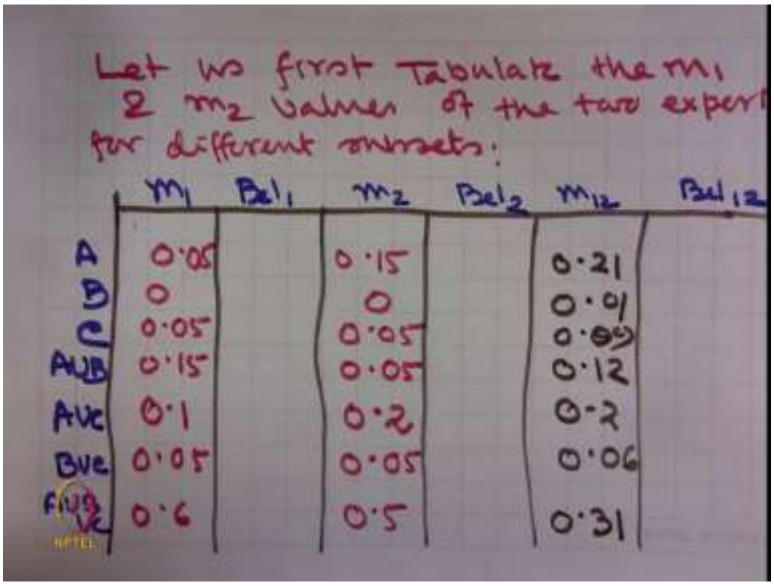

So, let us first tabulate the m_1 and m_2 values of the two experts for different subsets:

	m_1	Bel_1	m_2	Bel_2	m_{12}	Bel_{12}
A	0.05		0.15			
B	0		0			
C	0.05		0.05			
$A \cup B$	0.15		0.05			
$A \cup C$	0.1		0.2			
$B \cup C$	0.05		0.05			
$A \cup B \cup C$	0.6		0.5			

So, this shows that the evidences collected by the two doctors give rise to different $Basic\ Probability\ Assignment$

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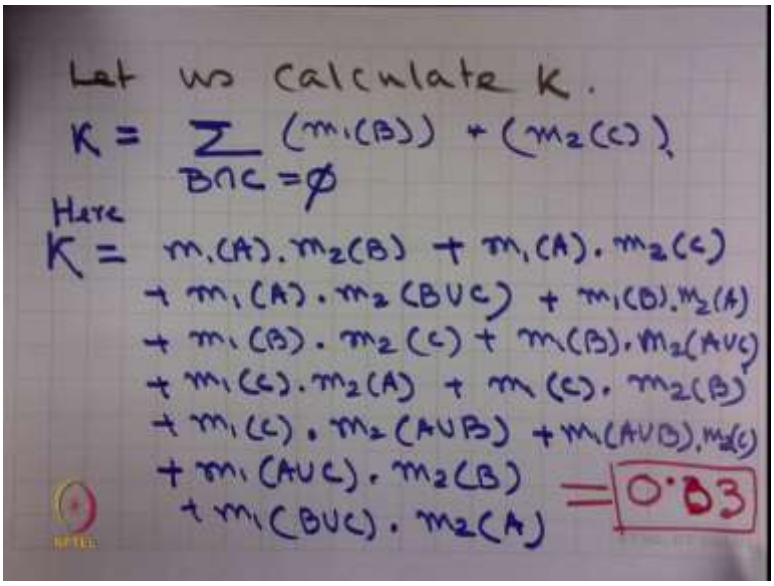

So, let us first calculate K

$$K = \sum_{B \cap C = \phi} m_1(B) \times m_2(C)$$

So, for this case we are going to have

$$K = m_1(A) \times m_2(B) + m_1(A) \times m_2(C) + m_1(A) \times m_2(B \cup C)$$

$$+ m_1(B) \times m_2(A) + m_1(B) \times m_2(C) + m_1(B) \times m_2(A \cup C)$$

$$+ m_1(C) \times m_2(A) + m_1(C) \times m_2(B) + m_1(C) \times m_2(A \cup B)$$

$$+ m_1(A \cup B) \times m_2(C) + m_1(A \cup C) \times m_2(B) + m_1(B \cup C) \times m_2(A)$$

So, you can see that the overall expression is pretty lengthy because corresponding to each subset of $P(X)$ we will have to look at all possible subsets, which have zero intersection with it and like that we have got all the possible things.

However, the overall value in this case is $K = 0.03$

So, that is going to be our K for the computation.

With this, we shall compute the different m_1 , m_2.

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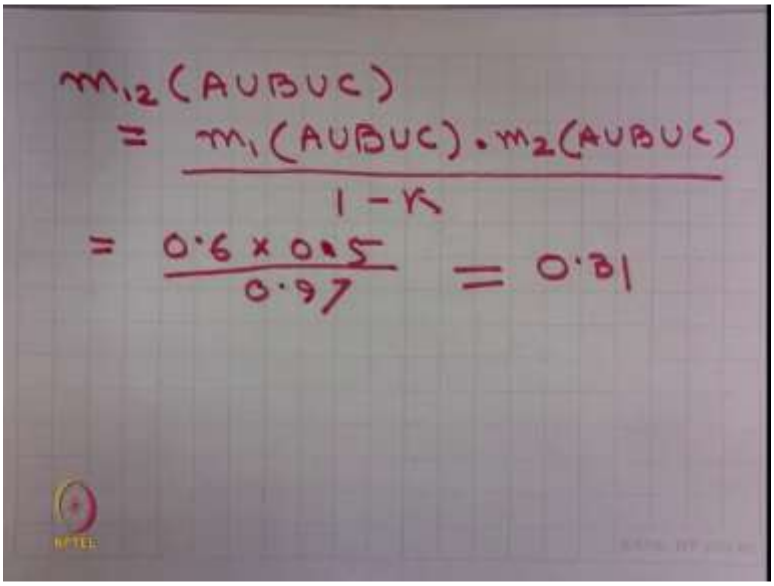

For example, let me illustrate

$$m_{12}(A \cup B \cup C) = \frac{m_1(A \cup B \cup C) \times m_2(A \cup B \cup C)}{1 - K} = \frac{0.6 \times 0.5}{0.97} = 0.31$$

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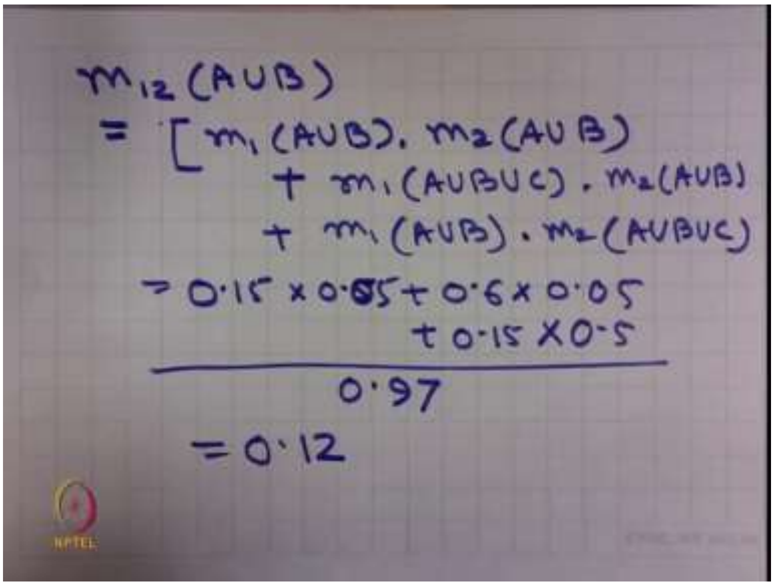

In a similar way,

$m_{12}(A \cup B)$

$$= \frac{m_1(A \cup B) \times m_2(A \cup B) + m_1(A \cup B \cup C) \times m_2(A \cup B) + m_1(A \cup B) \times m_2(A \cup B \cup C)}{1 - K}$$

$$= \frac{0.15 \times 0.05 + 0.6 \times 0.05 + 0.15 \times 0.5}{0.97} = 0.12$$

In a similar way, if we calculate all the values will get the following

	m_1	Bel_1	m_2	Bel_2	m_{12}	Bel_{12}
A	0.05		0.15		0.21	
B	0		0		0.01	
C	0.05		0.05		0.09	
$A \cup B$	0.15		0.05		0.12	
$A \cup C$	0.1		0.2		0.2	
$B \cup C$	0.05		0.05		0.06	
$A \cup B \cup C$	0.6		0.5		0.31	

Okay friends. I stop here today. In the next class, I shall tell you how to compute the belief from m and using that will calculate these three beliefs and then I shall continue with the concept of necessity and possibility. Thank you so much.