# Introduction of Fuzzy Sets Arithmetic & Logic Prof. Niladri Chatterjee Department of Mathematics Indian Institute of Technology-Delhi

# Lecture-29 Fuzzy Sets Arithmetic & Logic

Welcome students, To the MOOCs series of lectures on Fuzzy Sets Arithmetic and Logic. This is lecture number 29.

### (Refer Slide Time: 00:32)

So for anot are have done we handled uncertainty giviz membership net elements. in inhuitive com theory

So, far what we have done, we handled uncertainty by giving membership to different set elements.

- This is intuitively very easy to comprehend.
- And we have seen that a huge theory has been developed using Fuzzy Sets, Logic and Mathematics.

Over the last 28 lectures, we have discussed these things in detail.

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reality there may be 12UT IN mper 8 other uncertainty. person have a propecto Dengr Doctor evider in above. passibili he

But in reality there may be many other types of uncertainty. For example: Suppose a person has fever. And Doctor suspects:

- it can be Viral,
- it can be Dengue
- it can be Malaria.

Now, what the doctor will do, doctor will collect evidence in support of the above possibilities.

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the evidence is strong one a makin man RS the encen an a Grive G

If the evidence is very strong in favour of one of them then decision making is easy.

Say for example, if this is viral, this is malaria, and this is a dengue, and these are the different set of tests that he has conducted and if presence of this is means that disease is viral presence

of this means that disease is malaria and presence of this means it is that dengue then there is no problem.

But, many a time it may happen that the evidences are not so conclusive.

Question is how to arrive at a decision?

I hope you understand the point.

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Suppose a person has Faculty an institute. Suppose the applied formation lechnoto An expert will eval

Let me give one more example

Suppose a person has applied for faculty position in an institute.

And suppose he applied for

- computer science,
- mathematics and
- information technology

An expert will evaluate the application to take to assess the candidate's possible appointment. Now, depending upon the application.

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Depending upon the application there may be neveral options : Moninate for Comp Sc. Nominate (or math IT both CSEIT Like that for all powerble subsets of Ecompse., maths, IT3

There may be several options:

- nominate for computer science
- nominate for maths
- similarly, nomination for IT
- he may nominate for both CS and IT

Like that for all possible subsets of in this case the universal set is three items,

### {Computers Science, Maths, IT}

For each such subset of disparate set the evaluator has to find the possibility of nominating for one of them

### (Refer Slide Time: 09:22)

the possibility of nominating for one of them. Thus given X we are looking at P(x) and trying to gather powerbulity of clansifying an object into one of them.

Thus given X the universal set we are looking at P(X) and trying to gather possibility of classifying an object which in our case is a person into one of them.

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Note that: The decision boundaries are crist. The focus is not on to obbin the membership value for each subset of x i.e for each A = P(X) Pathen the aim is to collect evidence in mpport of each  $(A \in P(X))$ 

Note that

- The decision boundaries are crisp. It is one of these possible subsets that has to be nominated.
- The focus is not on to obtain the membership value for each subset of X that is for each
   A ∈ P(X)
- Rather the aim is to collect evidence in support of each  $A \in P(X)$

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The problem becomes more complicated if there are more than one person giving the judgement. Ex: Patient may consult multiple doctors. · Selection panel may have number of experts . In a court trial there may be particularly members. (-)

The problem becomes more complicated if there are more than one person giving the judgment. Example:

- Patient may consult multiple doctors
- Selection panel may have number of experts
- In a court trial there may be several jury members.

In each of this case, each concern person will have some opinion and they may differ among themselves. And therefore, we need to somehow combine the opinions or evidence is collected by the different experts to come into a conclusion.

So, that is the problem that we are taking up in this lecture. And, I will continue with that in the next class as well.

# (Refer Slide Time: 14:44)

Evidence Theory based upon two Belief Plausibility (PI) each member AEP -97 to calculate nociand coith

The theory that I will be discussing today is called Evidence Theory

- This is based upon two measures.
  - Belief which I will write as, Bel
  - *Plausibility* which I will write as *Pl*.

For each member  $A \in P(X)$ . We need to calculate the belief associated with it for each expert.

### (Refer Slide Time: 16:17)

TO, I n.+ member

So, let us talk about *Belief* Belief is a mapping from  $P(X) \rightarrow [0, 1]$  such that

 $Bel(\phi) = 0$ Bel(X) = 1

This is because  $\phi$  is also a member of P(X) and we cannot nominate  $\phi$  as the possible classification of the object.

For example:

- If the patient has a disease the doctor cannot say, No he does not have any disease.
- If a person is on trial, then the judge will have to take decision whether he is guilty or not guilty. That is the corresponding set here is only two members guilty and not guilty. Therefore, the *P*(*X*) will have 4 elements φ, guilty, not guilty and both guilty and not guilty.

Now, the judge will have to take a decision, it can be one of guilty not guilty and that can certainly not be  $\phi$  That is the judge cannot say that I won't take any decision. Just like a doctor cannot say no, this patient does not have any ailment. Therefore, corresponding to  $\phi$  the *Belief* associated is 0.

In a very similar line, since, the decision has to be one of the possible members of X or some subset of X, the belief of the entire set it has to be one.

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However Belief is not addition That in Bel(A) & Bel(B) + Bel(AUB) For example: A personia anitability for Maths may be sery high. Pont it does not mean that rouitability for Mathsor comp. Sc. in also high.

However, Belief s not additive.

That is,

$$Bel(A \cup B) \neq Bel(A) + Bel(B)$$

For example,

A person's suitability for maths maybe very high.

But, it does not mean that suitability for maths or computing or computer science is also high, which we can easily understand.

#### (Refer Slide Time: 21:08)

Therefore, Bel (Maths ) may be more than Bel (Mather U computing) The fact Bel((2A, A23) 7 Bel((2A13)) The fact Bel((2A1, A23)) The fact Bel((2A1, A23)) The fact Bel((2A1, A23)) The fact Bel((2A1, A23)) 2) Bel ({ A13) + Bel ({ A23) white - Bel ({ A13 ( { A23 })

And therefore *Bel*(*Maths*) maybe more than *Bel*(*Maths* ∪ *Computing*)

I hope the concept is clear and

Therefore,  $Bel(\{A_1, A_2\}) \neq Bel(\{A_1\}) + Bel(\{A_2\})$ In fact,  $Bel(\{A_1, A_2\}) \geq Bel(\{A_1\}) + Bel(\{A_2\}) - Bel(\{A_1\} \cap \{A_2\})$  You have seen similar formula with respect to probability, but in that case, it was an equality. But, in case of evidence, this is inequality. And because of that, belief is super additive, not just additive, it is super additive, because it exceeds that value.

#### (Refer Slide Time: 23:38)

- general: el(A, UA2U····UAn) Z Bel (Az) Z Bel (Az n As). Z Bel (Az n As). + C) Bel (AnnAs --- nA)

In general,

$$Bel(A_1 \cup A_2 \cup \dots \cup A_n)$$
  

$$\geq \sum_i Bel(A_i) - \sum_{i < j} Bel(A_i \cap A_j) \dots + (-1)^{n-1} Bel(A_1 \cap A_2 \cap \dots \cap A_n)$$

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Thus Belief function associates with each subset of X a numeric value 0 81 which denotes degree of belief on the barris of the collected evidence

Thus, belief function associates with each subset of X, a numeric value between 0 and 1 which denotes the degree of belief on the basis of the collected evidence.

#### (Refer Slide Time: 25:54)

Some proper his AS B them BU(B) > DU(A) Since ACB 3 C AUC = B Bel(B) = Belief (AVC) Bel (A) + Bel (C) - Bel (A n c)

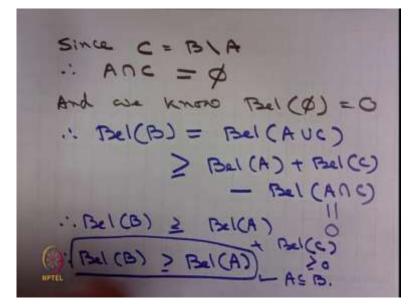
Some properties,

1. If  $A \subseteq B$  then,  $Bel(B) \ge Bel(A)$ 

Proof:

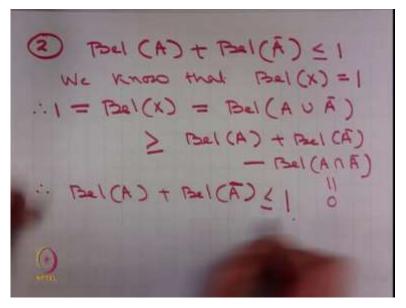
Since  $A \subset B \quad \exists C$  such that  $A \cup C = B$  $\therefore Bel(B) = Bel(A \cup C) \ge Bel(A) + Bel(C) - Bel(A \cap C)$ 

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Since  $C = B \setminus A$ Therefore,  $A \cap C = \phi$ And we know that  $Bel(\phi) = 0$ Therefore,  $Bel(B) = Bel(A \cup C) \ge Bel(A) + Bel(C) - Bel(A \cap C)$ Since,  $Bel(A \cap C) = 0$  Therefore,  $Bel(B) \ge Bel(A) + Bel(C)$ Now,  $Bel(C) \ge 0$ Therefore,  $Bel(B) \ge Bel(A)$ Note, we started with  $A \subseteq B$ 

### (Refer Slide Time: 29:00)



2.  $Bel(A) + Bel(\bar{A}) \le 1$ 

We know that Bel(X) = 1Therefore,  $1 = Bel(X) = Bel(A \cup \overline{A}) \ge Bel(A) + Bel(\overline{A}) - Bel(A \cap \overline{A})$ Since,  $Bel(A \cap \overline{A}) = Bel(\phi) = 0$ Therefore,  $Bel(A) + Bel(\overline{A}) \le 1$ 

(Refer Slide Time: 30:18)

An appociated term with Belief in Plannihility (PI)  $P(A) = I - Bal(\overline{A})$  $P(\overline{A}) = I - Bal(\overline{A}).$ Thus Belief and Plausibility are closely related to each other FTER

An associated, term with *Belief* is *Plausibility*. As I indicated earlier we will denote it by *Pl*. By definition:

 $Pl(A) = 1 - Bel(\overline{A})$  and  $Pl(\overline{A}) = 1 - Bel(A)$ 

Thus, *Belief* and *Plausibility* are closely related to each other.

(Refer Slide Time: 31:40)

Pumili:  
PI(ANB) 
$$\leq$$
 PI(A) + PI(B) - PI(AVB)  
PI: (Bel(AUB)  $\geq$  Bel(A) + Bel(B)  
- Bel(ANB)  
Replacing A mg  $\overline{A} \otimes B$  mg  $\overline{B}$   
Del( $\overline{A}U\overline{B}$ )  $\geq$  Del( $\overline{A}$ ) + Bel( $\overline{B}$ )  
- Bel( $\overline{A}N\overline{B}$ )  
 $=$  Bel( $\overline{A}N\overline{B}$ )  
 $=$  Del( $\overline{A}N\overline{B}$ )  
 $=$  Del( $\overline{A}N\overline{B}$ )  
 $=$  Del( $\overline{A}U\overline{B}$ )

Result:

 $Pl(A \cap B) \leq Pl(A) + Pl(B) - Pl(A \cup B)$ Proof: We have  $Bel(A \cup B) \geq Bel(A) + Bel(B) - Bel(A \cap B)$ Replacing A by  $\overline{A}$  and B by  $\overline{B}$  $Bel(\overline{A} \cup \overline{B}) \geq Bel(\overline{A}) + Bel(\overline{B}) - Bel(\overline{A} \cap \overline{B})$ Using De Morgan's Law

$$Bel(\overline{A \cap B}) \ge Bel(\overline{A}) + Bel(\overline{B}) - Bel(\overline{A \cup B})$$

(Refer Slide Time: 34:01)

.. Since PI(A) = 1 - Bel(A) From the above use get 1-PICANB) 2 1-PICAD + I - PI(B) - (I - PI(AUB)) - PI(AOD) + PI(A) + PI(B)> PI(AUB) or PI(ANB) ≤ PI(A) + PI(B) - PI (AUB)

Therefore, Since  $Pl(A) = 1 - Bel(\overline{A})$ 

From the above, we get

$$1 - Pl(A \cap B) \ge 1 - Pl(A) + 1 - Pl(B) - (1 - Pl(A \cup B))$$
  
Or  $-Pl(A \cap B) + Pl(A) + Pl(B) \ge Pl(A \cup B)$   
Or  $Pl(A \cap B) \le Pl(A) + Pl(B) - Pl(A \cup B)$ 

(Refer Slide Time: 35:50)

The above semit can be generally  
an:  

$$PI(A, \cap A_2 \cap \cdots \cap A_n)$$
  
 $\leq \sum PI(A_1)$   
 $= \sum P(A_1 \cup A_2)$   
 $= \sum P(A_1 \cup A_2)$   
 $= \sum I (A_1 \cup A_2)$   
 $= I$ 

The above result can be generalized as

$$Pl(A_1 \cap A_2 \cap \dots \cap A_n)$$
  
$$\leq \sum_i Pl(A_i) - \sum_{i < j} Pl(A_i \cup A_j) + \dots + (-1)^{n-1} Pl(A_1 \cup A_2 \cup \dots \cup A_n)$$

(Refer Slide Time: 36:54)

Note that C) PICA) + PICA) > 1  $(\phi) = 1 - Bel(x)$ = 1 - Bel (\$) 3) PI(X)

Note that

- 1.  $Pl(A) + Pl(\overline{A}) \ge 1$
- 2.  $Pl(\phi) = 1 Bel(X) = 0$
- 3.  $Pl(X) = 1 Bel(\phi) = 1$

(Refer Slide Time: 37:37)

Budshim is have to compute Bel 2 Pl for any given romonat A of X. The barrie concept in of " Basic Probability Anignment" ashich typically in denoted m AE POX) (x) -> [0, [] () =0 -

Question is How to compute *Belief* and *Plausibility* for any given subset A of X.

The basic concept here is of *Basic Probability Assignment* which typically is denoted by m such that:

 $m {:} P(X) \to [0,1]$ 

 $m(\phi) = 0$  and  $\sum_{A \in P(X)} m(A) = 1$ 

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Diskinckan bet" pdf on X 2 Barric Probabilits anoignment. that the latter on 5 anociated with all possible subset ashile odd amolian usill elements all

The distinction should be very clear any probability distribution function on X and basic probability assignment is that the latter one is associated with all possible subsets of X. While a probability distribution function is associated with all the elements of X.

So, this has to be remembered very carefully.

### (Refer Slide Time: 40:35)

.. m(x) is not necessarily = 1 If ACD => m(A) <m(B) m(A) & m(A) do no have any selational differs J Probatoils ty

And therefore, m(X) is not necessarily equal to 1.

- $m(X) \neq 1$
- If  $A \subset B \Rightarrow m(A) \le m(B)$
- m(A) and  $m(\overline{A})$  do not have any relationship.

Although with respect to probability, we know that:

- P(X) = 1
- If  $A \subset B \Rightarrow P(A) \le P(B)$
- $P(A) + P(\overline{A}) = 1$

So, m differs from probability, we have to remember this thing because this is conceptually different from what you have learned in your probability class.

(Refer Slide Time: 42:11)

How to define But & PI.  $Bel(A) = \sum_{B \subseteq A} m(B)$   $B(A) = \sum_{B \cap A \neq \emptyset} m(B)$   $B|B \cap A \neq \emptyset$ WFTEL

So, how to define *Belief* and *Plausibility* that definition is as follows:

$$Bel(A) = \sum_{B \subseteq A} m(B)$$
$$Pl(A) = \sum_{B \mid B \cap A \neq \phi} m(B)$$

(Refer Slide Time: 43:02)

Therefore; m(A) characterizes the degree of Belief that the object belongs to the net Pel(A) characterizes the total idence the object that some of its or evidence that object some swos A, and OVETIMPIY

Therefore,

- *m*(*A*) characterizes the degree of *Belief* that the object to be classified belongs to the set *A*
- Bel(A) characterizes the total evidence that the object belongs to A or some of its subset
- *Pl(A)* gives evidence that object belongs to some subset of *A* and additional *Belief* or any set overlapping with *A*.

So, that is the basic characterization of the 3 measures:

- Plausibility
- Belief
- *m*

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The question in hero to comprise if use have evidences from multiple pources. For example : suppose m, 2 m2 are barrie probability arrighment given by two experts. tors to combine them?

The question is how to combine if we have evidences from multiple sources? For example:

Suppose  $m_1$  and  $m_2$  are the *Basic Probability Assignment* given by two experts. Our problem is how to combine them?

(Refer Slide Time: 46:59)

Let the basic joint anignment be miz (A) ashere A E P(X)  $\frac{1}{2}(A) = \sum_{m_1(B) * m_2(c)} Bnc = A$ BAC = A k is a normalizatim factor which in word to dormalize the roum of miz for il pubblets = 1

So, let the basic joint assignment be  $m_{12}(A)$  where,  $A \in P(X)$ 

$$\therefore m_{12}(A) = \frac{\sum_{B \cap C = A} m_1(B) \times m_2(C)}{1 - K}$$

What is *K*?

K is a normalization factor which is used to normalize that the sum of  $m_{12}$  for all subsets is equal to 1.

#### (Refer Slide Time: 48:43)

This normalization is required since I miz(A) should be 1 Since AE POX) AZØ " are calculate K m(B) \* m2(c)

We know that this normalization is required.

$$: \sum_{A \in P(X) \mid A \neq \phi} m_{12}(A) = 1$$

Therefore, we calculate

$$K = \sum_{B \cap C = \phi} m_1(B) \times m_2(C)$$

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Example two doctors Suppose rowspect a patient has one of that the three diseases: or B or C A Both the doctors nave 0 their baric probabilis appignment for all non-empt losels of AUSB3 VECZ

Example:

Suppose two doctors suspect that a patient has one of the three diseases

A or B or C

Both the doctors have their basic probability assignment for all non-empty subsets of  $X = \{A, B, C\}$ 

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From there we calculate their beliets for each nonempty owned of X= SM, B, C3 2 Them compoins them using the above formula which is called: Dempotro's Theory Combination

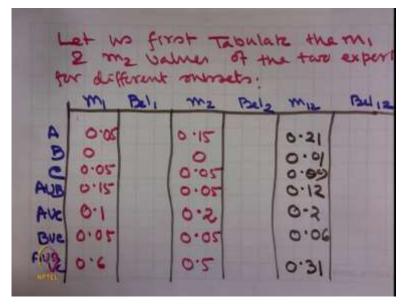
From there we calculate their *Beliefs* for each non empty subset of the whole set  $X = \{A, B, C\}$ .

And then combine them using the about formula which is called

Dempster's Theory Of Combination.

There are other ways of combining also, but Dempster's Theory is one of the most popular one.

(Refer Slide Time: 53:14)



	$m_1$	$Bel_1$	$m_2$	$Bel_2$	$m_{12}$	$Bel_{12}$
Α	0.05		0.15			
В	0		0			
С	0.05		0.05			
$A \cup B$	0.15		0.05			
$A \cup C$	0.1		0.2			
$B \cup C$	0.05		0.05			
$A \cup B \cup C$	0.6		0.5			

So, let us first tabulate the  $m_1$  and  $m_2$  values of the two experts for different subsets:

So, this shows that the evidences collected by the two doctors give rise to different *Basic Probability Assignment* 

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et us calculate K.  $C = Z (m(B)) + (m_2(C)),$ BRC = pm.(A). m2(B) + m, (A). m2(c) + m, (A). m2(BUC) + m, (D). m2(A) + m. (B). m2 (c) + m. (B). m2 (AUC) + m(c). m2(A) + m (c). m2(B) + mile) . ma (AUB) + milAUD), males + mi (AUC). m2(B) + mi (BUC). m2(A)

So, let us first calculate K

$$K = \sum_{B \cap C = \phi} m_1(B) \times m_2(C)$$

So, for this case we are going to have

$$K = m_1(A) \times m_2(B) + m_1(A) \times m_2(C) + m_1(A) \times m_2(B \cup C)$$
  
+ $m_1(B) \times m_2(A) + m_1(B) \times m_2(C) + m_1(B) \times m_2(A \cup C)$   
+ $m_1(C) \times m_2(A) + m_1(C) \times m_2(B) + m_1(C) \times m_2(A \cup B)$   
+ $m_1(A \cup B) \times m_2(C) + m_1(A \cup C) \times m_2(B) + m_1(B \cup C) \times m_2(A)$ 

So, you can see that the overall expression is pretty lengthy because corresponding to each subset of P(X) we will have to look at all possible subsets, which have zero intersection with it and like that we have got all the possible things. However, the overall value in this case is K = 0.03So, that is going to be our *K* for the computation.

With this, we shall compute the different  $m_1$ ,  $m_2$ .

#### (Refer Slide Time: 59:48)

(AUBUC) m, (AUBUC) . m2 (AUBUC) I-K 0.6 × 0.5

For example, let me illustrate

$$m_{12}(A \cup B \cup C) = \frac{m_1(A \cup B \cup C) \times m_2(A \cup B \cup C)}{1 - K} = \frac{0.6 \times 0.5}{0.97} = 0.31$$

(Refer Slide Time: 01:01:13)

miz (AUB) = [m, (AUB). m2 (AUB) + m, (AUBUC). m2 (AUB) + m. (AUB) . M. (AUBUC) = 0.12 × 0.02+ 0.6× 0.02 2-0X 21-0 + 0.97 = 0.12

In a similar way,

$$\begin{split} m_{12}(A \cup B) \\ &= \frac{m_1(A \cup B) \times m_2(A \cup B) + m_1(A \cup B \cup C) \times m_2(A \cup B) + m_1(A \cup B) \times m_2(A \cup B \cup C)}{1 - K} \\ &= \frac{0.15 \times 0.05 + 0.6 \times 0.05 + 0.15 \times 0.5}{0.97} = 0.12 \end{split}$$

	$m_1$	$Bel_1$	$m_2$	$Bel_2$	$m_{12}$	$Bel_{12}$
Α	0.05		0.15		0.21	
В	0		0		0.01	
С	0.05		0.05		0.09	
$A \cup B$	0.15		0.05		0.12	
$A \cup C$	0.1		0.2		0.2	
$B \cup C$	0.05		0.05		0.06	
$A \cup B \cup C$	0.6		0.5		0.31	

In a similar way, if we calculate all the values will get the following

Okay friends. I stop here today. In the next class, I shall tell you how to compute the belief from m and using that will calculate these three beliefs and then I shall continue with the concept of necessity and possibility. Thank you so much.