**Introduction to Fuzzy Sets Arithmetic & Logic**
**Prof. Niladri Chetterjee**
**Department of Mathematics**
**Indian Institute of Technology – Delhi**

**Lecture – 28**
**Fuzzy Sets Arithmetic & Logic**

Welcome students to MOOCS lecture on Fuzzy Sets Arithmetic and Logic.

This is lecture number 28. And I discussed earlier that in today's class, we shall discuss Mamdani style Fuzzy inferencing.

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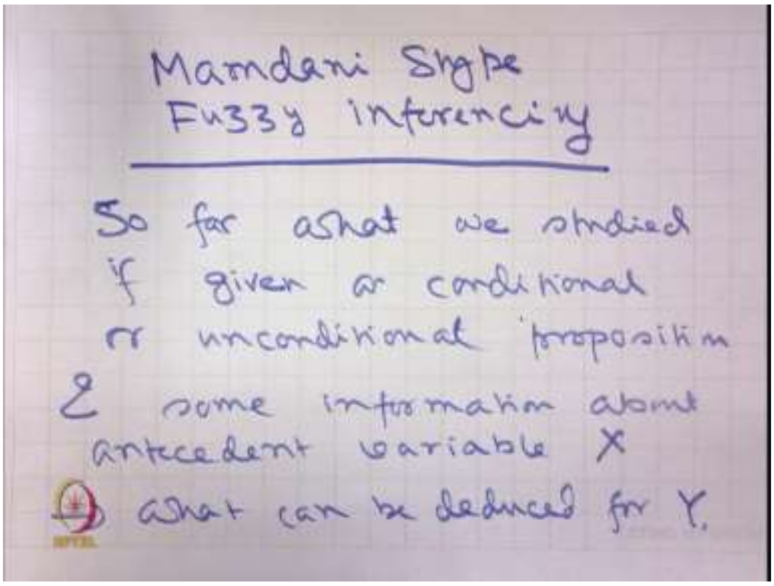


If you recall, so far what we studied is given a conditional or unconditional proposition and some information about antecedent variable $X$, what can be deduced for $Y$.

And we obtained a fuzzy set $B'$ for corresponding consequent variable $Y$.

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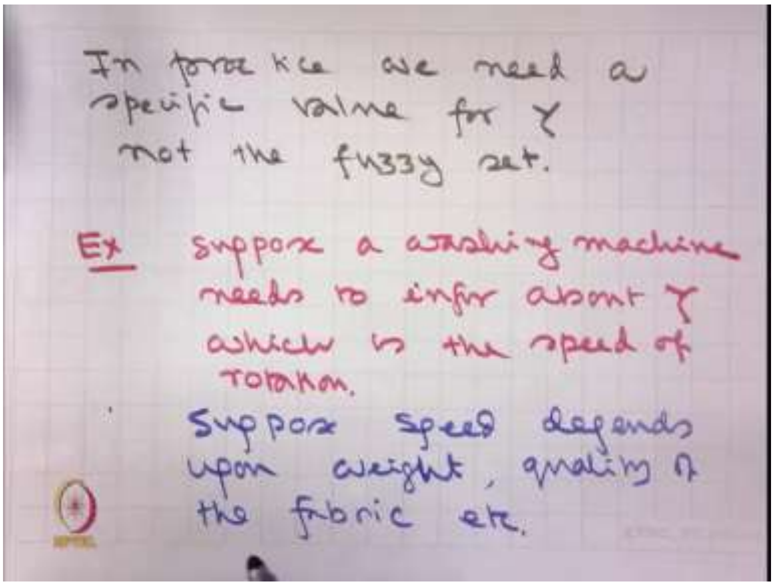


In practice, we need a specific value for $Y$, not the fuzzy set.

For example: Suppose a washing machine needs to infer about $Y$, which is the speed of rotation.

And suppose it depends upon several factors such as weight of cloth, quality of the fabric, etc.

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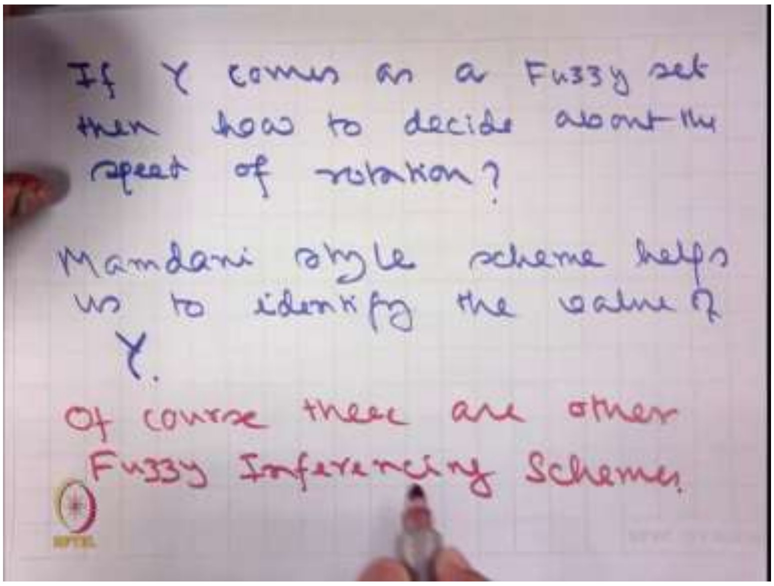


If $Y$ comes as a fuzzy set then, how to decide about the speed of rotation.

So, you understand the problem we got from the influencing a fuzzy set $B'$, which should give the membership for different values of $Y$, but we need to choose one.

How to choose that?

So, Mamdani style scheme helps us to identify the value of $Y$.

Of course, there are other Fuzzy Inferencing schemes.

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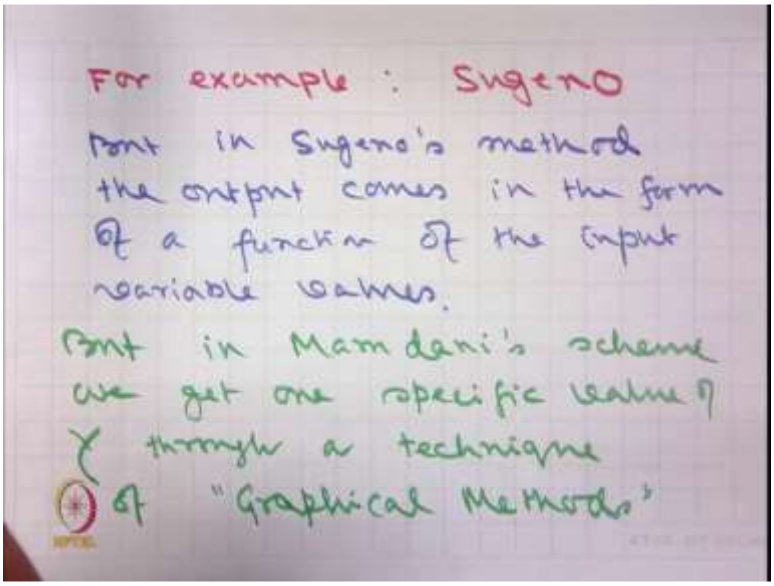


For example, Sugeno’s methods is very well known one but, in Sugeno’s method the output comes in the form of a function of the input variable values.

But in Mamdani’s scheme we get one specific value of $Y$ through a technique of graphical methods.

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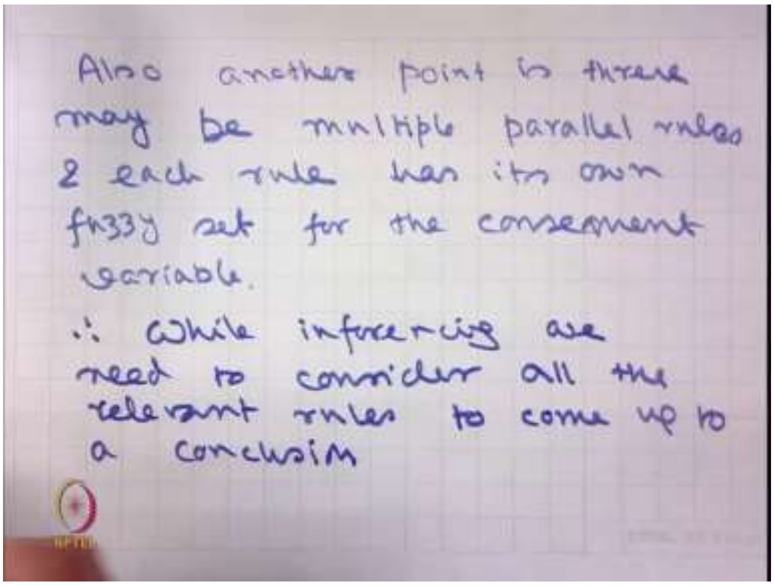


Also, another point is there may be multiple parallel rules and each rule has its own fuzzy set for the consequent variable.

Therefore, while inferencing we need to consider all the relevant rules to come up to a conclusion.

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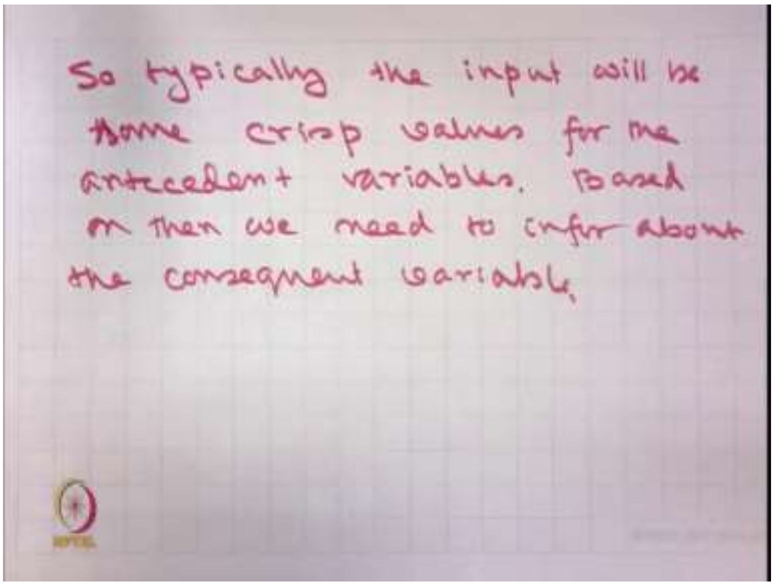


So, typically the input will be the crisp values for the antecedent variables. Based on them we need to infer about the consequent variable.

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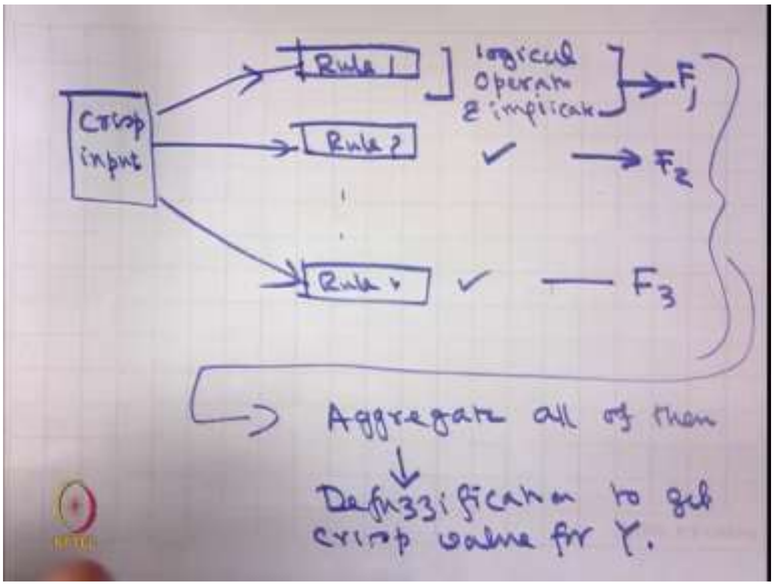


Pictorially, we have crisp inputs there may be rule 1 rule 2… rule n and in each rule we need logical operator and implication. Same is here, same is for them.

Based on that we get some fuzzy membership for some set $F_1$, this give for some set $F_2$ and this gives for some set $F_n$.

Then we need to aggregate all of them and from there we use defuzzification to get it crisp value for that consequent variable $Y$.

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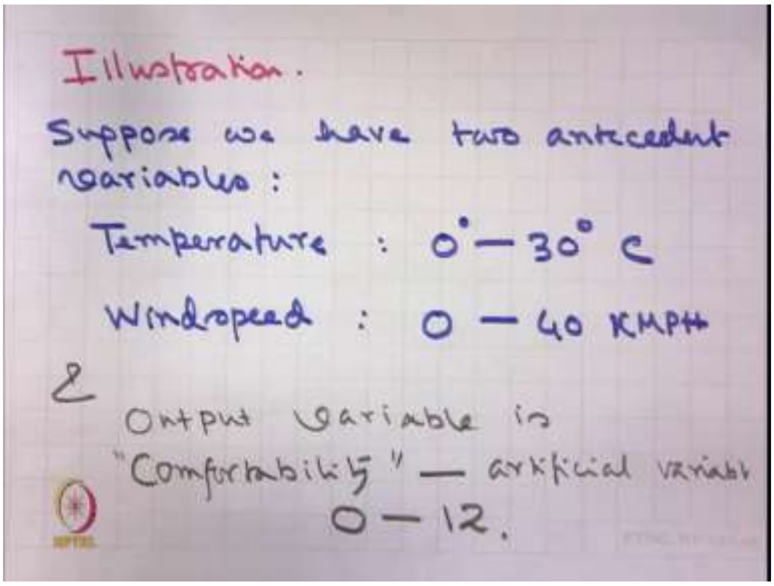


So, let us illustrate with an example.

Suppose we have two antecedent variables one is the $temperature$ which we assume is in the range, $0 - 30$℃, another is $wind\ speed$ and which can set between $0 - 40\ Kmph$ and output variable is what we call $comfortability$ of the weather which is a artificial variable, taking values in the range $0 - 12$

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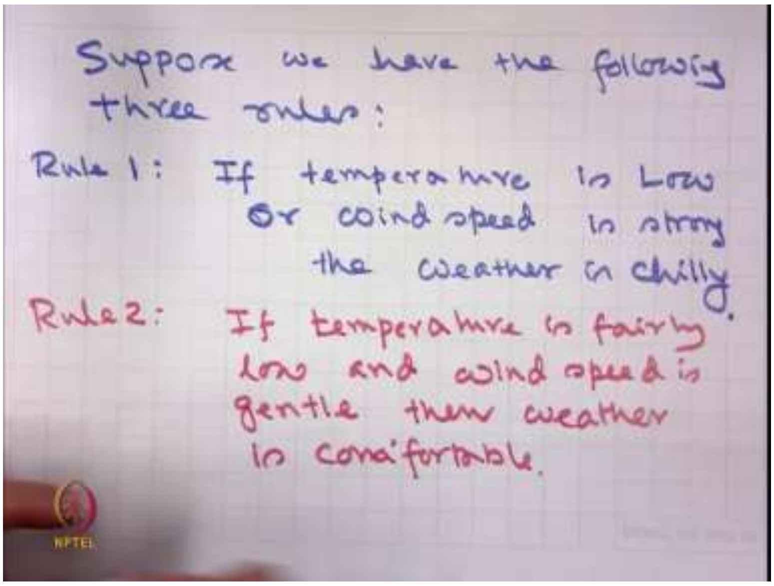


Suppose, we have the following three rules:

- Rule 1: *If temperature is low or wind speed is strong then, weather is chilly.*
- Rule 2: *If temperature is fairly low and wind speed is gentle then weather is comfortable.*

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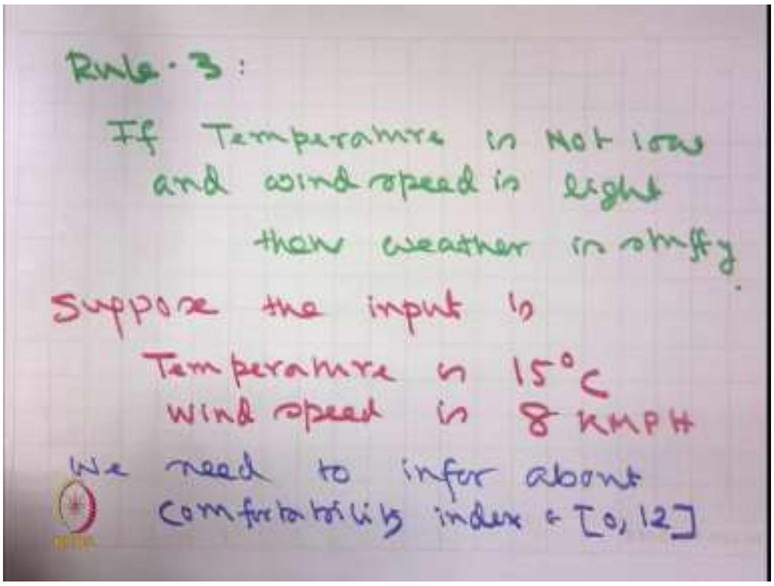


- Rule 3: *If temperature is not low and wind speed is light then weather is stuffy.*

Suppose the given input is temperature is 15℃, wind speed is 8 $Kmph$ and we need to infer about the comfortability index ∈ $[0, 12]$. So, I hope the problem is clear.

We have the input temperature 15℃ and wind speed is 8 $Kmph$.

The problem is we have three different rules.

Each rule has something to say about temperature and wind speed and each rule gives a value for the consequent variable which is the weather comfortability index and that comes in different rules in different ways.

Therefore, question comes, how to infer for that given input?

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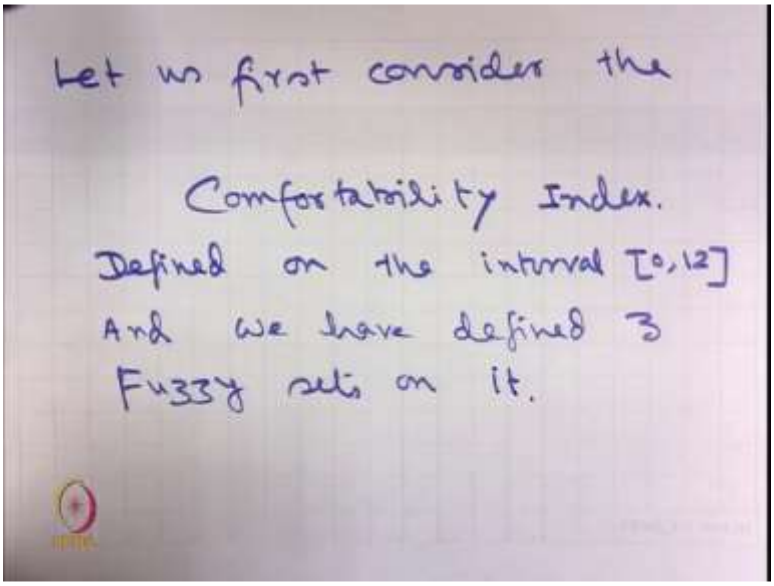


So, let us first consider *comfortability index*. It is defined on the interval $[0, 12]$ and we have defined three Fuzzy sets on it.

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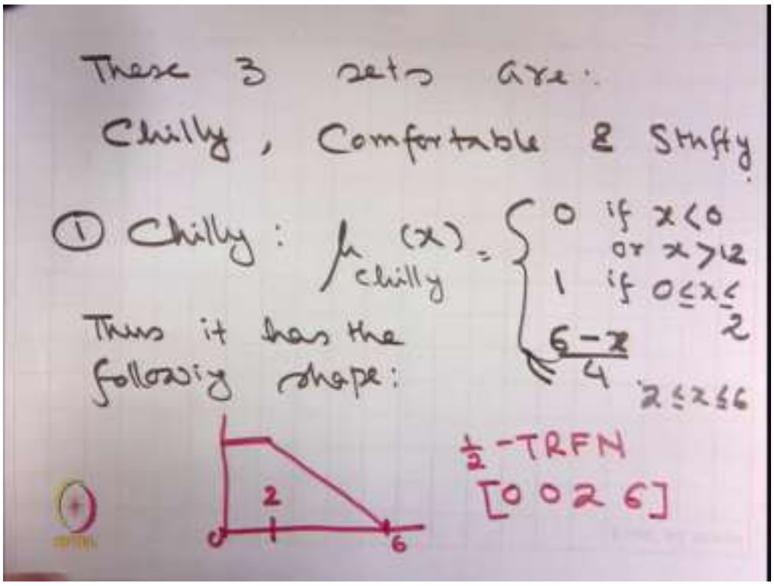


These 3 sets are $chilly, comfortable$ and $stuffy$.

So, let us define

1. $chilly$

$$\mu_{chilly}(x) = \begin{cases} 0 & if\ x < 0\ or\ x > 6 \\ 1 & if\ 0 \leq x \leq 2 \\ \frac{6-x}{4} & if\ 2 \leq x \leq 6 \end{cases}$$

We shall call it $half\ TrFN$ and we may denote it as [0  0  2  6] .

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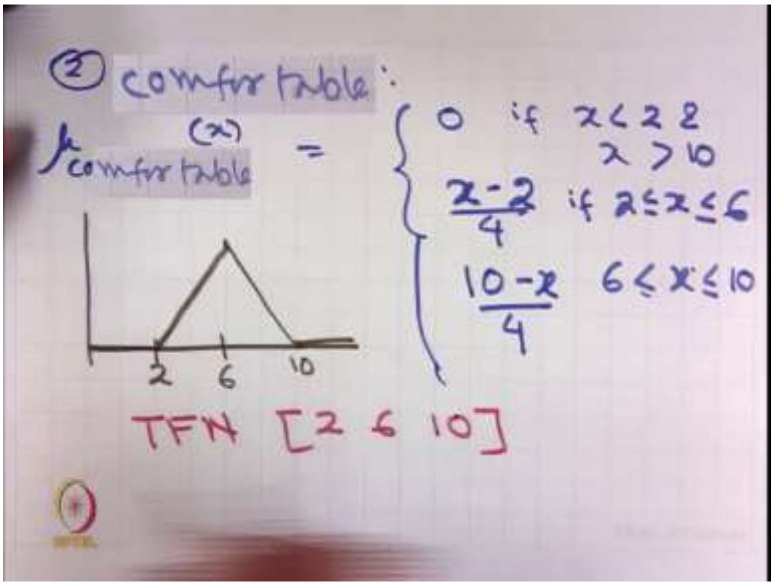


In a similar way,

2. $comfortable$

$$\mu_{comfortable}(x) = \begin{cases} 0 & if\ x < 2\ or\ x > 10 \\ \frac{x-2}{4} & if\ 2 \le x \le 6 \\ \frac{10-x}{4} & if\ 6 \le x \le 10 \end{cases}$$

So, this is going to be $TFN[2 \quad 6 \quad 10]$

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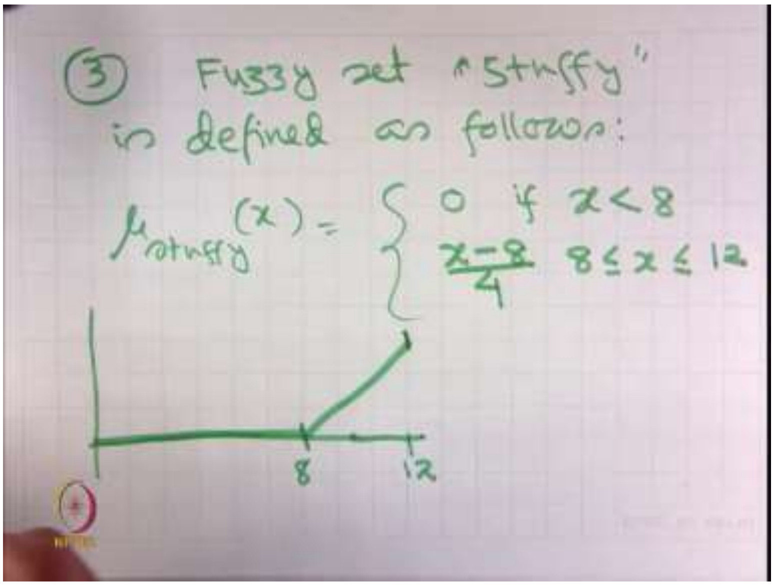


3. The fuzzy set $stuffy$ is defined as follows:

$$\mu_{stuffy}(x) = \begin{cases} 0 & if\ x < 8 \\ \frac{x-8}{4} & 8 \le x \le 12 \end{cases}$$

It is going to be linearly increasing in the interval $[8, 12]$. Therefore, it has the shape something like this.

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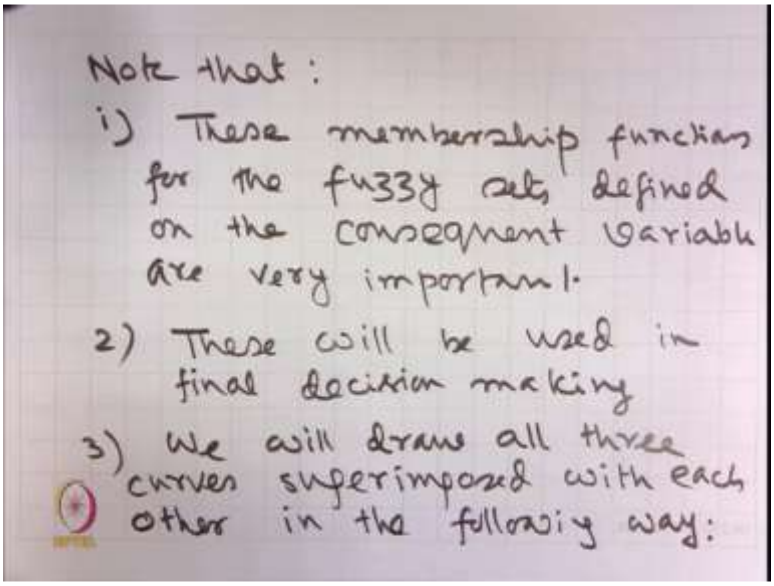


Note that:

i. These membership functions for the fuzzy sets defined on the consequent variable are very important
ii. These will be used in the final decision making.
iii. We will draw all three curves super imposed with each other in the following way:

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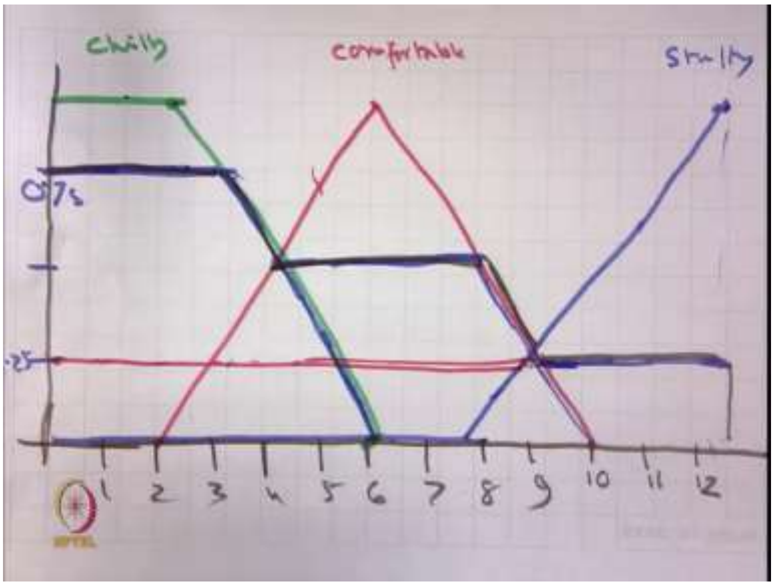


So, let us now draw the three curves.

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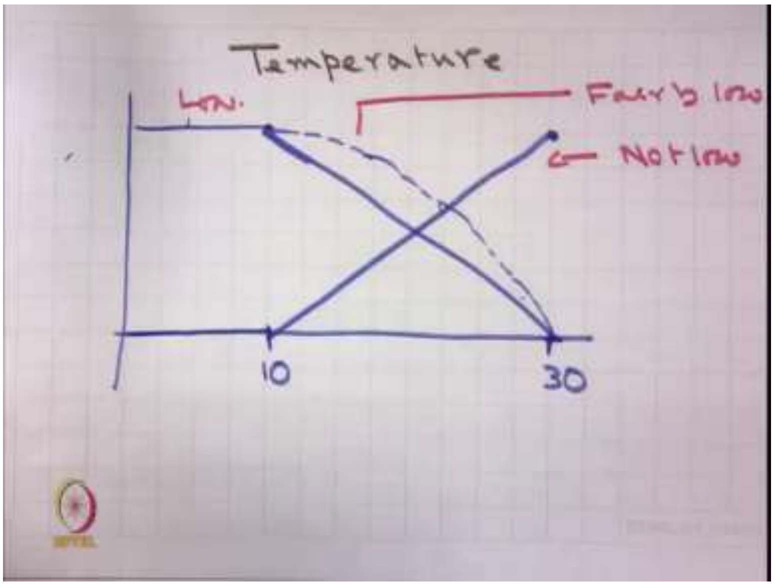


Let us now look at temperature and suppose we have the following:

Between 0 to 10 for low it is 1 and from 10 to 30 it is linear.

Therefore, fairly low as we know, we can use the square root and we may get something like this and not low this line.

So this is not low, this is fairly low and this is low.

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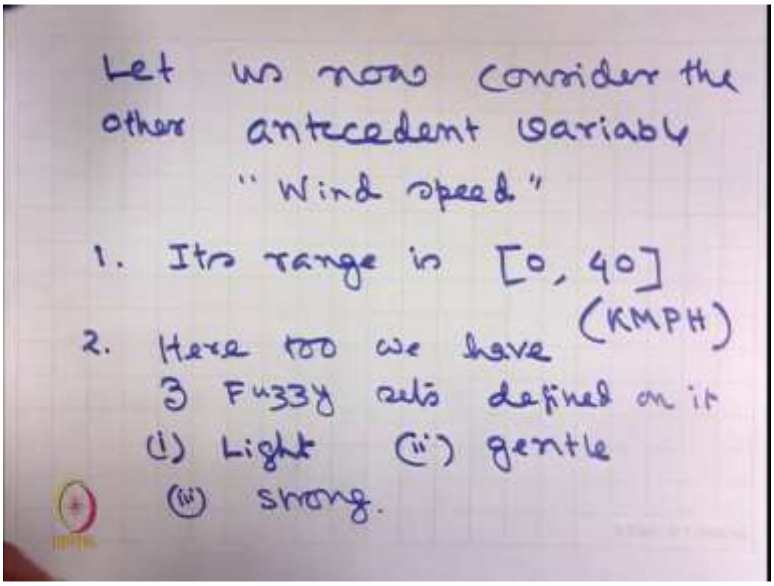


Let us now consider, the other antecedent variable, $wind\ speed$

1) its range is $[0, 40]$. Basically they are kilometres per hour.
2) Here, too we have 3 fuzzy sets defined on it which are
    a. light
    b. gentle
    c. strong.

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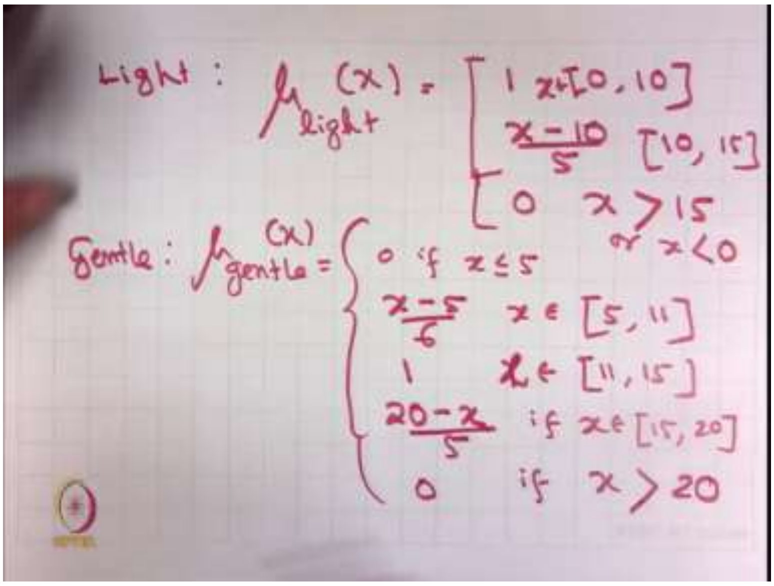


Let us now, look at this suppose, $light$ is defined as follows.

$$\mu_{light}(x) = \begin{cases} 0 & if\ x < 0\ or\ x > 15 \\ 1 & if\ 0 \le x \le 10 \\ \frac{x-10}{5} & if\ 10 \le x \le 15 \end{cases}$$

Similarly, $gentle$ is defined as follows

$$\mu_{gentle}(x) = \begin{cases} 0 & if\ x \leq 5\ or\ x > 20 \\ \frac{x-5}{6} & if\ \ 5 \leq x \leq 11 \\ 1 & if\ 11 \leq x \leq 15 \\ \frac{20-x}{5} & if\ 15 \leq x \leq 20 \end{cases}$$

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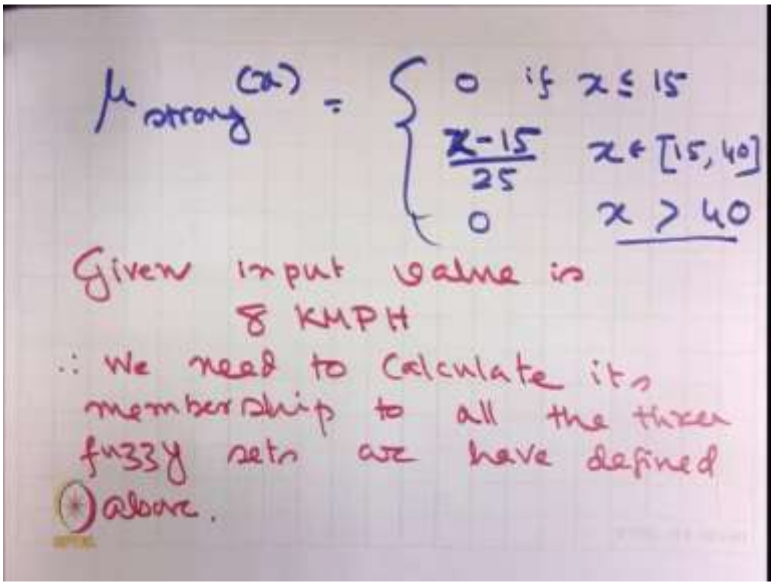


$$\mu_{strong}(x) = \begin{cases} 0 & if\ x \leq 15 \\ 0 & if\ x \geq 40 \\ \frac{x-15}{25} & if\ 15 \leq x \leq 40 \end{cases}$$

Now, our given input value is $8Kmph$. Therefore, we need to calculate its membership to all the three fuzzy sets we have defined above.

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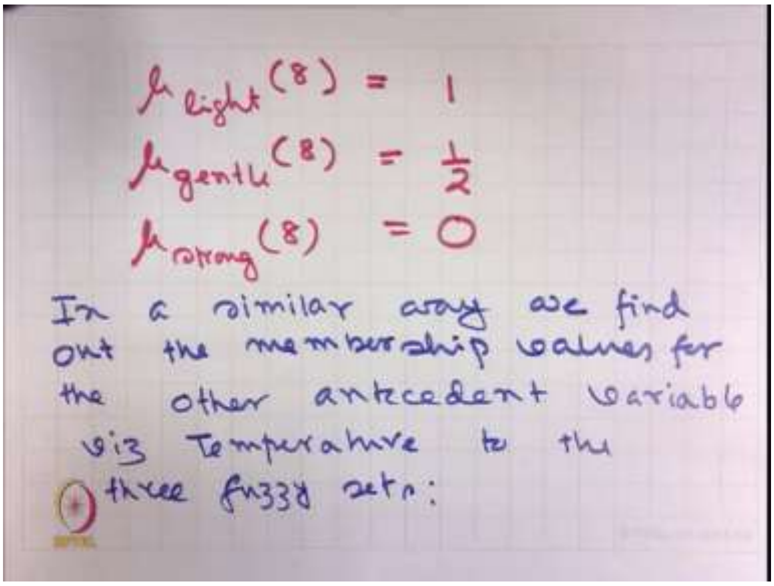

And we have

$$\mu_{light}(8) = 1$$

$$\mu_{gentle}(8) = \frac{1}{2}$$

$$\mu_{strong}(8) = 0$$

In a similar way, we find out the membership values for the antecedent variable namely $temperature$ to the three fuzzy sets.

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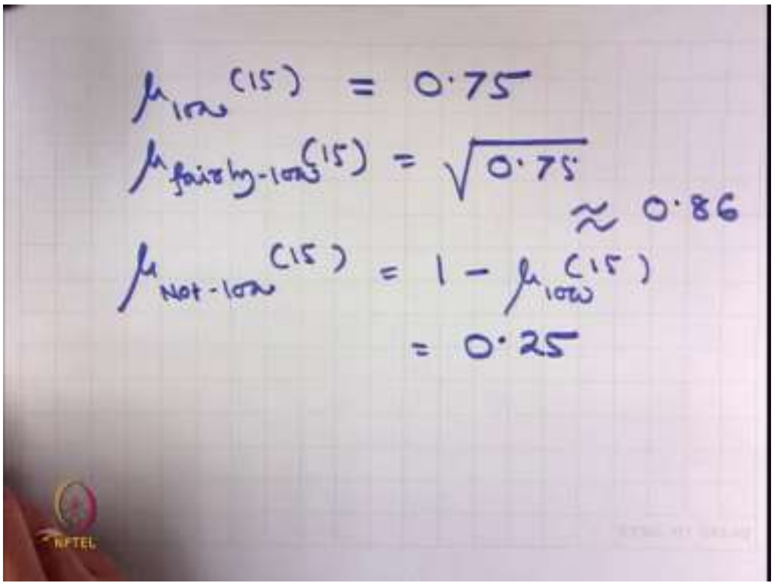


And we have the following

$$\mu_{low}(15) = 0.75$$

$$\mu_{fairly\ low}(15) = \sqrt{0.75} \approx 0.86$$

$$\mu_{not\ low}(15) = 1 - \mu_{low}(15) = 0.25$$

Now, we use these values for the 3 rules that we have in the domain and let us see how we apply Mamdani's scheme for coming into a decision.

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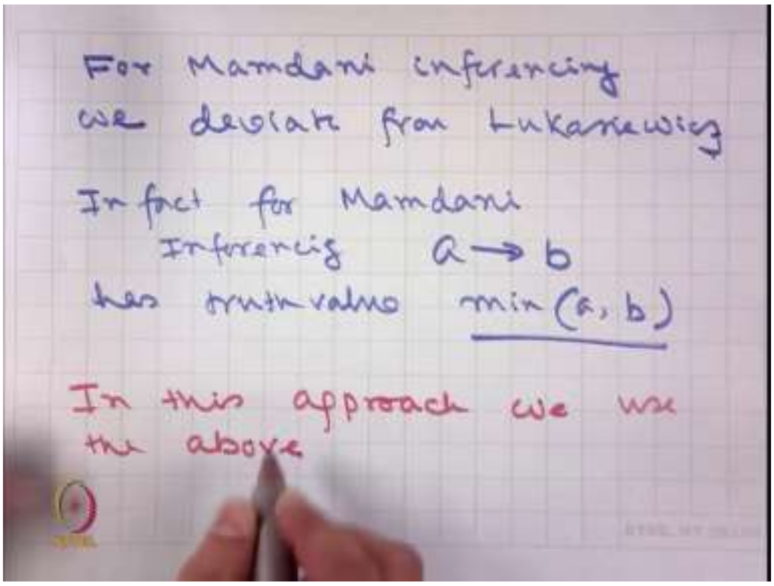


For, Mamdani inferencing we deviate from Lukasiewicz.

In fact, for Mamdani Inferencing

$a \to b$ has truth value $\min(a, b)$

So, in this approach we use the above.

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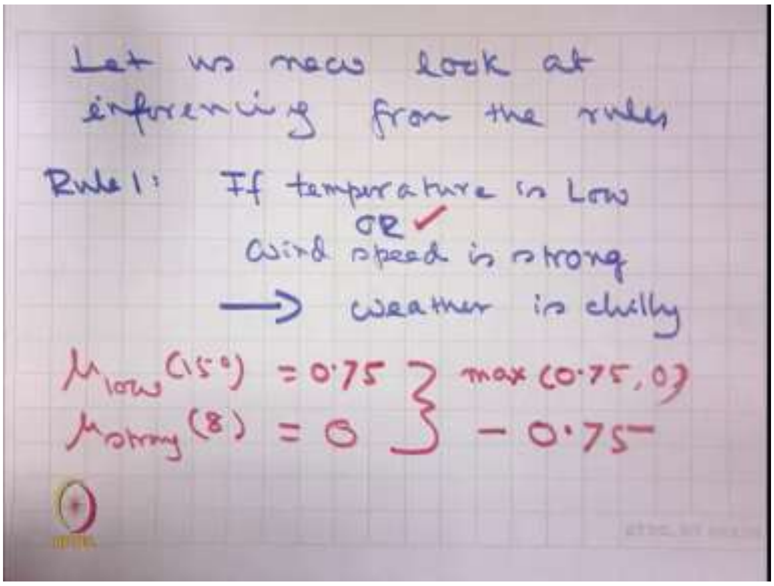


Let us down look at inferencing from the rules.

So, what is the Rule 1?

Rule 1 says that, $If\ temperature\ is\ low\ OR\ wind\ speed\ is\ strong \to\ weather\ is\ chilly.$

Now, $\mu_{low}(15°C) = 0.75$, $\mu_{strong}(8) = 0$

Now, we have $OR$ therefore we take maximum of them $\max(0.75, 0) = 0.75$

And therefore, when we apply the Mamdani inferencing. We get from the first rule that the original membership is clipped like this.

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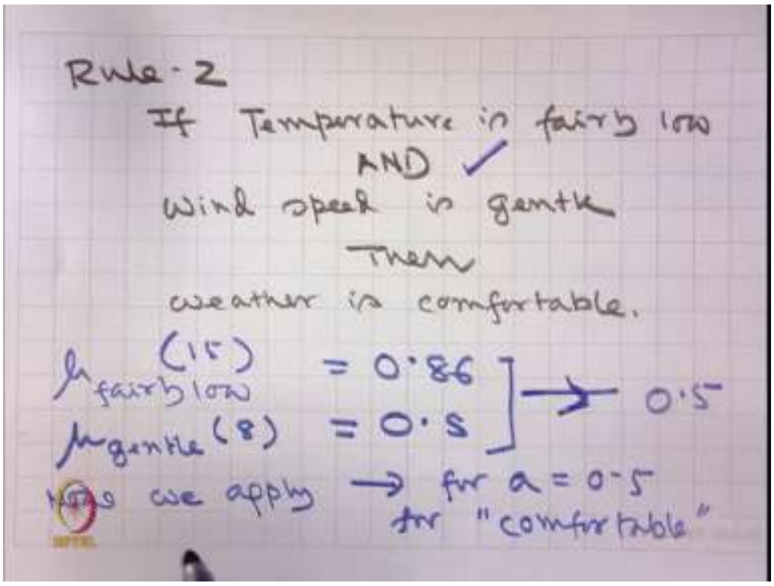


Let us down look at the second rule which suggests that

$If\ temperature\ is\ fairly\ low\ AND\ wind\ speed\ is\ gentle\ THEN, weather\ is\ comfortable.$

Now, we obtained that $\mu_{faily\ low}(15°C) = 0.86$, $\mu_{gentle}(8) = 0.5$.

Since, we have a conjunction and so, we combine them with the minimum. Therefore, the value coming as an $a$ from the antecedent variables is 0.5. Therefore, now we apply implication for $a = 0.5$ for comfortable.

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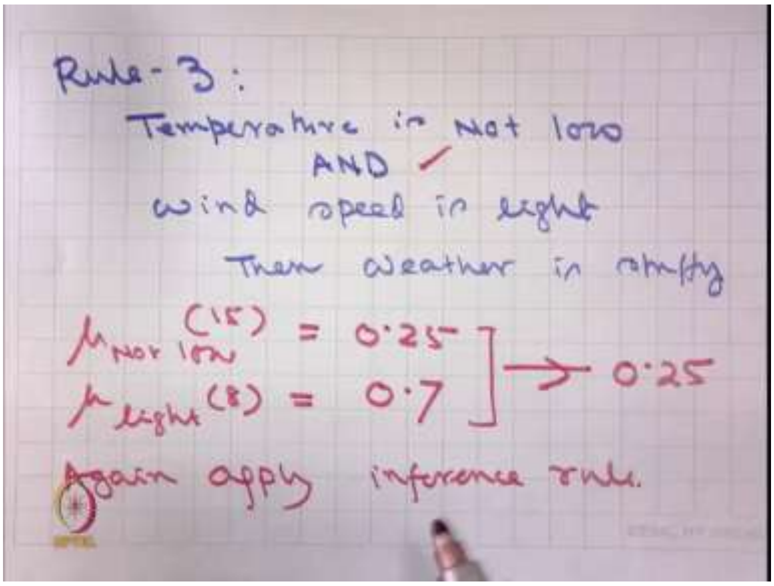


Let us now look at rule 3 which says that

$If\ temperature\ is\ not\ low\ AND\ wind\ speed\ is\ light\ THEN\ weather\ is\ stuffy.$

As before, $\mu_{not\ low}(15°C) = 0.25$, $\mu_{light}(8) = 0.7$

Since, it is a conjunction, we use minimum and therefore, the output coming out from these two is 0.25. Now, we again apply inference rule.

Therefore, everything together after all the clipping, we get these to be the fuzzy set. So this is the fuzzy set that we obtained by using 3 rules. So we summarize this finding as follows.

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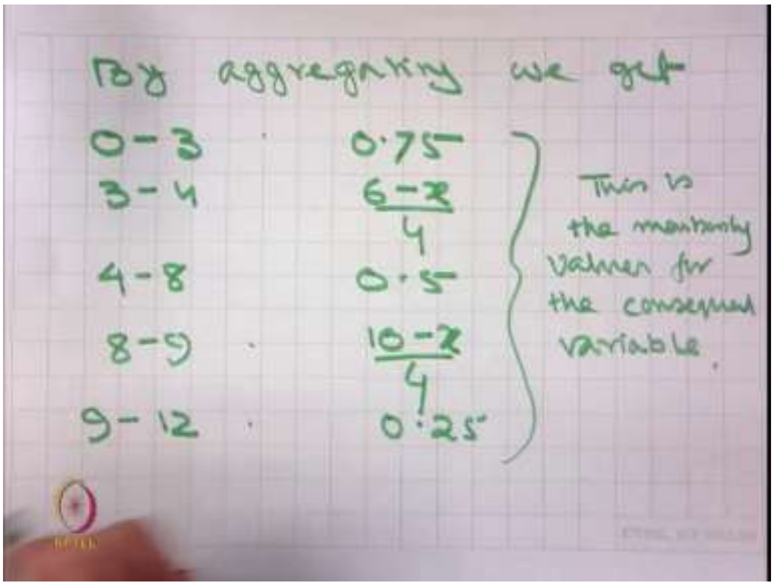


By aggregating, we get

| | |
|---|---|
| $0-3$ | $0.75$ |
| $3-4$ | $\frac{6-x}{4}$ |
| $4-8$ | $0.5$ |
| $8-9$ | $\frac{10-x}{4}$ |
| $9-12$ | $0.25$ |

So, this is the membership values for the consequent variable.

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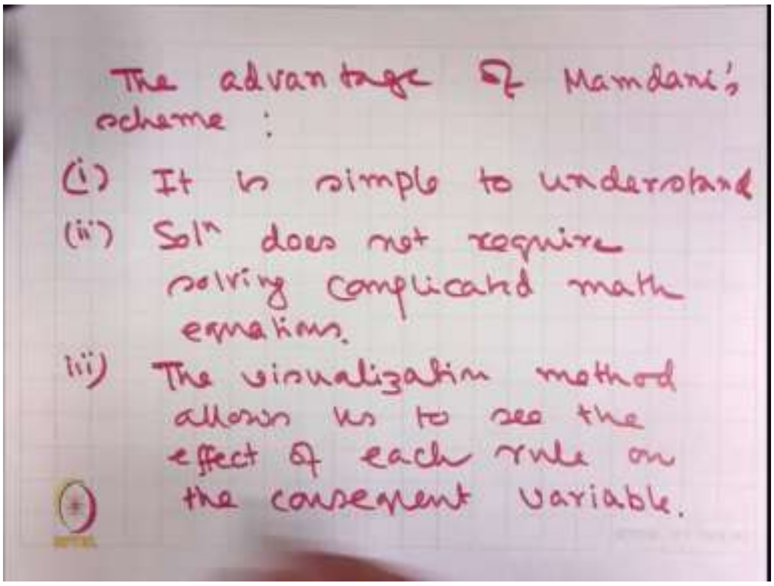


The advantage of Mamdani's scheme are the following:

1) It is simple to understand,
2) Solution does not require solving complicated mathematical equations.
3) The visualization method allows us to see the effect of each rule on the consequent variable.

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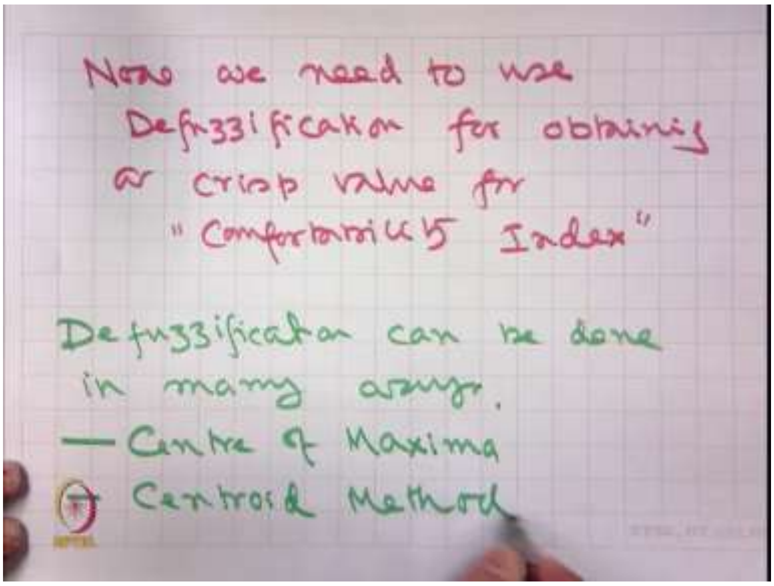


Now, we need to use Defuzzification for obtaining a crisp value for $comfortability\ index$.

Now, Defuzzification can be done in many ways

We discuss to what is called

- centre of maxima
- centroid method.

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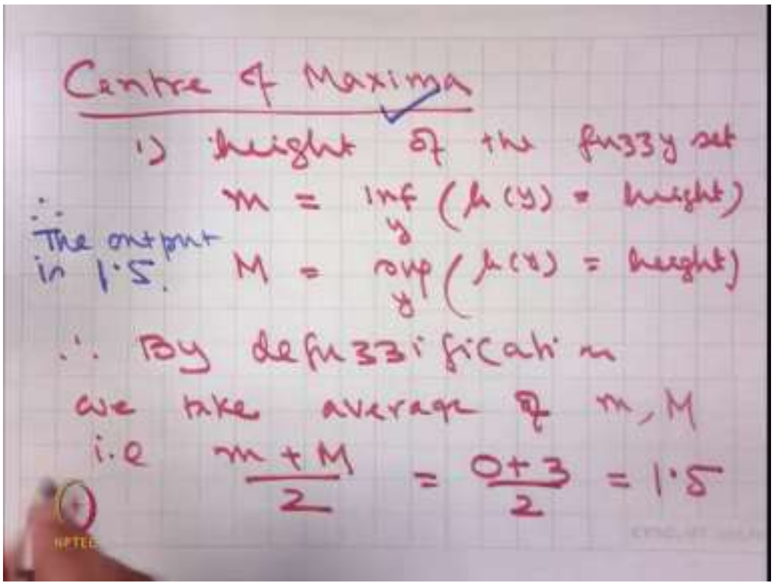


Centre of Maxima

Here, first we consider the height of the fuzzy set then we look at

$$m = \inf_{y}(\mu(y) = height)$$

$$M = \sup_{y}(\mu(y) = height)$$

We know that the height of the clipped fuzzy set is 0.75 and the infimum is 0 and the supremum is 3. Therefore, by defuzzification we take average of $m$ and $M$ that is

$$\frac{m+M}{2} = \frac{0+3}{2} = 1.5$$

Therefore, if we use center of maxima, then the output is 1.5.

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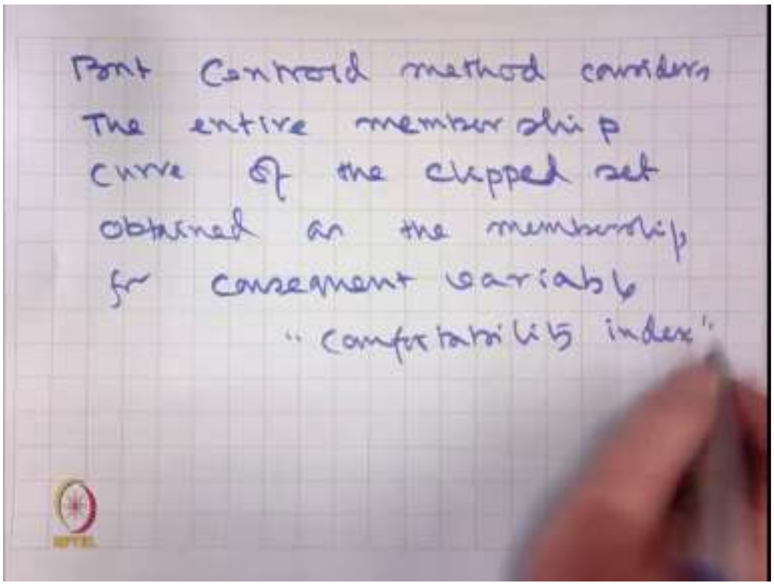

But, centroid method considers the entire membership curve of the clipped set obtained as the membership for consequent variable which in our case is the $comfortability\ index$.

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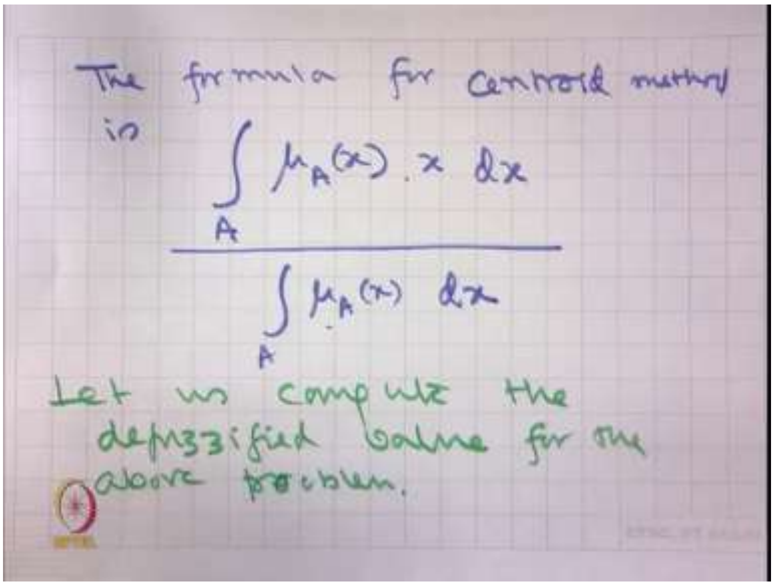


The formula for centroid method is on the entire fuzzy set $A$ we take the weighted average

$$\frac{\int_A \mu_A(x) x dx}{\int_A \mu_A(x) dx}$$

So, let us compute the defuzzified value for the above problem. So, this is the membership function, we are going to use it for the defuzzification.

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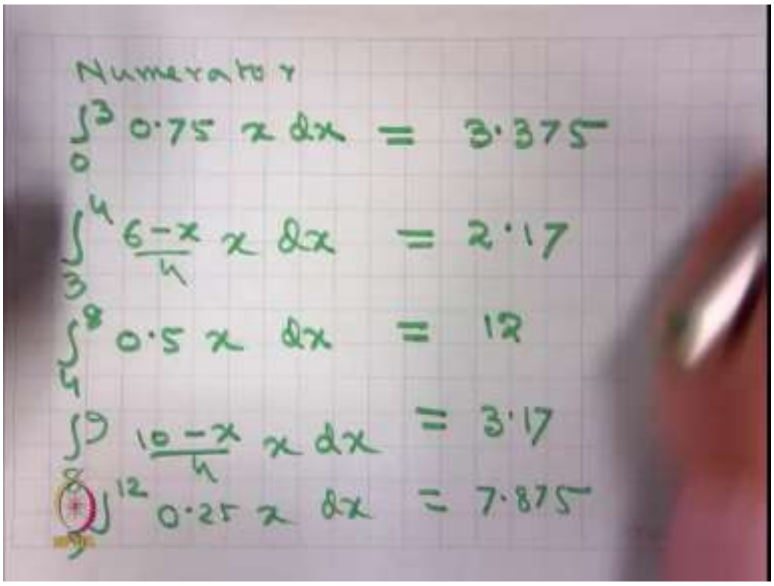


So, numerator is

$$\int_0^3 0.75 \, x dx = 3.375$$

$$\int_3^4 \frac{6-x}{4} \, x dx = 2.17$$

$$\int_4^8 0.5\, xdx = 12$$

$$\int_8^9 \frac{10 - x}{4}\, xdx = 3.17$$

$$\int_9^{12} 0.25\, xdx = 7.875$$

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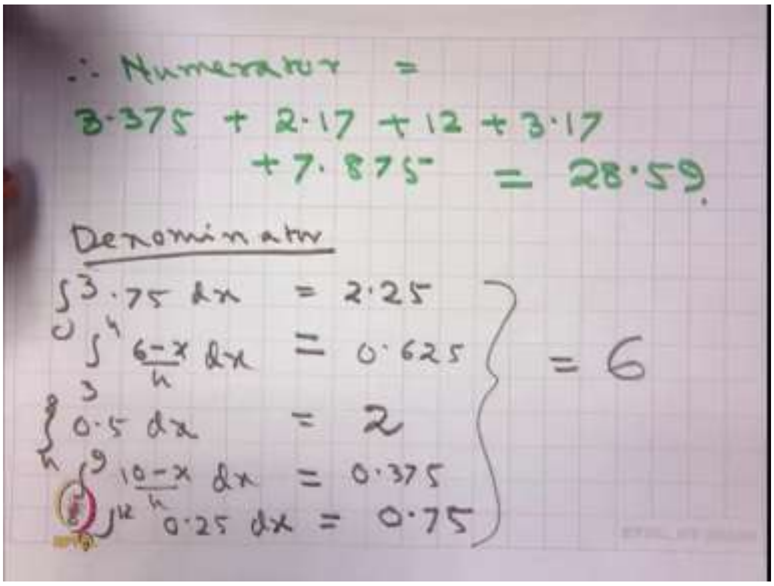


Therefore, Numerator = $3.375 + 2.17 + 12 + 3.17 + 7.875 = 28.59$

In a similar way, we calculate the denominator which is

$$\int_0^3 0.75\, dx = 2.25$$

$$\int_3^4 \frac{6 - x}{4}\, dx = 0.625$$

$$\int_4^8 0.5\, dx = 2$$

$$\int_8^9 \frac{10 - x}{4}\, dx = 0.375$$

$$\int_9^{12} 0.25\, dx = 0.75$$

So together, when we sum them up, we get the value is equal to 6.

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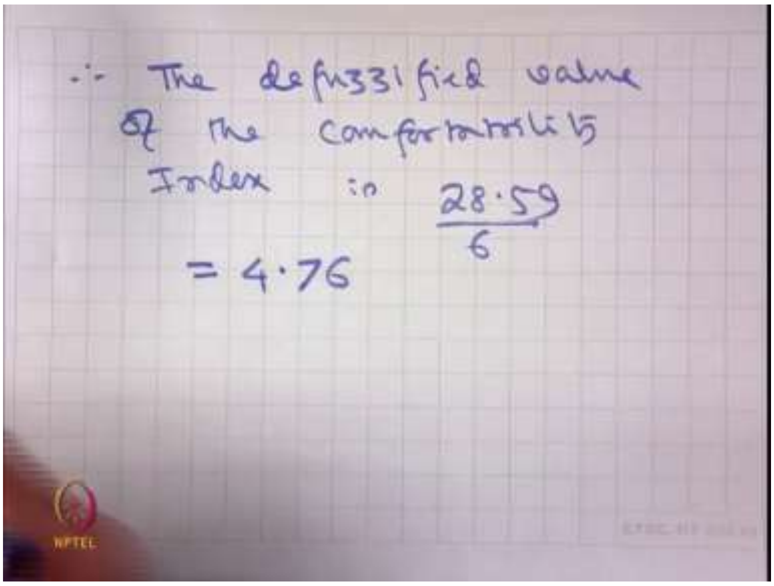


Therefore, that defuzzified value of the $comfortability\ index$ is

$$\frac{28.59}{6} = 4.76$$

As you can while understand that the centroid method is more computationally intensive in comparison with the centre of maximum method, but this gives a much more comprehensive value for the consequent variable.

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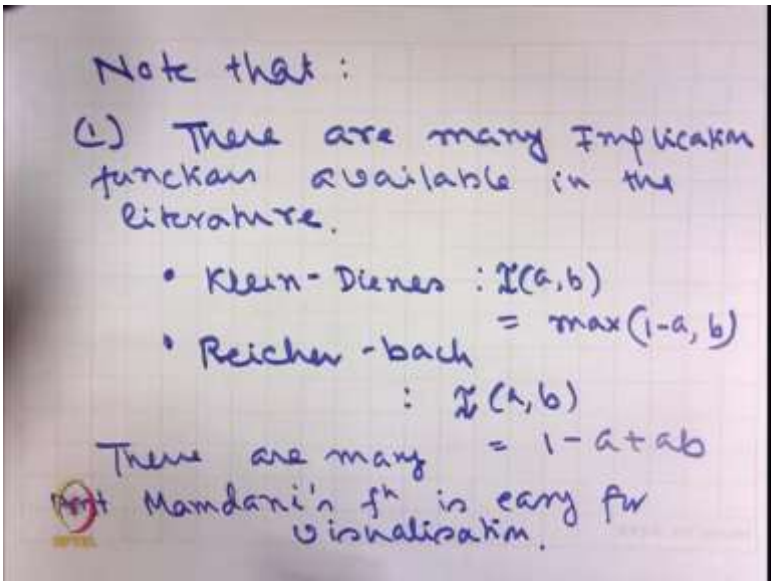


Note that

1) There are many implication functions available in the literature as I have discussed before but still I give you a few name
   - Kleene-Dienes :$\mathfrak{I}(a,b) = \max(1 - a, b)$
   - Reichen-bach: $\mathfrak{I}(a,b) = 1 - a + ab$

Similarly, there are many but Mamdani's function is easy for visualization.

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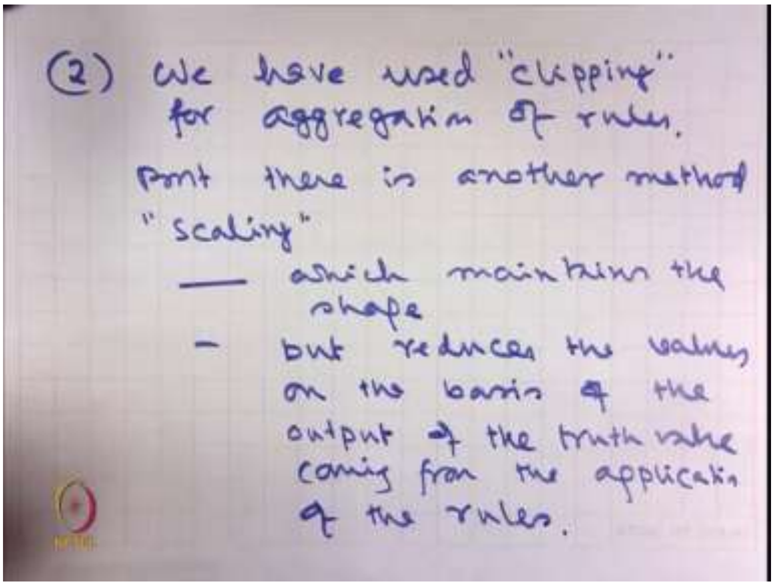


Another important thing is that

2) We have used clipping for aggregation of rules. But, there is another method called scaling
    a. which maintains the shape
    b. but reduces the values on the basis of the output of the truth value coming from the application of the rules.

Okay friends. I stop here today. With this I conclude my topics on fuzzy logic in the next class. I shall start the concept of belief and possibility. Thank you so much.