Introduction to Fuzzy Sets Arithmetic & Logic
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Lecture – 27
Fuzzy Sets Arithmetic & Logic

Welcome students to the MOOCs is goes on Fuzzy Sets Arithmetic and Logic. This is lecture number 27. If you remember, in the last class we started Fuzzy Quantifiers and in today's class, I shall continue with that.

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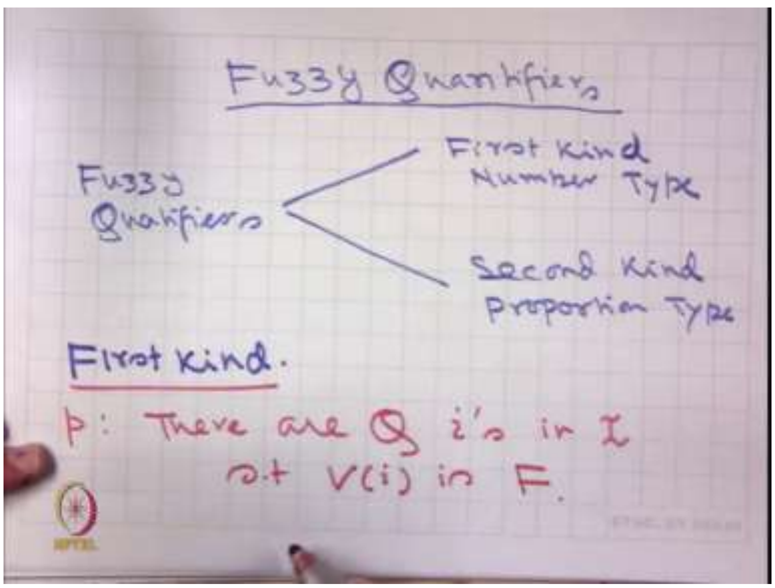

We know that fuzzy quantifiers are of two types,
one is a first kind that is number type
and other is the second kind which is proportion type.
And we were discussing first kind and we have looked at propositions such as:
p: There are Q $i's$ in some set I such that $V(i)$ is F where, F is a fuzzy set and Q is a fuzzy quantifier of first kind.

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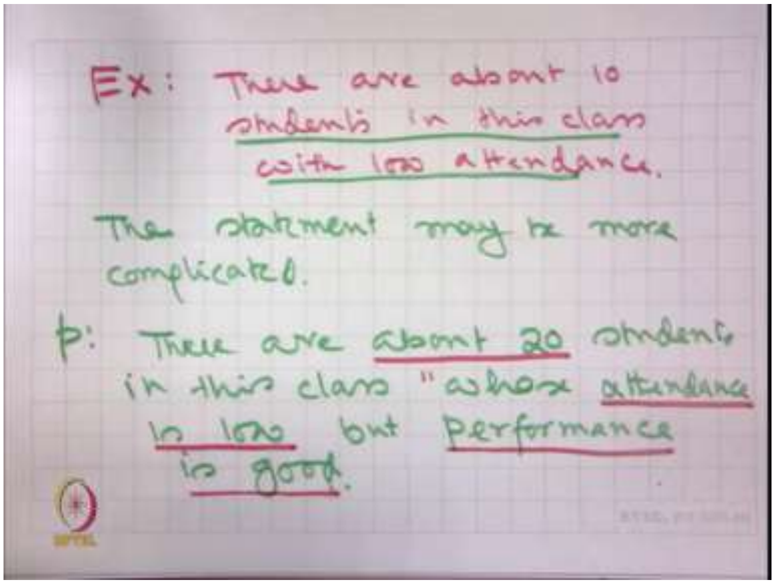

And the example that we studied was:

There are about 10 students in this class with low attendance.

In practice this variable or say, this statement may be more complicated.

Say, for example, suppose we write

p: *There are about 20 students in this class whose attendance is low but, performance is good*.

So, it is a more general version we have one fuzzy quantifier.

We are looking at essentially two different fuzzy sets, which give attributes to the students.

One is attendance is low and the other one is performance is good.

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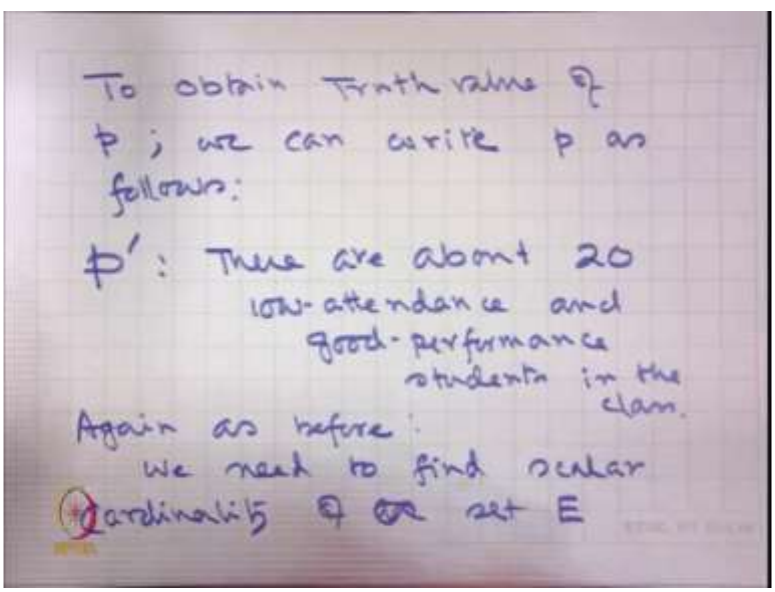

So, to obtain to the truth value of p; we can write p as follows.

Let me call it p':

There are about 20 low attendance and good performance student in the class.

And therefore, again as before we need to find scalar cardinality of a set E.

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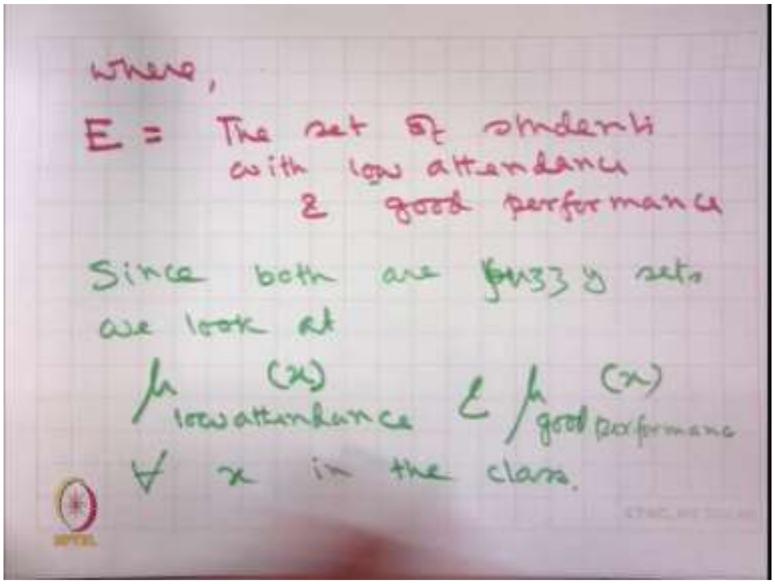

Where, E is the set of students with *low attendance and good performance*.

Since, both are fuzzy sets we look at

$\mu_{low\ attendance}(x)$ and $\mu_{good\ performance}\ (x)$ for all x in the class.

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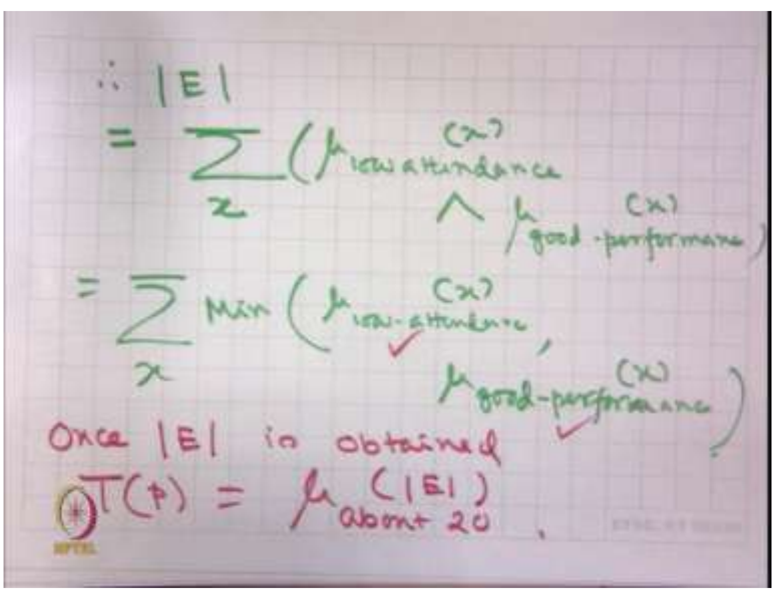

Therefore,

$$|E| = \Sigma_x \left(\mu_{low\ attendance}(x) \wedge \ \mu_{good\ performance}\ (x)\right)$$

$$= \Sigma_x \min\left(\mu_{low\ attendance}(x), \mu_{good\ performance}\ (x)\right)$$

Therefore, if we can define these two sets and we can compute the membership value for each student to both the classes by using this summation, we can get $|E|$.

And once $|E|$ is obtained $T(p) = \mu_{about\ 20}(|E|)$, this part is very similar to our earlier example.

So, I hope that you understood how to solve a problem of this type with that background.

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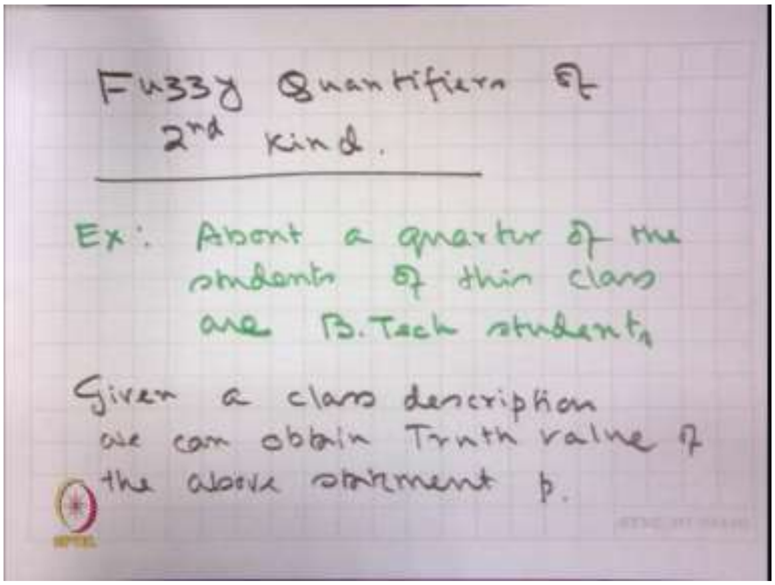

Let us now focus on fuzzy quantifiers of 2nd kind.

Example:

About a quarter of the students of this class are B. Tech students.

Given a class description, we can obtain truth value of the above statement p.

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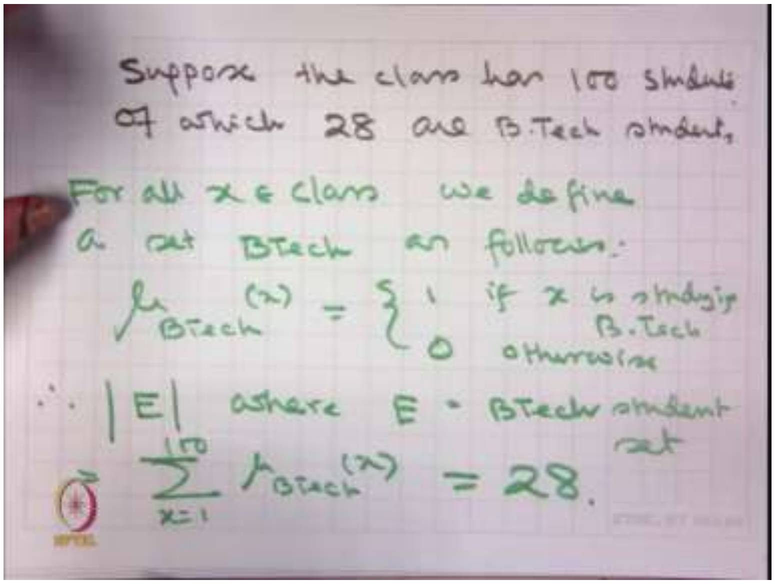

Suppose the class has 100 students of which 28 are B. Tech students.

Therefore, for all $x \in$ class. We define a set $B.Tech$ as follows:

$$\mu_{B.Tech} = \begin{cases} 1 & \text{if } x \text{ is studying B. Tech} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $|E|$ where $E \Rightarrow$ B. Tech student set $= \Sigma_{x=1}^{100} \mu_{B.tech}(x) = 28$

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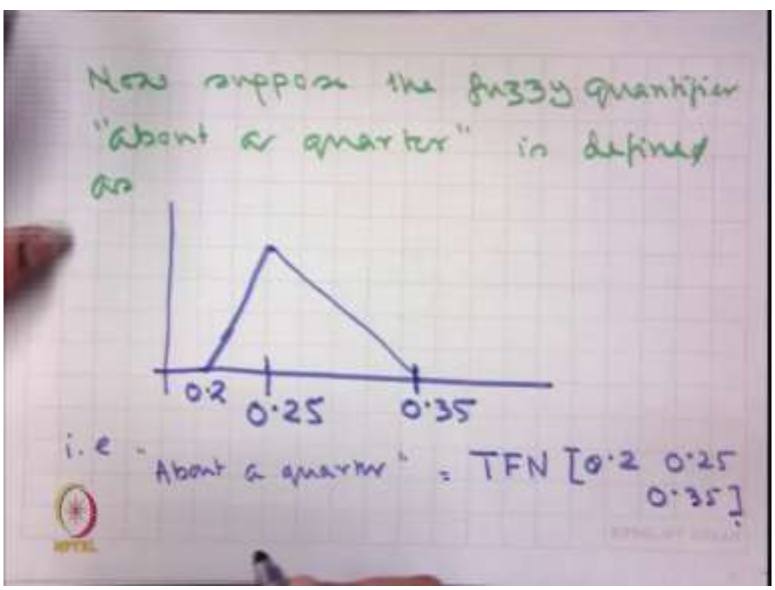

Now, suppose the fuzzy quantifier '$about\ a\ quarter$' is defined as this set that is,

$$About\ a\ quarter = TFN[0.2 \quad 0.25 \quad 0.35]$$

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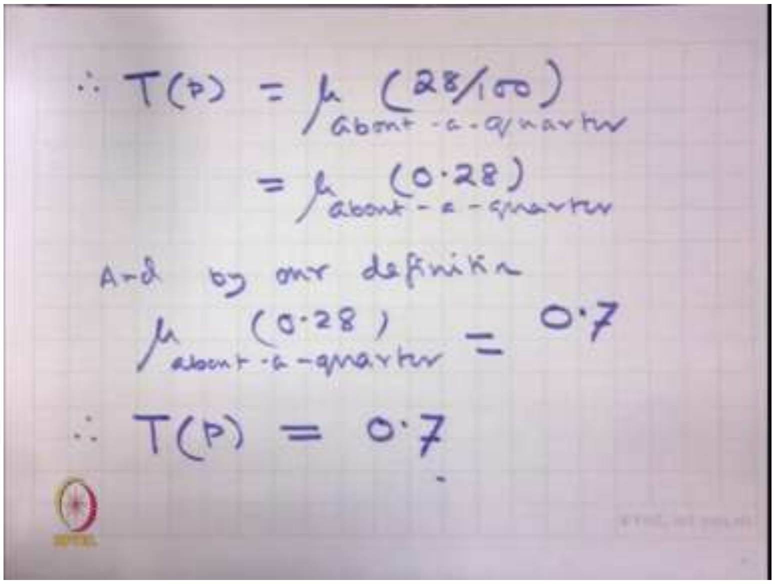

Therefore, $T(p) = \mu_{about\ a\ quarter}\left(\frac{28}{100}\right) = \mu_{about\ a\ quarter}(0.28)$

$\frac{28}{10}$ is the proportion of student who are studying B. Tech.

And by the definition that we have given

$$\mu_{about\ a\ quarter}(0.28) = 0.7$$

Therefore, truth value of the above statement can be determined as $T(p) = 0.7$

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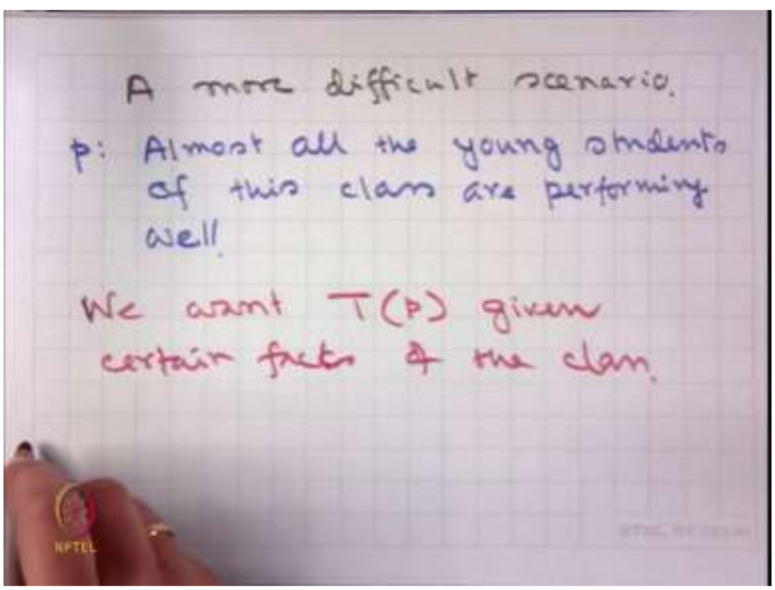

Let us consider slightly more difficult scenario.

Suppose the proposition is

p: *Almost all the young students of this class are performing well*.

And we want $T(p)$ given certain facts of the class and suppose we have the following fact.

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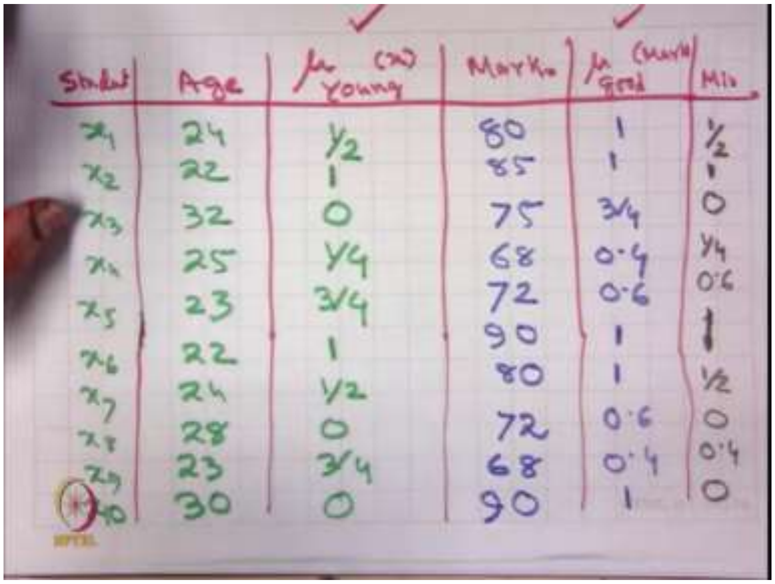

Student	Age	$\mu_{Young}(x)$	Marks	$\mu_{Good}(marks)$	Min
x_1	24		80		
x_2	22		85		
x_3	32		75		
x_4	25		68		
x_5	23		72		
x_6	22		90		
x_7	24		80		
x_8	28		72		
x_9	23		68		
x_{10}	30		90		

Now, to obtain $\mu_{Young}(x)$ and $\mu_{Good}(marks)$, we have to define the fuzzy set $young$ and the fuzzy set $good\ performance$. Suppose we define them as follows.

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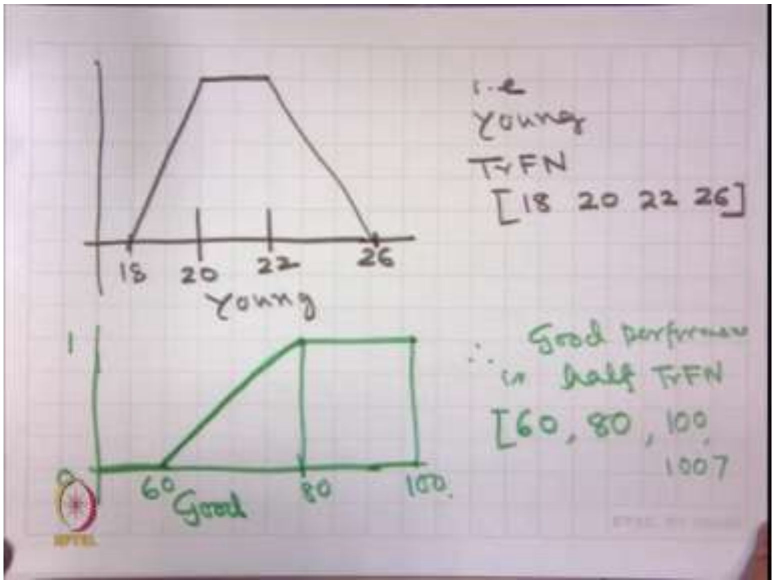

$Young$ is a trapezoidal fuzzy number $TrFN$ [18 20 22 26] and $good\ performance$ is a half $TrFN$ [60 80 100 100]

With these two fuzzy sets define like that, let us now fill in the above table.

Student	Age	$\mu_{Young}(x)$	Marks	$\mu_{Good}(marks)$	Min
x_1	24	$\frac{1}{2}$	80	1	$\frac{1}{2}$
x_2	22	1	85	1	1
x_3	32	0	75	$\frac{3}{4}$	0
x_4	25	$\frac{1}{4}$	68	0.4	$\frac{1}{4}$
x_5	23	$\frac{3}{4}$	72	0.6	0.6
x_6	22	1	90	1	1
x_7	24	$\frac{1}{2}$	80	1	$\frac{1}{2}$
x_8	28	0	72	0.6	0
x_9	23	$\frac{3}{4}$	68	0.4	0.4
x_{10}	30	0	90	1	0

Therefore, now we need to look at the minimum because we are looking at both $young$ as well as $good\ performance$. Therefore, we look at the conjunction of this and therefore, by standard intersection or by standard fuzzy t-norm, we are using the minimum.

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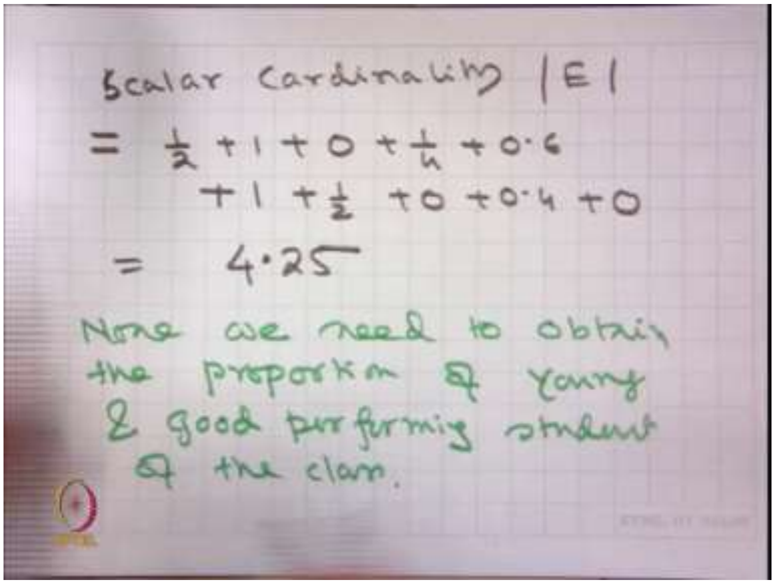

Therefore, $|E| = \frac{1}{2} + 1 + 0 + \frac{1}{4} + 0.6 + 1 + \frac{1}{2} + 0 + 0.4 + 0 = 4.25$

Now, we need to obtain the proportion of young and good performing student of the class.

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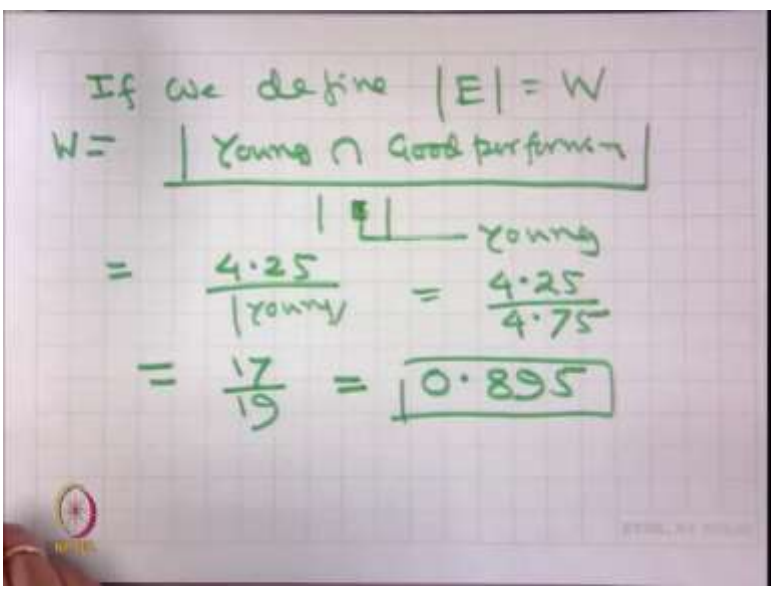

So, if we define $|E| = W$ then

$$W = \frac{|Young \ \cap \ Good \ performance|}{|Young|} = \frac{4.25}{|young|} = \frac{4.25}{4.75} = \frac{17}{19} = 0.895$$

Therefore, given the fact of the class, the truth value of the statement that, *almost the all young students of the class are performing good* is 0.895.

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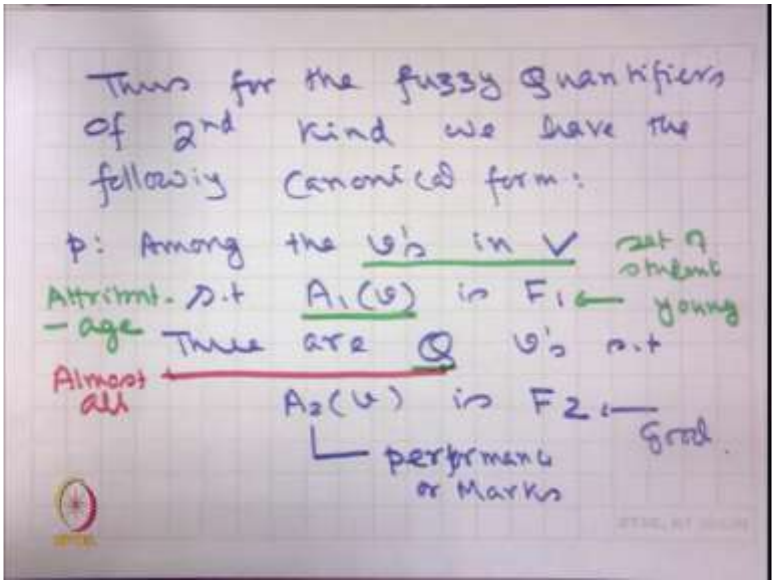

Thus, for the fuzzy Quantifiers of second kind, we have the following canonical form.

p: *Among the* v*'s in* V *such that* $A_1(v)$ *is* F_1 *there are* Q v*'s such that* $A_2(v)$ *is* F_2.

This is apparently complicated, but let us compare with the example just I have given.

V is the set of students, $A_1(v)$ is that attribute age, F_1 is the fuzzy set young, Q is the quantifier almost all and $A_2(v)$ is that tribute performance or marks and F_2 is the fuzzy set good.

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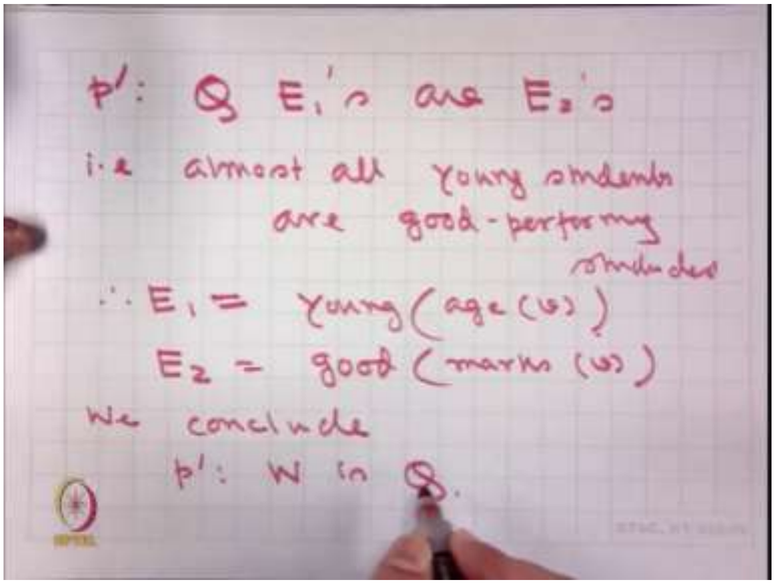

So, when we have such a canonical form, we convert it into a form like this:

$$p': Q\ E_1\text{'s are } E_2\text{'s}$$

that is $almost\ all\ young\ students\ are\ good\ performing\ students$.

Therefore,

$$E_1 = Young\big(age(v)\big)$$

$$E_2 = Good\big(marks(v)\big)$$

And we conclude $p': W\ is\ Q$

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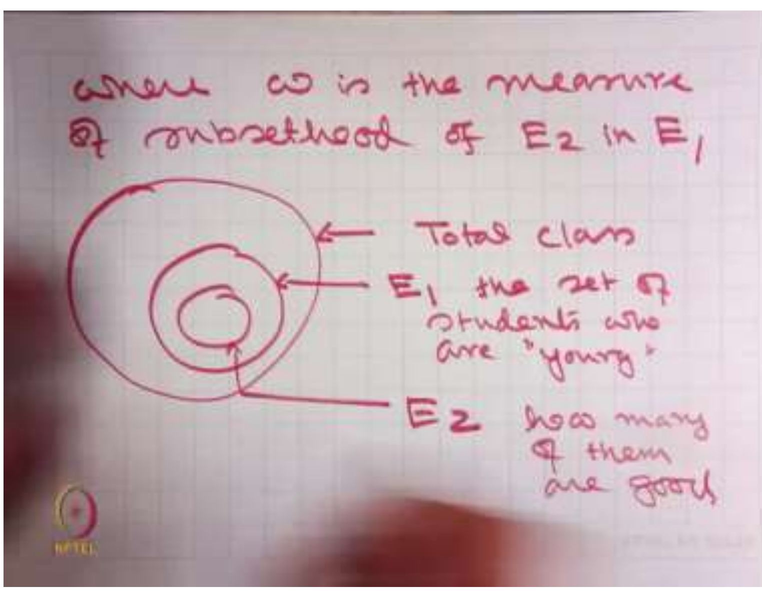

Where W is the measure of subsethood of E_2 in E_1.

Graphically suppose, this is the total class this is E_1 the set of students who are young and E_2 is how many of them are good.

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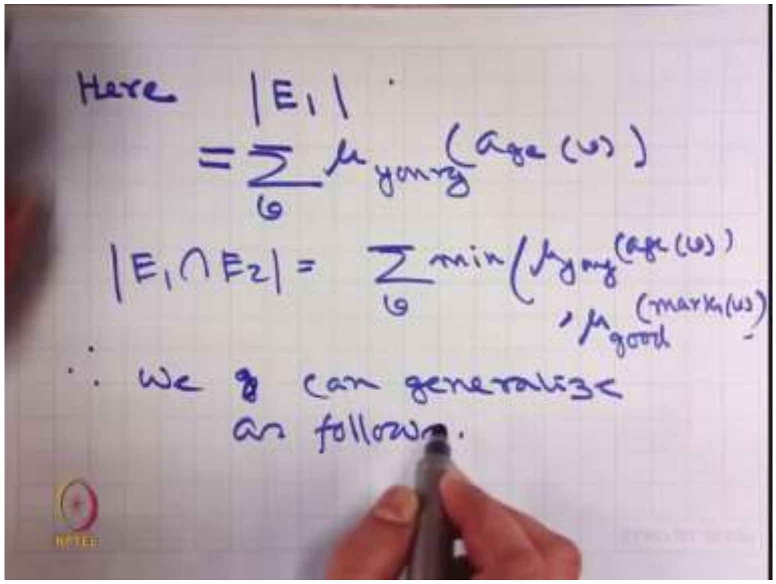

Here

$$|E_1| = \Sigma_v \mu_{Young}\big(age(v)\big)$$

$$|E_1 \cap E_2| = \Sigma_v \min\Big(\mu_{Young}\big(age(v)\big), \mu_{Good}\big(marks(v)\big)\Big)$$

Therefore, we can generalize as follows.

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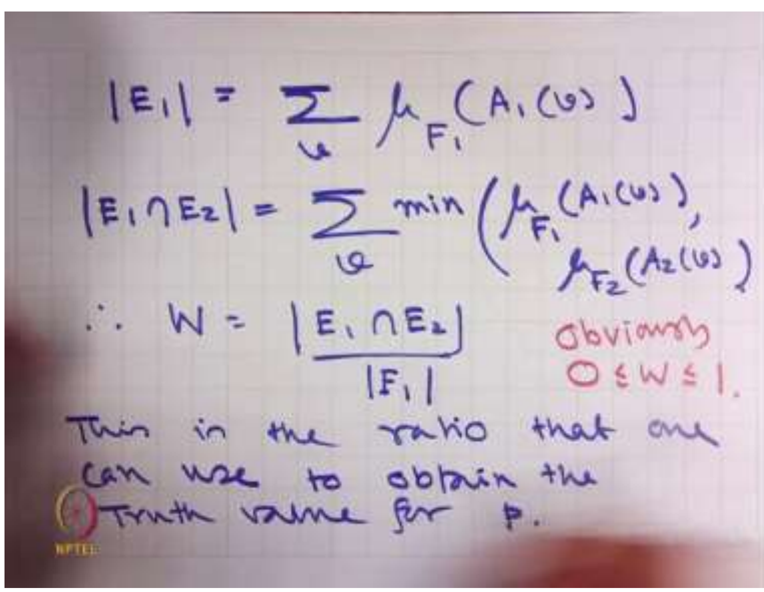

$$|E_1| = \Sigma_v \mu_{F_1}\big(A_1(v)\big)$$

$$|E_1 \cap E_2| = \Sigma_v \min\left(\mu_{F_1}\big(A_1(v)\big), \mu_{F_2}\big(A_2(v)\big)\right)$$

$$\therefore W = \frac{|E_1 \cap E_2|}{|E_1|}$$

This is the ratio that one can use to obtain the truth value for p.

Obviously, $0 \leq W \leq 1$

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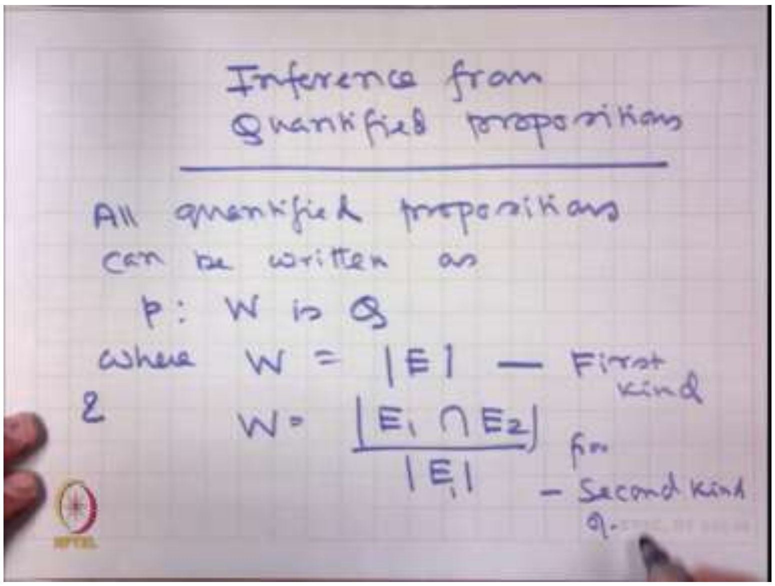

Now let us look at inference from quantified propositions.

We have seen that all quantified propositions can be written as

$$p: W\ is\ Q$$

where $W = |E|$ for first kind that is number type of fuzzy sets or $W = \frac{|E_1 \cap E_2|}{|E_1|}$ for second kind of quantifiers.

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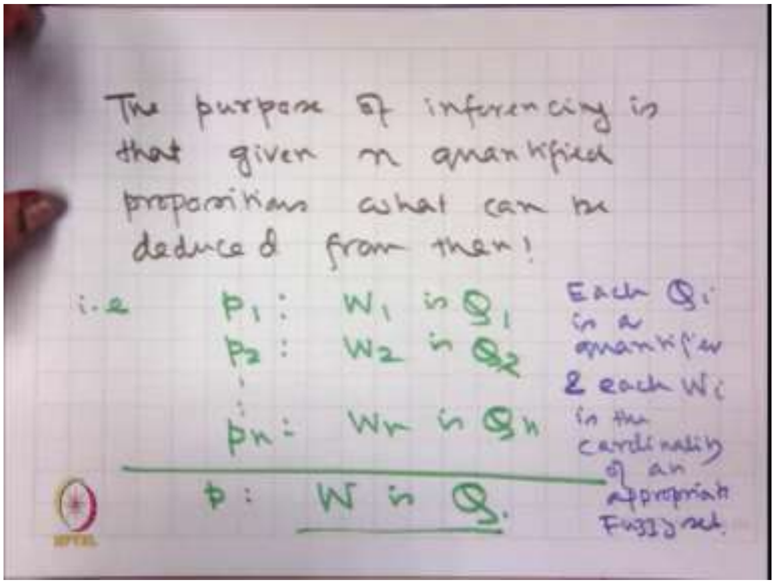

Now the purpose of inferencing is that, given n quantified propositions, what we can infer or what can be deduced from them.

That is, if we have

$$p_1 : W_1 \;\; is \; Q_1$$

$$p_2 : W_2 \;\; is \; Q_2$$

$$\vdots$$

$$p_n : W_n \;\; is \; Q_n$$

And our aim is to find a statement p: $W \; is \; Q$ and we try to obtain it truth value.

Here. Each Q_i is a quantifier and each W_i is the cardinality of an appropriate fuzzy set.

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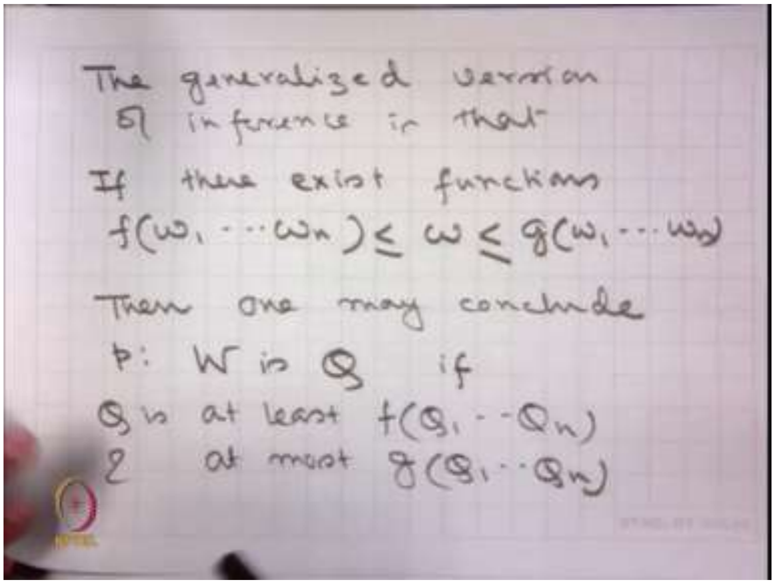

Now, the generalized version of inference is that:

If there exists functions $f(W_1, W_2 \ldots W_n) \le W \le g(W_1, W_2 \ldots W_n)$

Then one may conclude $p: W\ is\ Q$

if Q is at least $f(Q_1, \ldots Q_n)$ and at most $g(Q_1, \ldots Q_n)$. Therefore, if we could obtain such functions, then we can conclude $W\ is\ Q$.

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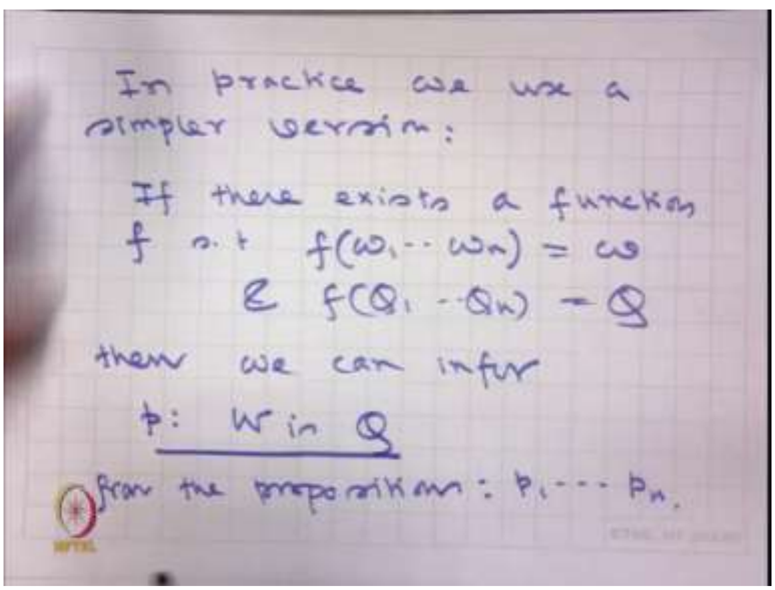

In practice we use a simpler version, which says that

If there exists a function f such that $f(W_1, \ldots W_n) = W$ and $f(Q_1, \ldots Q_n) = Q$. Then we can infer that $p: W\ is\ Q$ from the propositions $p_1, p_2, \ldots p_n$.

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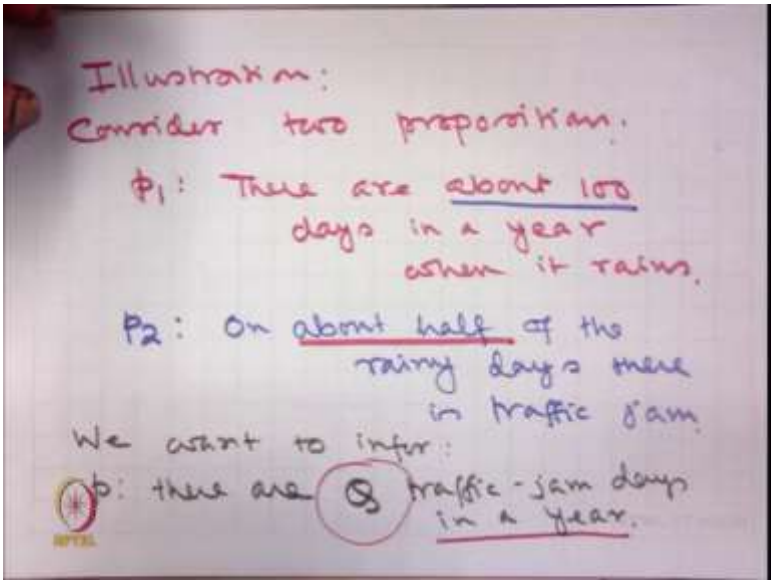

For illustration, considered these two propositions:

$$p_1: There\ are\ about\ 100\ days\ in\ a\ year\ when\ it\ rains.$$

So, this is a statement with a fuzzy quantifier of first kind $about$ 100.

$$p_2: On\ about\ half\ of\ the\ rainy\ days\ there\ is\ traffic\ jam$$

So, this is a fuzzy quantifier of the second kind.

So, we want to infer p: $There\ are\ Q\ traffic\ jam\ days\ in\ a\ year$.

I think the idea is clear. We have some fuzzy quantifier related statement about on how many days there is rain and a proportional statement on how many days there is a traffic jam. So, we want to infer about on a how many days in a year there is a traffic jam. We want to have an idea of this fuzzy quantifier Q.

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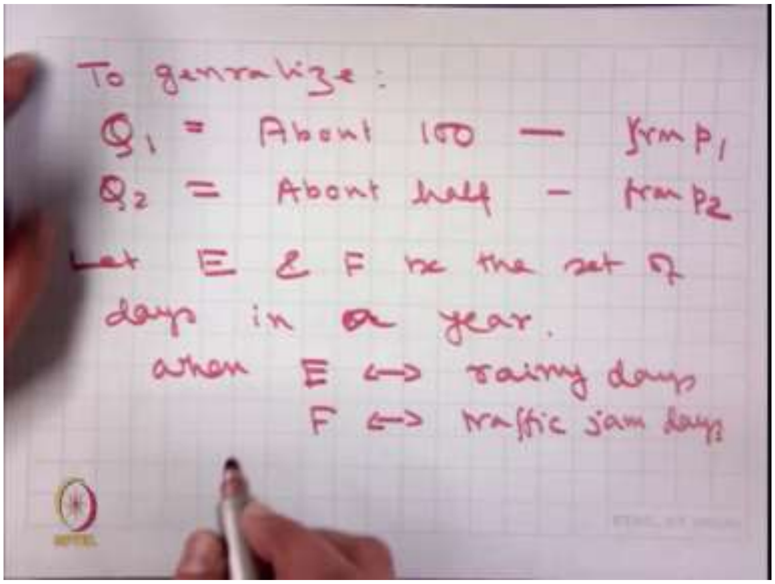

So to generalize this we have

$$Q_1 = About\ 100 \quad - \text{from } p_1$$

$$Q_2 = About\ half \quad - \text{from } p_2$$

And let E and F be the set of days in a year when E corresponds to rainy days and F corresponds to traffic jam days.

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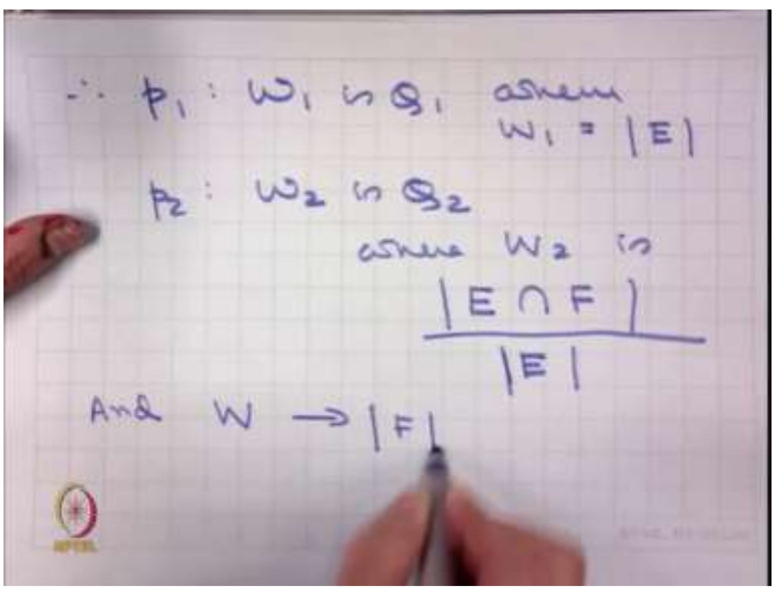

Therefore, we have $p_1{:}\, W_1\ \ is\ Q_1$ where $W_1 = |E|$

and $p_2{:}\, W_2\ \ is\ Q_2$ where, W_2 is the proportion $\frac{|E \cap F|}{|E|}$

Assume that traffic jam causes only with rainy days. So, $\frac{|E \cap F|}{|E|}$ gives us the proportion on how many of the rainy days there is going to be traffic jam. And W is going to be the cardinality of F.

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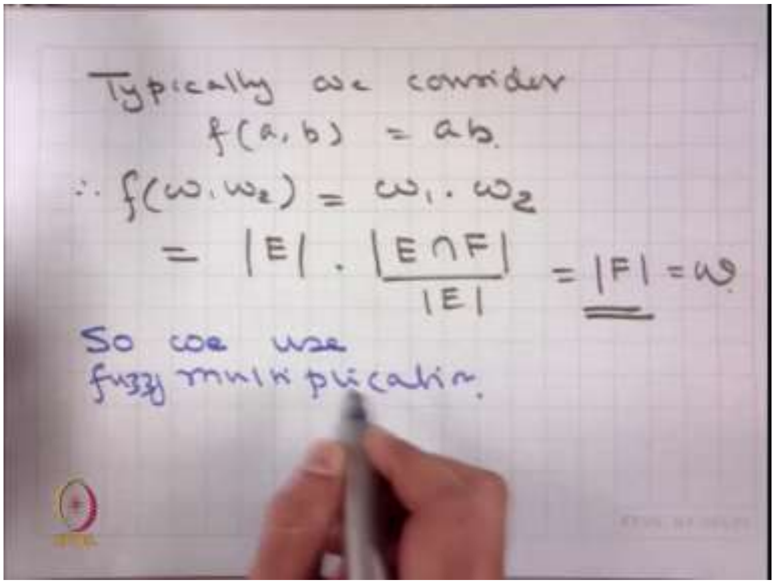

We can consider typically, we consider $f(a, b) = ab$

If you remember, we have stated this that from the given statements, these are the quantifiers, we are trying to get F and W_1, W_2 ... W_n are the different fuzzy sets defined on the set of elements in our case the set of days.

Therefore, $f(W_1, W_2) = W_1 W_2 = |E| \frac{|E \cap F|}{|E|} = |E \cap F| = W$

That is, this is the number of traffic jam days among the rainy days. So, what we do fuzzy multiplication.

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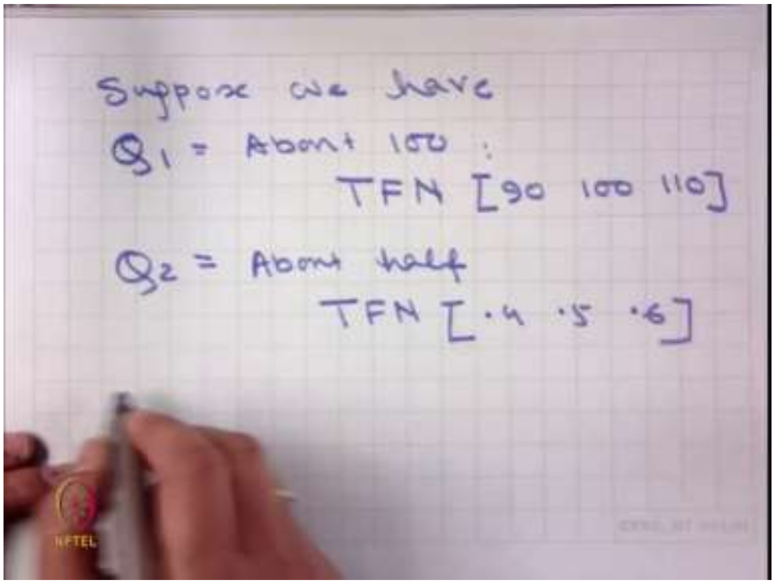

Suppose, we have

$$Q_1 = About\ 100 : TFN[90 \quad 100 \quad 110]$$

$$Q_2 = About\ half : TFN[0.4 \quad 0.5 \quad 0.6]$$

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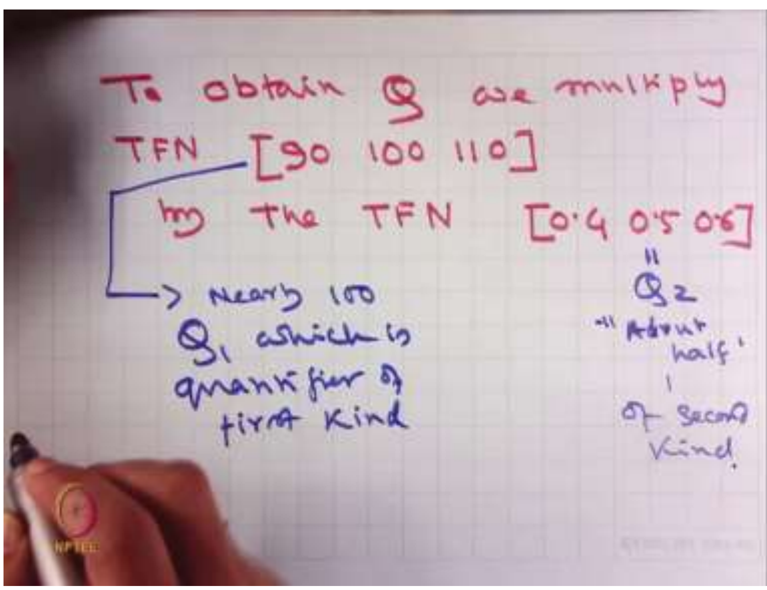

To obtain Q we multiply the $TFN[90 \quad 100 \quad 110]$ by the $TFN[0.4 \quad 0.5 \quad 0.6]$.

$Q_1 = nearly$ 100 which is a quantifier of first kind and $Q_2 = about\ half$ is of second kind.

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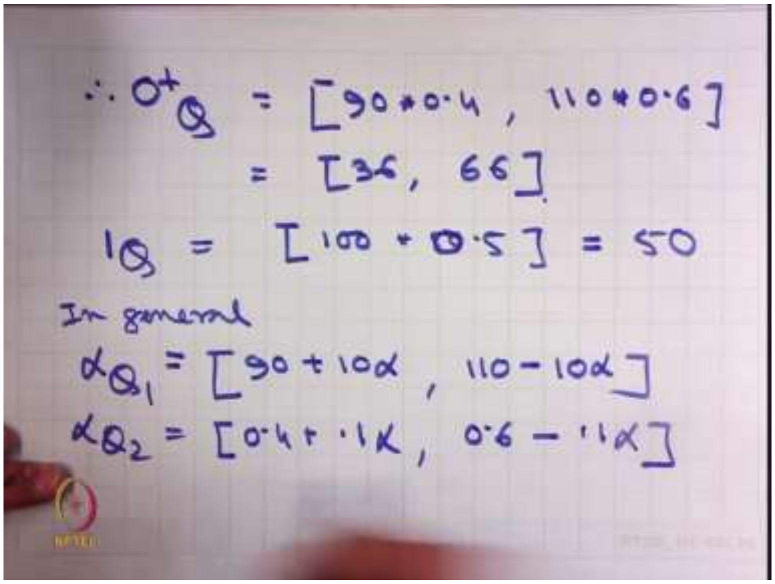

Therefore, $^{0+}Q = [90 * 0.4,\ 110 * 0.6] = [36,\ 66]$

$$^{1}Q = [100 * 0.5] = 50$$

In general

$$^{\alpha}Q_1 = [90 + 10\alpha, 110 - 10\alpha]$$

$$^{\alpha}Q_2 = [0.4 + 0.1\alpha, 0.6 - 0.1\alpha]$$

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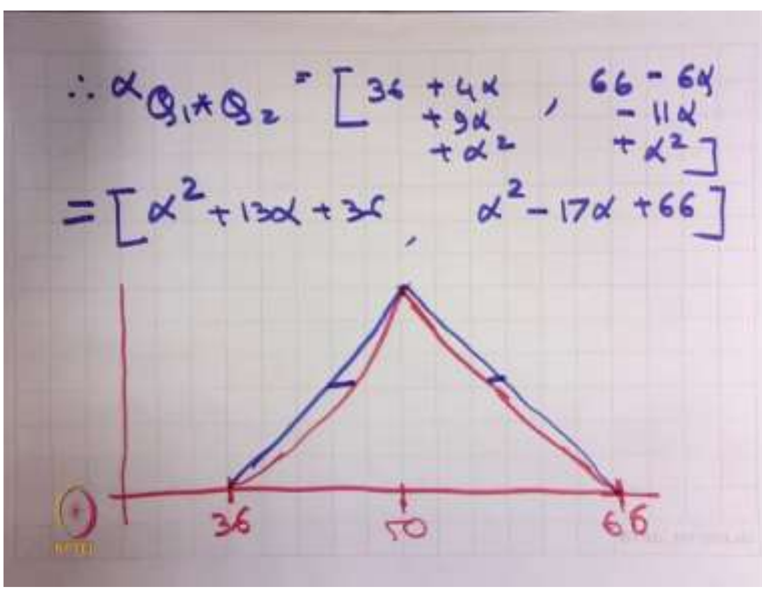

Therefore,

$$^{\alpha}(Q_1 * Q_2) = [36 + 4\alpha + 9\alpha + \alpha^2, 66 - 6\alpha - 11\alpha + \alpha^2\ ]$$

$$= [\alpha^2 + 13\alpha + 36, \alpha^2 - 17\alpha + 66]$$

Therefore, if we plot it, it appears like a triangular fuzzy number, but we know that it is not. In fact, it is going to be quadratic equation. So, we can expect a shape like this. And, we have already studied that such type of shapes can be approximated by triangular fuzzy number provided the maximum gap is within a threshold. So like that we can estimate the value of Q.

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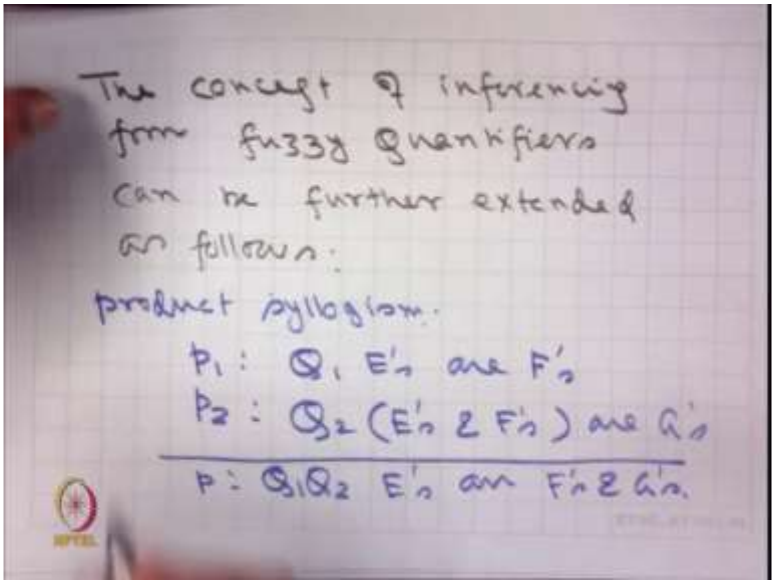

The concept of inferencing from fuzzy quantifiers can be further extended as follows. One of them is called product syllogism that is, suppose we have proposition

$$p_1: Q_1\ E's\ are\ F's$$

$$p_2: Q_2\ (E's\ and\ F's) are\ G's$$

From that to want to conclude by a similar way is $p: Q_1 Q_2\ E's\ are\ F's\ and\ G's$

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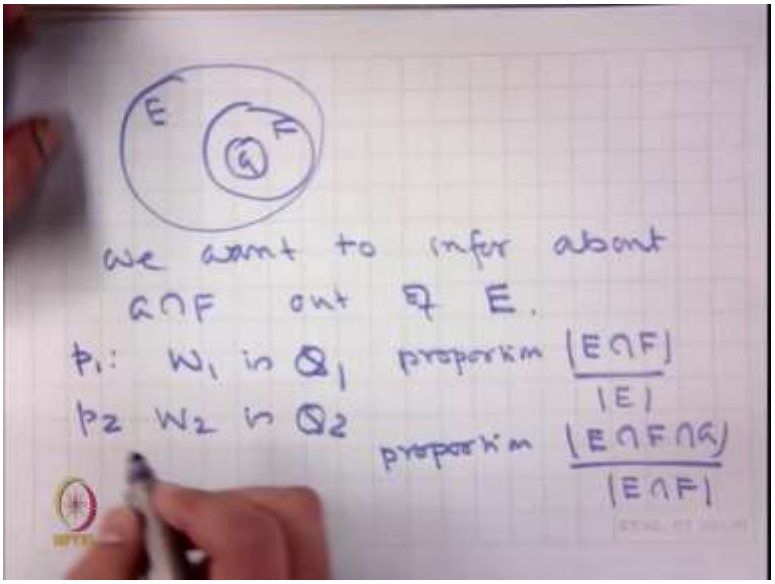

We want to infer about $G \cap F$ out of E. Therefore, we can write it as

$p_1\colon W_1\ \ is\ Q_1$ proportion $\frac{|E \cap F|}{|E|}$

$p_2\colon W_2\ \ is\ Q_2$ proportion $\frac{|E \cap F \cap G|}{|E \cap F|}$

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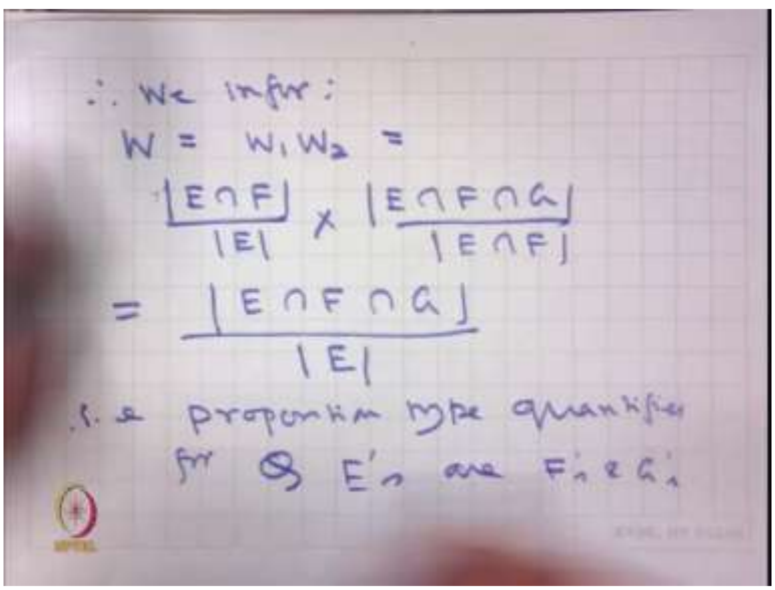

Therefore, we infer

$$W = W_1 W_2 = \frac{|E \cap F|}{|E|} \times \frac{|E \cap F \cap G|}{|E \cap F|} = \frac{|E \cap F \cap G|}{|E|}$$

. That is, it gives the proportion type quantifier for $Q\ E\text{'s are } F\text{'s and } G\text{'s}$.

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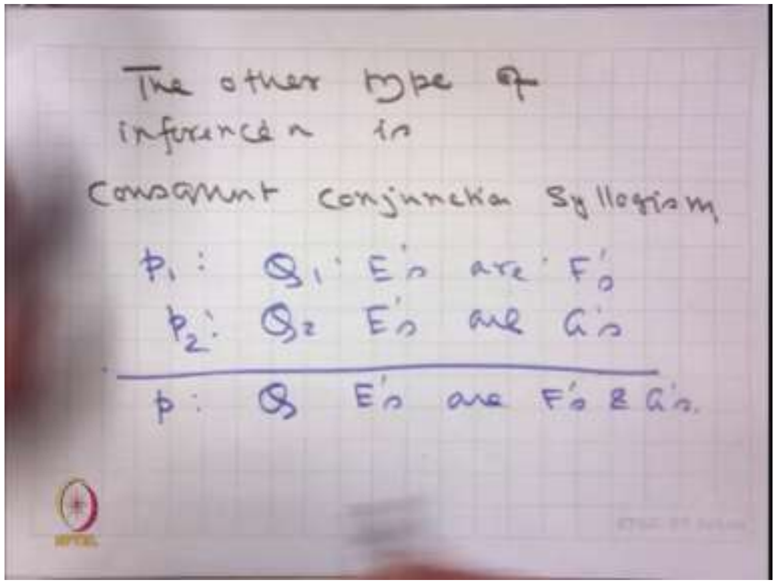

The other type of inferencing is called consequent conjunction syllogism where

$$p_1 : Q_1\ E's\ are\ F's$$

$$p_2 : Q_2\ E's\ are\ G's$$

and we want to infer

$$p : Q\ E's\ are\ F's\ \ and\ G's.$$

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We want to infer about how many of the $E's\ are\ F's\ and\ G's$.

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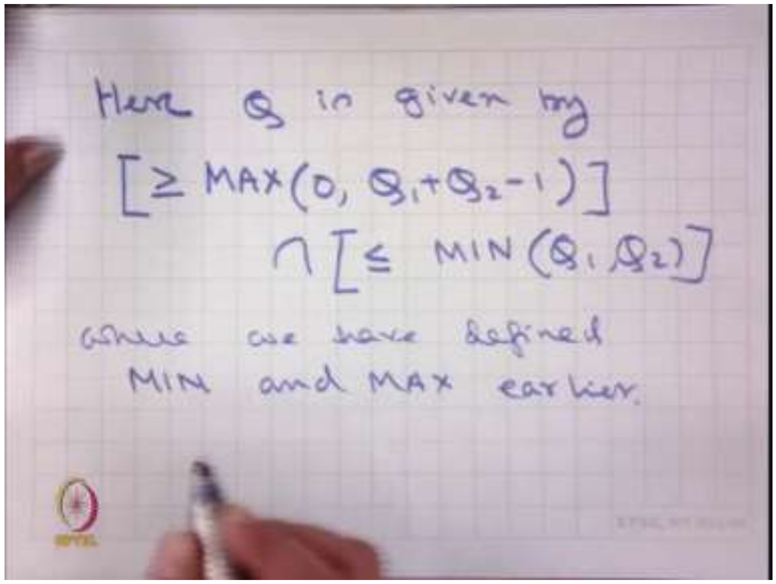

So, here Q is given by

$$[\geq MAX(0, Q_1 + Q_2 - 1)] \cap [\leq MIN(Q_1, Q_2)]$$

where we have defined MIN and MAX earlier.

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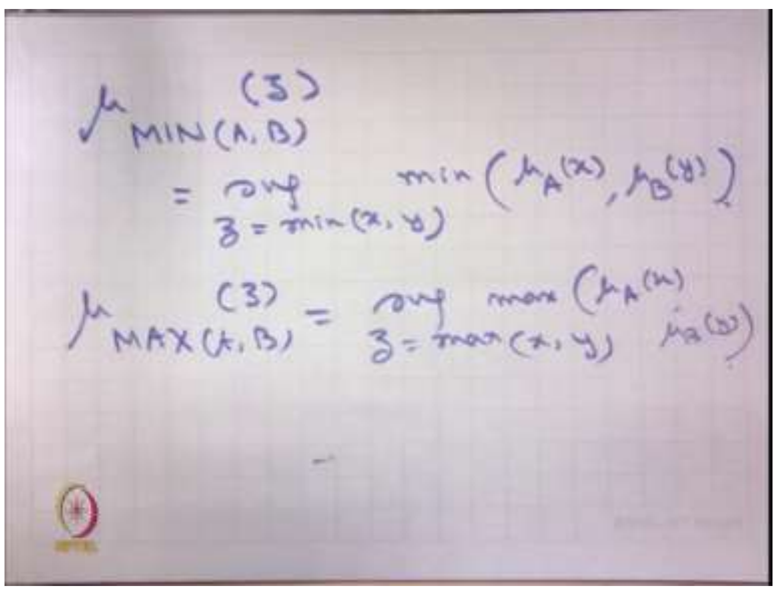

For your recollection,

$$\mu_{MIN(A,B)}(z) = \sup_{\text{z=min(x,y)}} \min\big(\mu_A(x), \mu_B(y)\big)$$

$$\mu_{MAX(A,B)}(z) = \sup_{\text{z=max(x,y)}} \min\big(\mu_A(x), \mu_B(y)\big)$$

Like that, when we are given different proportion type quantifiers we can infer about some other statement or other proposition p, when we are given $p_1, p_2 \ldots p_n$, that is n different quantifier propositions that are quantifier or the proportion.

Okay students, with that I stop fuzzy quantifier. In the next class, I shall look into very interesting type of fuzzy inference system, which is called Mamdani scheme for inferencing given a set of propositions or fuzzy propositions and an input variable. Till then thank you.