Introduction to Fuzzy Sets Arithmetic and Logic

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Lecture-25
Fuzzy Sets Arithmetic and Logic

Welcome students to the MOOCs course on fuzzy sets arithmetic and logic. This is lecture number 25.

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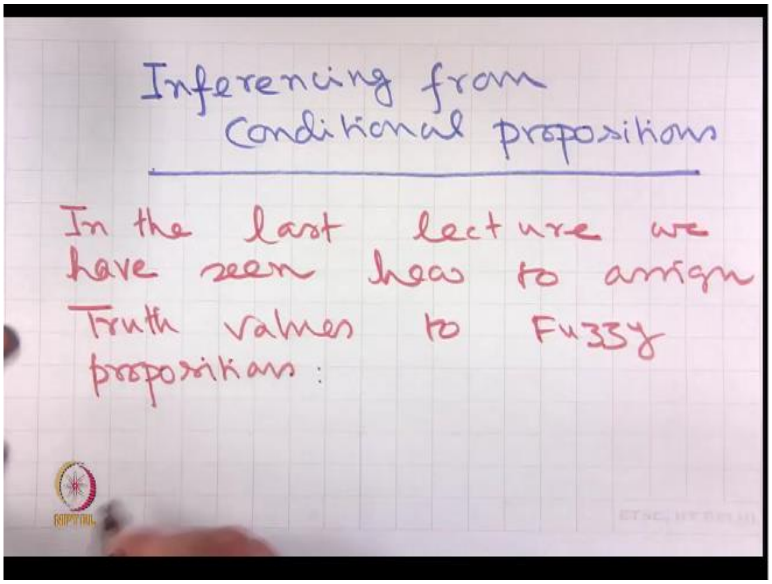

If you remember in the last lecture I said that in today's class I shall talk about Inferencing from Conditional Propositions.

Let us recall what we have done.

In the last lecture we have seen how to assign truth values to Fuzzy Propositions.

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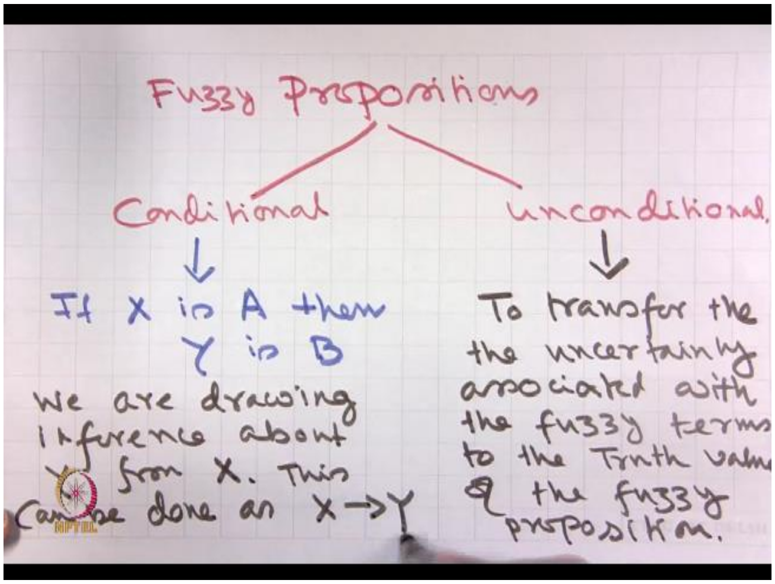

In particular, we have talked about the propositions of two types:

1. Conditional,
2. Unconditional.

Unconditional fuzzy propositions allowed us to transfer the uncertainty associated with the fuzzy terms to the truth value of the fuzzy proposition. So that gave us the foundation for the work.

On the other hand, in conditional what we have done we looked at statements of the form
If X is A, then Y is B, that is we are drawing inference about Y from X.
And this can be done as $X \rightarrow Y$.

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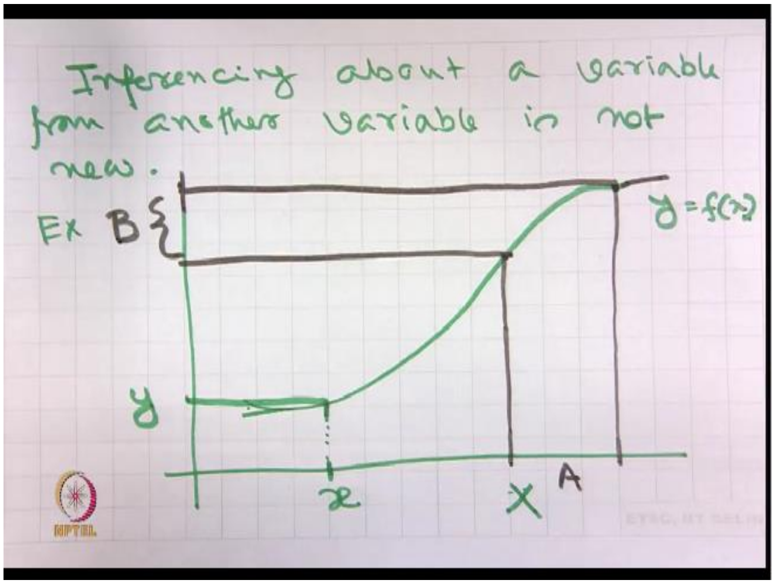

Now inferencing about a variable from another variable is not new.

For example:

Consider a function $y = f(x)$

What is the advantage?

If I give any value x for the variable X then we know from the function what is going to be the value for Y.

In fact, we can go even further suppose we give an interval on X and I call it A, then we can see that the value of Y is going to be in the interval B.

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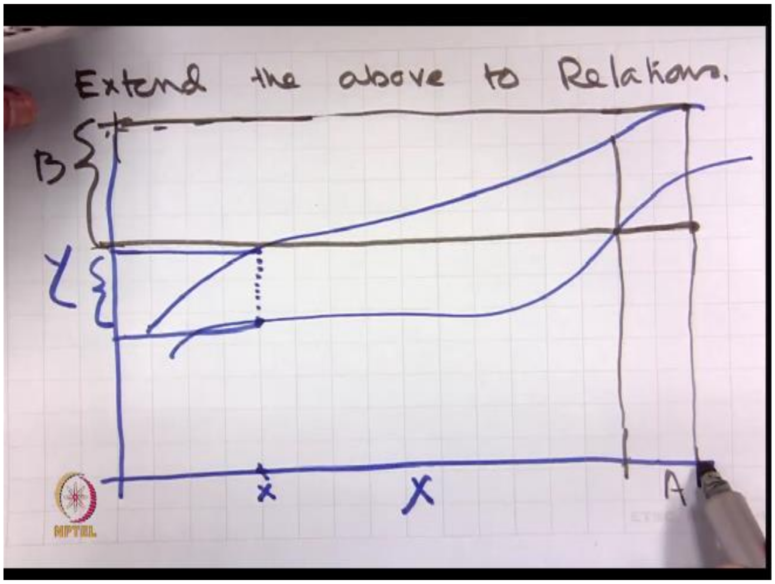

Suppose we extend it to relations.

There is relation for each x with several values of $Y's$ and each y may be related to several values of X. Corresponding to each x we know what are the y-values that are associated with. So, given x, I can identify the set of values which are associated with x.

On the other hand, if we consider an interval $[x_1, x_2]$ then, the set of values of y which are associated with this interval A on X, it does not mean that all y values in the interval B are associated with all values of x in the interval A. But we know that any y belonging to this interval B is associated with at least one x in this set A.

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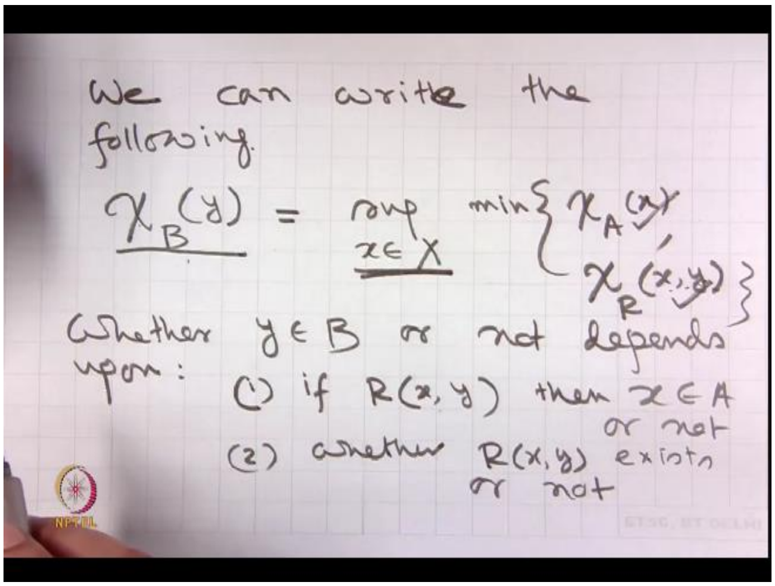

Therefore, we can write the following,

$$\chi_B(y) = \sup_{x \in X} (\min\{ \chi_A(x), \chi_R(x, y)\})$$

That is whether $y \in B$ or not depends upon:

1. If $R(x, y)$, then $x \in A$ or not
2. whether $R(x, y)$ exists or not.

So if both y is related to x and $x \in A$, then we see that $y \in B$.

Although here we have used supremum. We know that all the χ values are either 0 or 1. So when both of them are 1 then only we see that $y \in B$.

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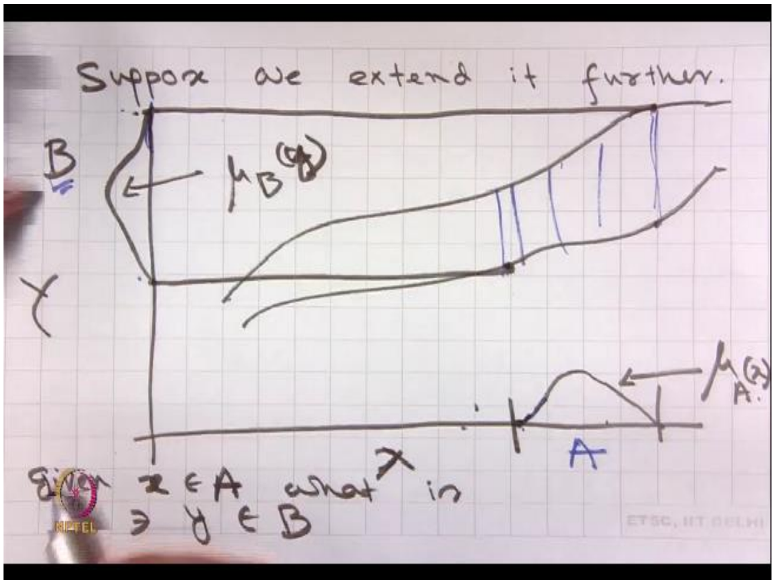

Now suppose we extend it further and we have the following relation a very similar type of drawing that I have drawn before. Suppose this gives us the relations of X and Y and we want to find given $x \in A$, where A is an interval. What is B?

Now we make further generalization, suppose A is a fuzzy set. Then as before we know that the set B which is associated with $x \in A$ and our target is to find $\mu_B(y)$ that describes the fuzzy set B. Or in other words, given the relation and the points which are associated with the corresponding x, we want to find out what is going to be the membership of an y belonging to interval B.

When we are looking at $x \to y$ and we are focusing on the subset A of X and we are trying to find out that membership of y to the fuzzy set B.

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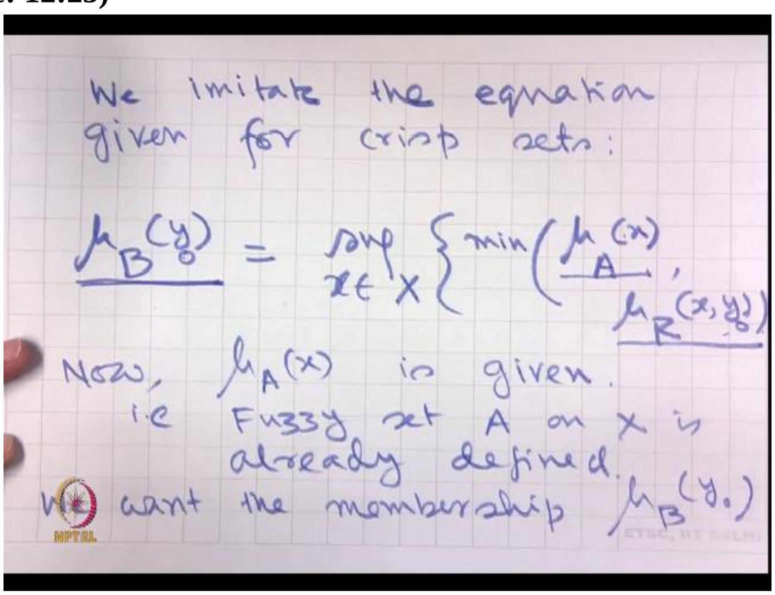

So how we proceed?

We imitate the equation given for crisp sets:

$$\mu_B(y_0) = \sup_{x \in X}\{\min(\mu_A(x), \mu_R(x, y_0))\}$$

We are just imitating what we did with respect to crisp sets and we can see that if we take the maximum or supremum of them that gives us $\mu_B(y_0)$. Now $\mu_A(x)$ is given for the fuzzy set A on X is already defined and we want the membership of y in B.

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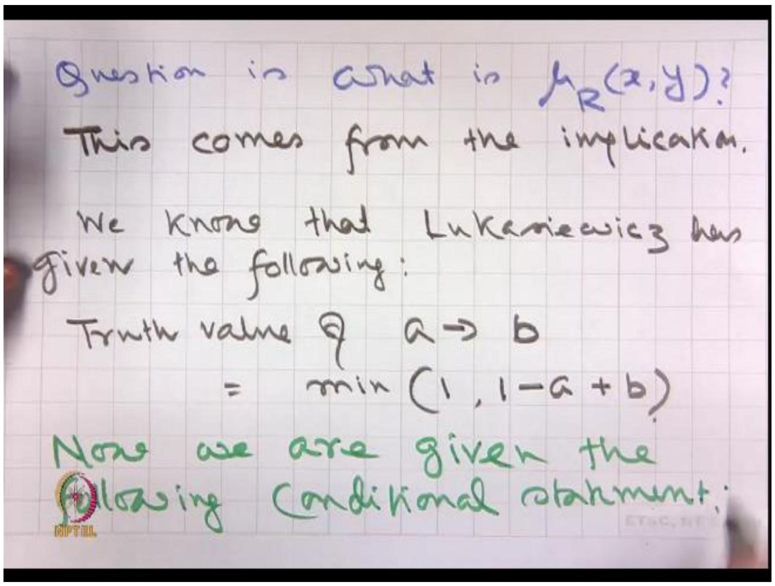

Question is what is $\mu_R(x, y)$?

This comes from the implication.

We know that Lukasiewicz has given the following:

Truth value of $a \to b = \min(1, 1 - a + b)$

Now we are given the following conditional statement.

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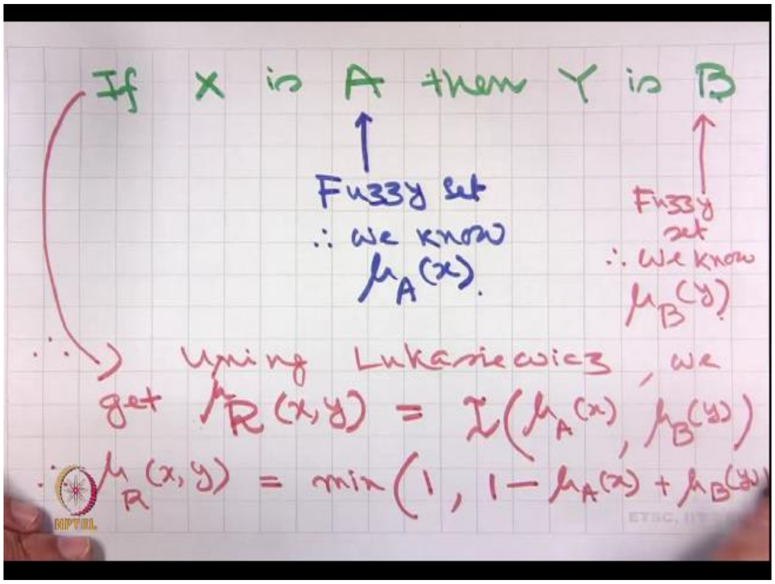

If X is A then Y is B, where A, B are fuzzy sets, therefore, we know $\mu_A(x)$ and $\mu_B(y)$.

Therefore from this above implication using Lukasiewicz we get

$$\mu_R(x,y) = \mathfrak{I}\big(\mu_A(x), \mu_B(y)\big)$$

$$\therefore \mu_R(x,y) = \min\big(1, 1 - \mu_A(x) + \mu_B(y)\big)$$

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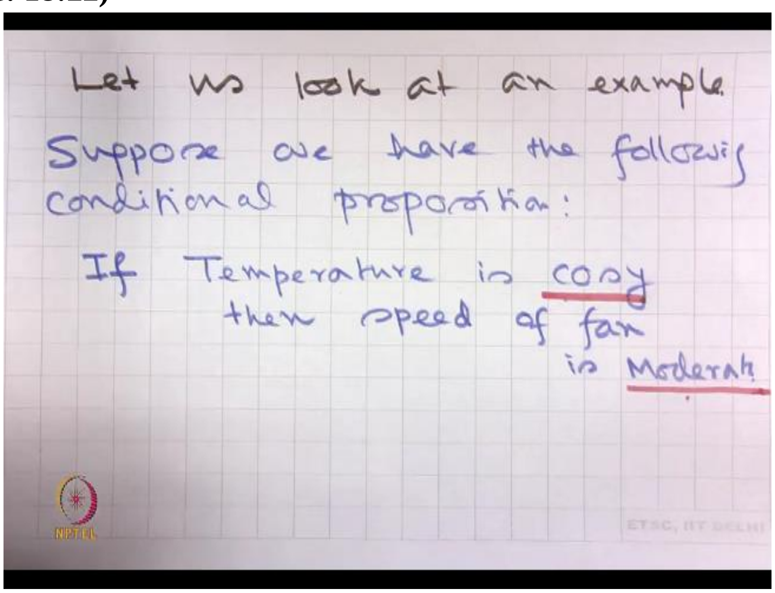

Let me now illustrate with an example, suppose we have the following conditional proposition:

$$If\ Temperature\ is\ Cosy\ then\ Speed\ of\ Fan\ is\ Moderate$$

So we can see that there are two fuzzy terms A and B where A is $Temperature\ is\ Cosy$ and B is with respect to the Speed of Fan that the $Speed\ is\ Moderate$.

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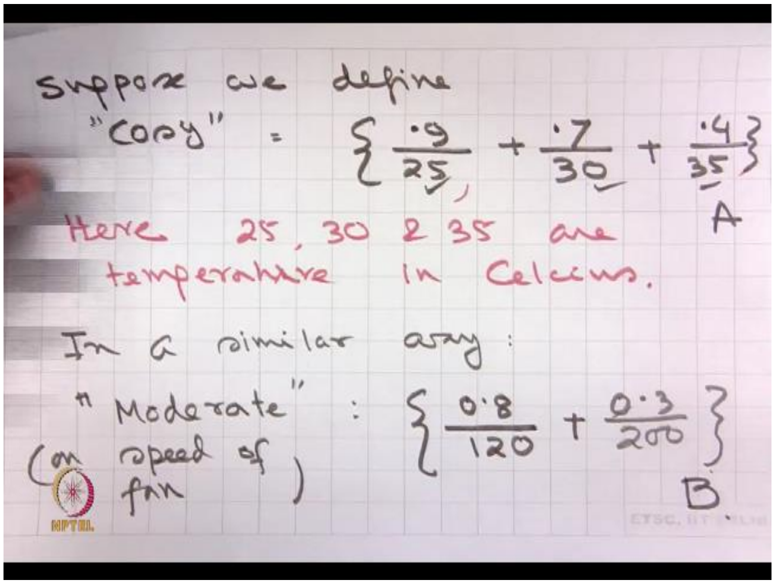

Suppose we define

$$A: \quad Cosy = \left\{\frac{0.9}{25} + \frac{0.7}{30} + \frac{0.4}{35}\right\}$$

Here $25, 30$ and 35 are temperature in ℃.

In other words, we are saying that if the temperature is 25℃ then membership to coziness is 0.9, similarly for 30℃ it is 0.7, for 35℃ it is 0.4.

So to keep it simple we have taken discrete values.

In practice, of course, one can take a continuous range of temperature and one can assign the membership values by defining appropriate function.

In a similar way suppose the fuzzy set $Moderate$ on speed of fan is defined as follows:

$$B: \ Moderate = \left\{\frac{0.8}{120} + \frac{0.3}{200}\right\}$$

Where 120 are 200 revolutions per minute.

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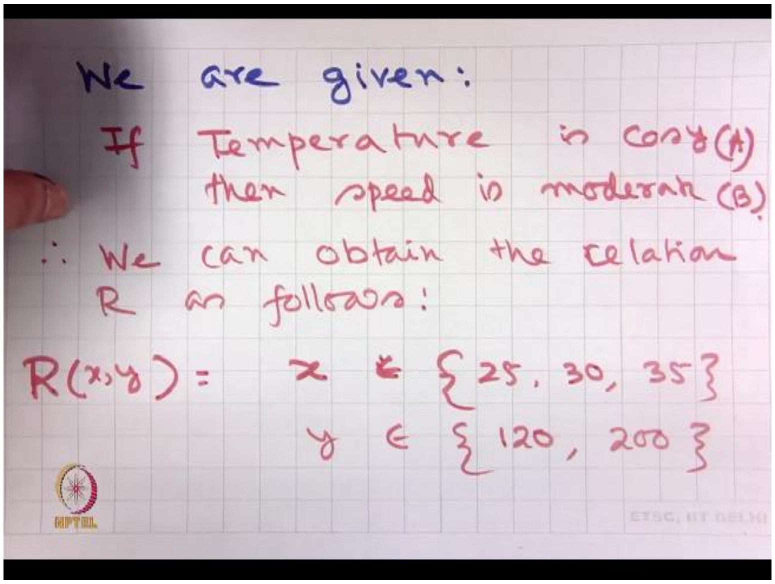

We are given:

$$If\ Temperature\ is\ Cosy\ (A)\ then, Speed\ is\ Moderate\ (B)$$

Therefore, we can obtain the relation R as follows:

$R(x, y)$ where $x \in \{25, 30, 35\}$ and $y \in \{120, 200\}$

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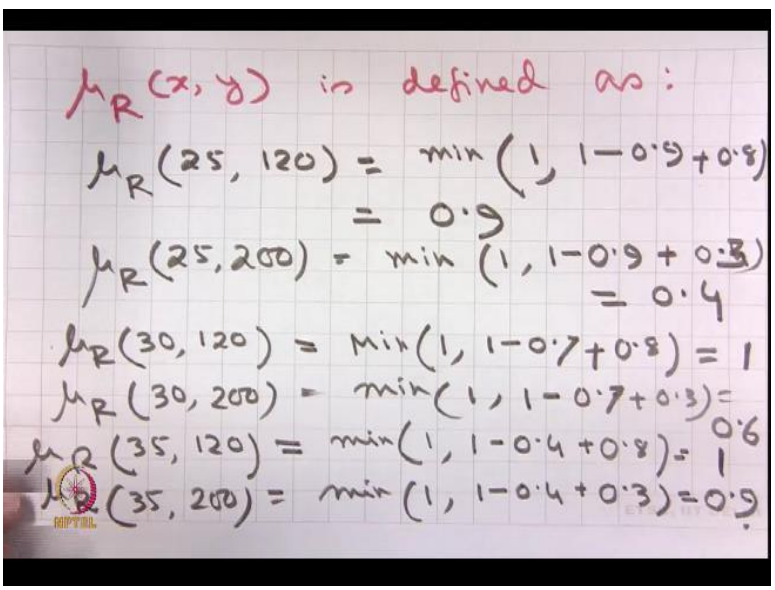

Therefore, $\mu_R(x, y)$ is defined as:

- $\mu_R(25, 120) = \min(1, 1 - 0.9 + 0.8) = 0.9$
- $\mu_R(25, 200) = \min(1, 1 - 0.9 + 0.3) = 0.4$
- $\mu_R(30, 120) = \min(1, 1 - 0.7 + 0.8) = 1$

- $\mu_R(30, 200) = \min(1, 1 - 0.7 + 0.3) = 0.6$
- $\mu_R(35, 120) = \min(1, 1 - 0.4 + 0.8) = 1$
- $\mu_R(35, 200) = \min(1, 1 - 0.4 + 0.3) = 0.9$

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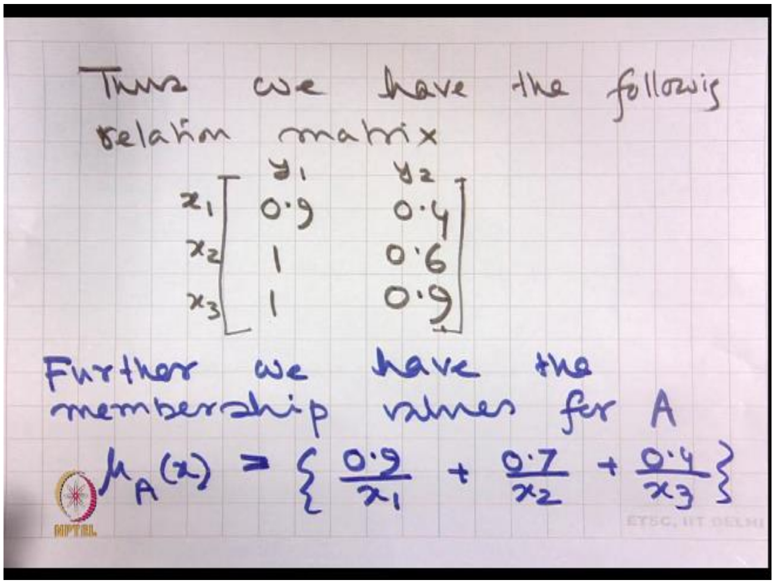

Thus we have the following relation matrix.

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.9 & 0.4 \\ 1 & 0.6 \\ 1 & 0.9 \end{bmatrix} \end{array}$$

Further we have the membership for the fuzzy set A

$$\mu_A(x) = \left\{\frac{0.9}{x_1} + \frac{0.7}{x_2} + \frac{0.4}{x_3}\right\}$$

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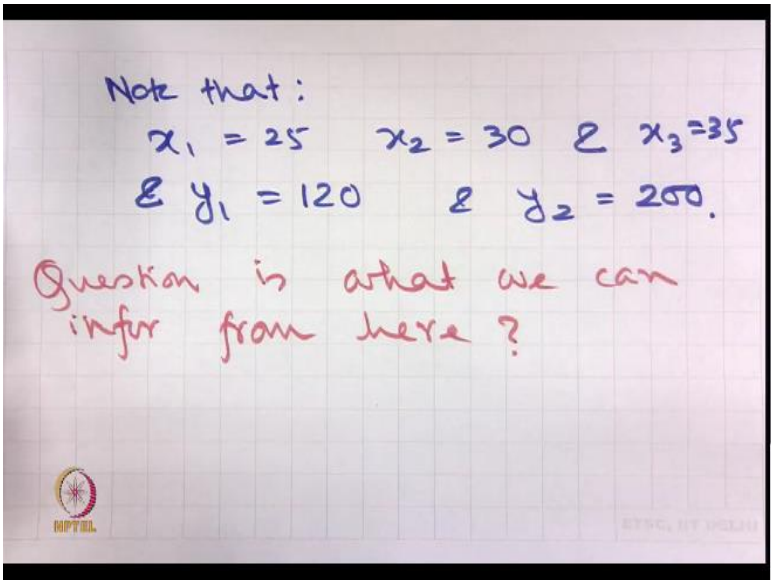

Note that:

$x_1 = 25,\ x_2 = 30$ and $x_3 = 35,$

$y_1 = 120$ and $y_2 = 200$

So question is, what we can infer from here?

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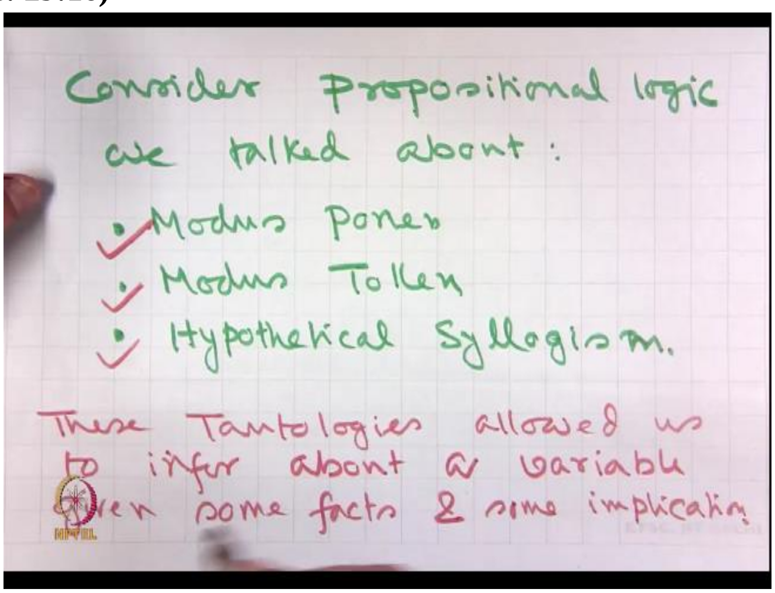

So let us look at the following:

In propositional logic we talked about

- Modus Ponen,
- Modus Tollen,
- Hypothetical Syllogism.

So, these 3 tautologies allowed us to infer about a variable, given some facts and some implication. So when we are talking about fuzzy we want to generalize this concept of Modus Ponen, Modus Tollen, and Hypothetical Syllogism.

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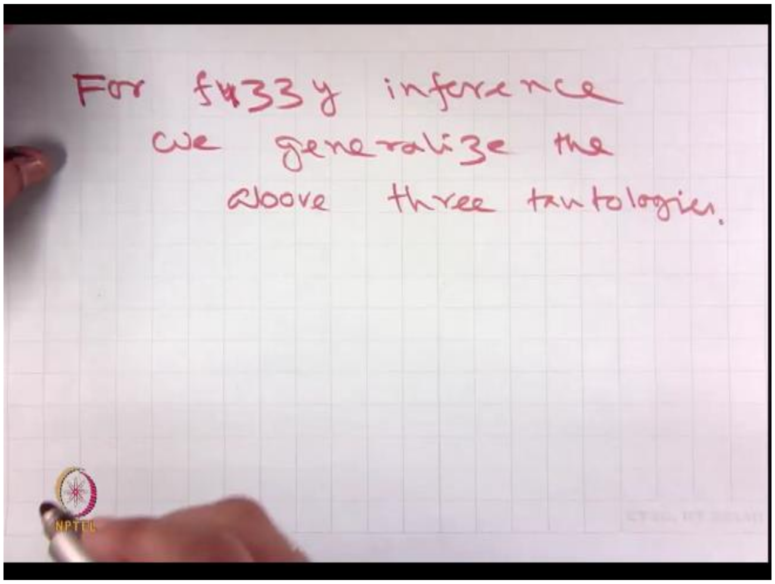

So for Fuzzy Inference we generalize the above three tautologies, so let me explain them.

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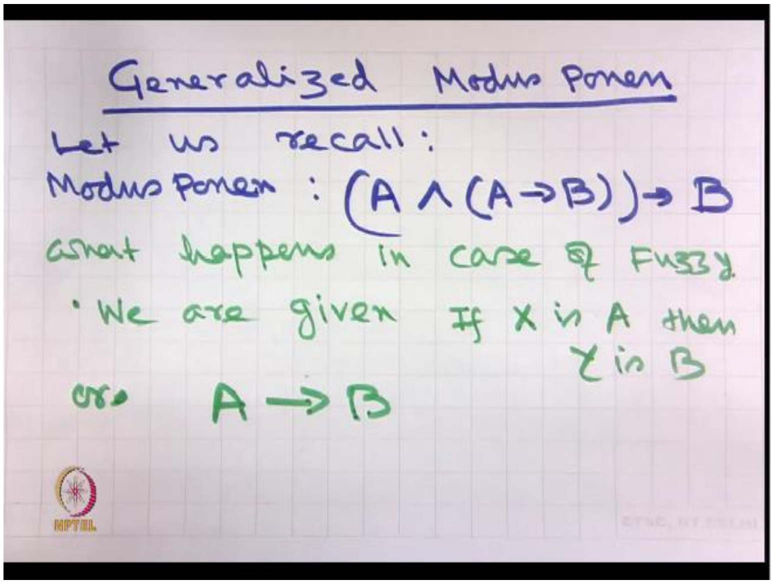

Consider Generalized Modus Ponen.

Let us recall:

Modus Ponen: $\left(A \wedge (A \rightarrow B)\right) \rightarrow B$

What happens in case of Fuzzy?

We are given

$If\ X\ is\ A, then\ Y\ is\ B$ or we can write it as $A \rightarrow B$

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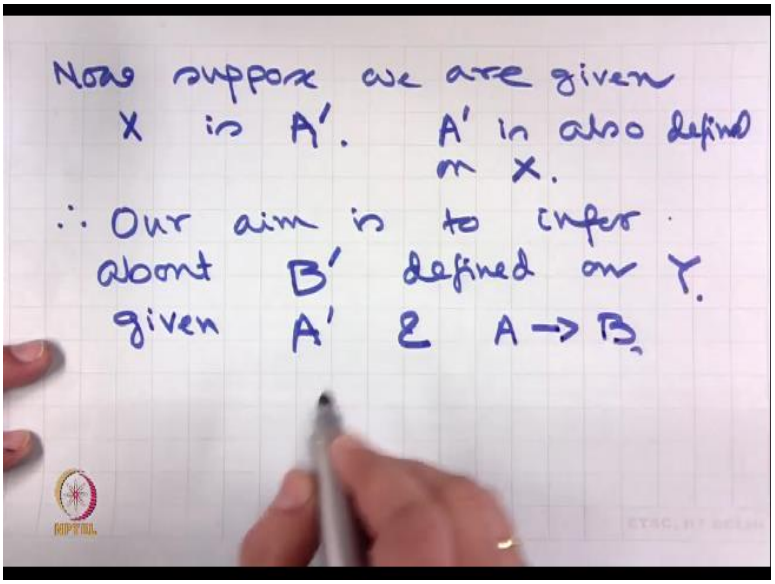

Now suppose we are given

X is A' , A' is also defined on X

Therefore, our aim is to infer about a fuzzy set B' defined on Y given A' and $A \rightarrow B$

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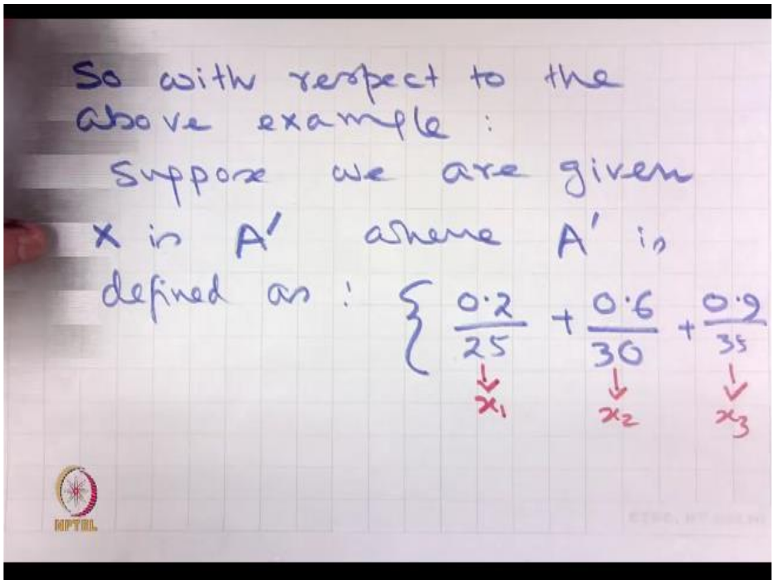

So with respect to the above example:

Suppose we are given X is A' where A' is defined as:

$$A' = \left\{\frac{0.2}{25} + \frac{0.6}{30} + \frac{0.9}{35}\right\}$$

If you remember we are calling $25, 30$ and 35 as x_1, x_2 and x_3 respectively.

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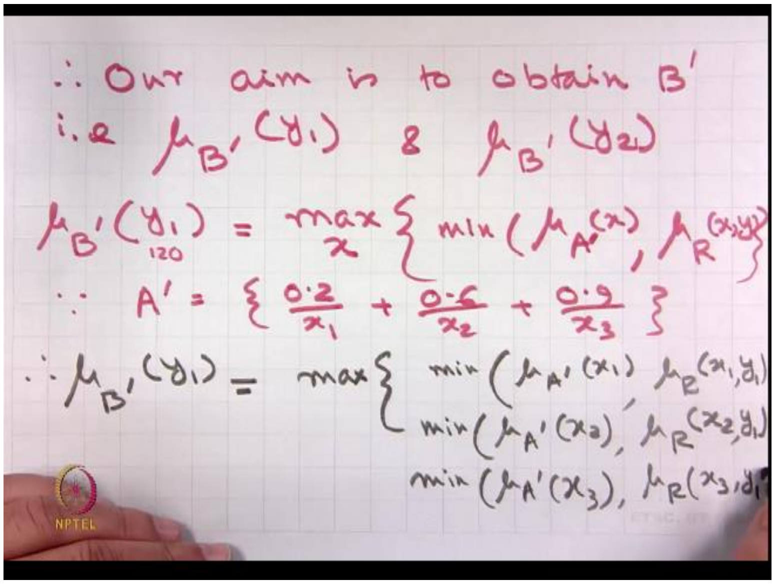

Therefore, our aim is to obtain B' that is $\mu_{B'}(y_1)$ and $\mu_{B'}(y_2)$

$$\mu_{B'}(y_1) = \max_x \{\min(\mu_{A'}(x), \mu_R(x, y_1))\}$$

$$\because\ A' = \left\{\frac{0.2}{x_1} + \frac{0.6}{x_2} + \frac{0.9}{x_3}\right\}$$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \left[\begin{array}{cc} 0.9 & 0.4 \\ 1 & 0.6 \\ 1 & 0.9 \end{array}\right] \end{array}$$

$$\therefore \mu_{B'}(y_1) = \max \left\{\begin{array}{l} \min(\mu_{A'}(x_1), \mu_R(x_1, y_1)) \\ \min(\mu_{A'}(x_2), \mu_R(x_2, y_1)) \\ \min(\mu_{A'}(x_3), \mu_R(x_3, y_1)) \end{array}\right\}$$

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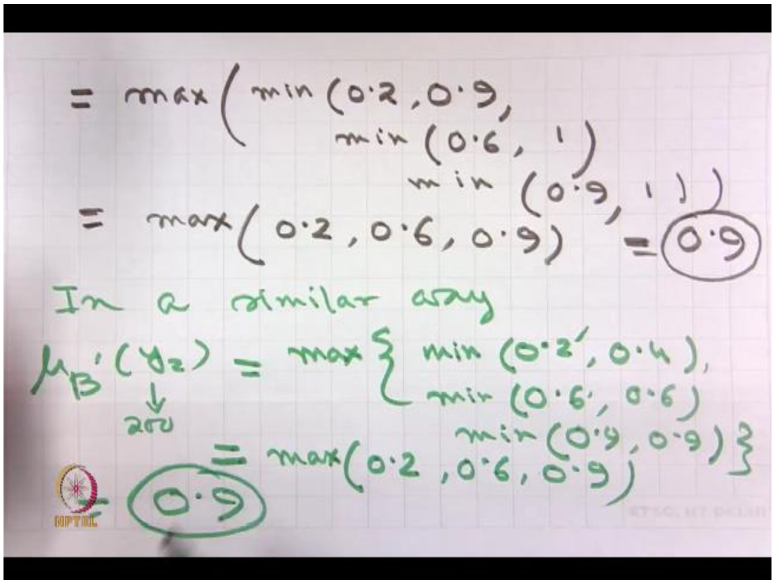

$$\Rightarrow \mu_{B'}(y_1) = \max \begin{Bmatrix} \min(0.2, 0.9) \\ \min(0.6, 1) \\ \min(0.9, 1) \end{Bmatrix} = \max\{0.2, 0.6, 0.9\} = 0.9$$

In a similar way,

$$\mu_{B'}(y_2) = \max \begin{Bmatrix} \min(0.2, 0.4) \\ \min(0.6, 0.6) \\ \min(0.9, 0.9) \end{Bmatrix} = \max\{0.2, 0.6, 0.9\} = 0.9$$

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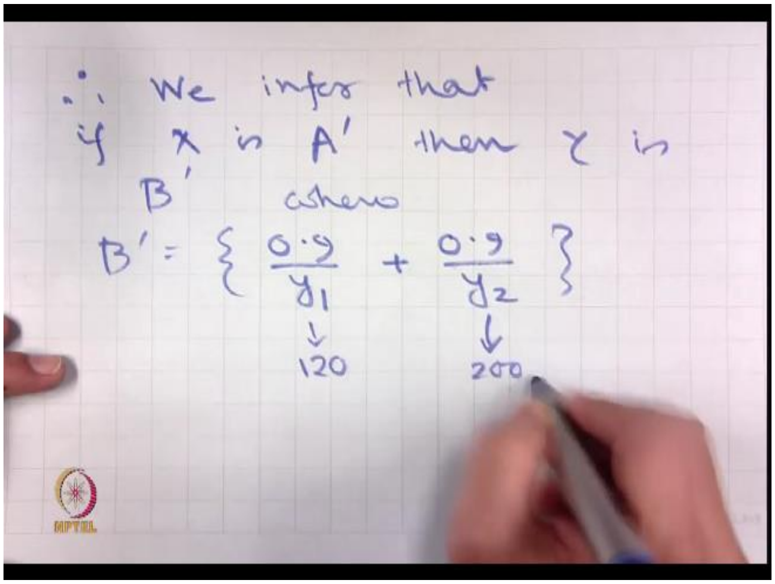

Therefore, we infer that

$$If\ X\ is\ A', then\ Y\ is\ B'$$

Where $B' = \left\{\frac{0.9}{y_1} + \frac{0.9}{y_2}\right\}$, $y_1 = 120$ and $y_2 = 200$

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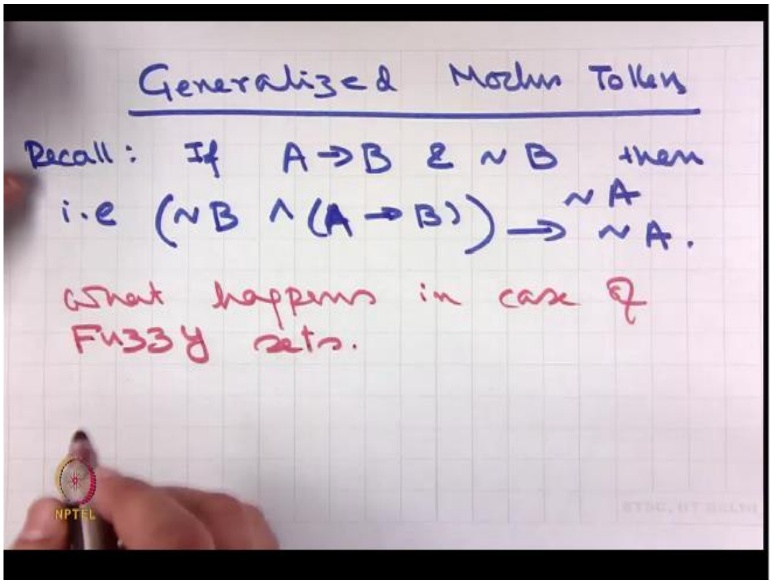

Let us now look at Generalized Modus Tollen.

So let us recall

Modus Tollen : If $(A \to B)$ and $\sim B$ then, $\sim A$ that is,

$$\left(\sim B \wedge (A \to B)\right) \to \sim A$$

That is if we are given the implication and we are also given that some information about B then we can derive some information about A.

Let us now look at in case of fuzzy sets.

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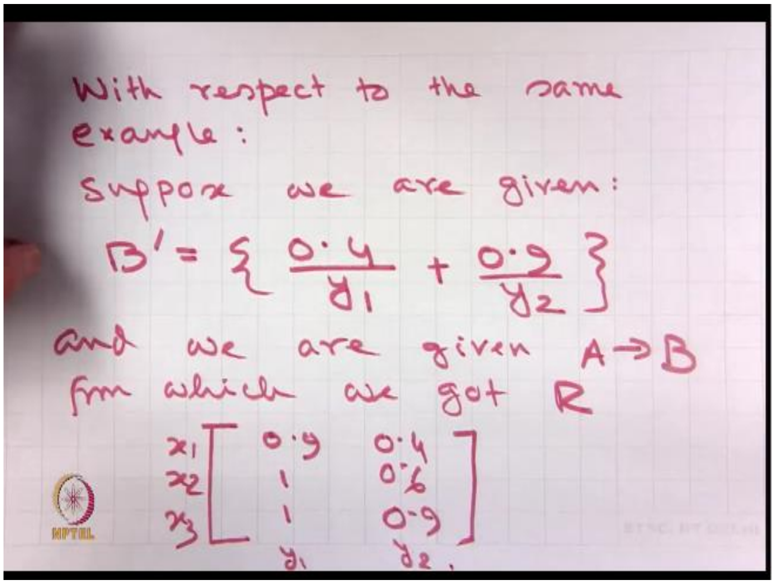

So with respect to the same example:

Suppose we are given:

$$B' = \left\{\frac{0.4}{y_1} + \frac{0.9}{y_2}\right\}$$

and we are given $A \rightarrow B$ from which we got the R.

What was that R?

Just for your recollection.

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.9 & 0.4 \\ 1 & 0.6 \\ 1 & 0.9 \end{bmatrix} \end{array}$$

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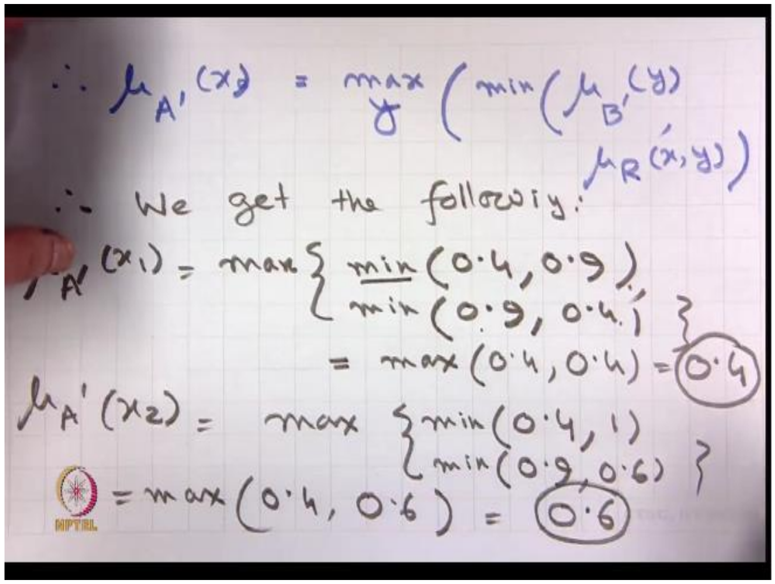

Therefore,

$$\mu_{A'}(x) = \max_{y}\{\min(\mu_{B'}(y), \mu_R(x, y))\}$$

Therefore, we get the following:

$$\mu_{A'}(x_1) = \max\begin{Bmatrix}\min(0.4, 0.9)\\ \min(0.9, 0.4)\end{Bmatrix} = \max\{0.4, 0.4\} = 0.4$$

$$\mu_{A'}(x_2) = \max\begin{Bmatrix}\min(0.4, 1)\\ \min(0.9, 0.6)\end{Bmatrix} = \max\{0.4, 0.6\} = 0.6$$

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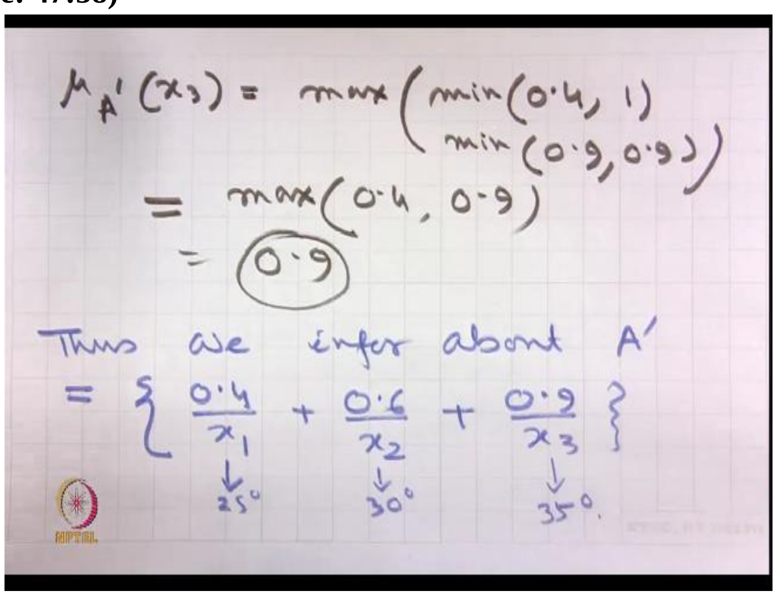

$$\mu_{A'}(x_3) = \max\begin{Bmatrix} \min(0.4, 1) \\ \min(0.9, 0.9) \end{Bmatrix} = \max\{0.4, 0.9\} = 0.9$$

Thus, we infer about A'

$$A' = \left\{\frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.9}{x_3}\right\}$$

Here, $25°C, 30°C$ and $35°C$ as x_1, x_2 and x_3 respectively.

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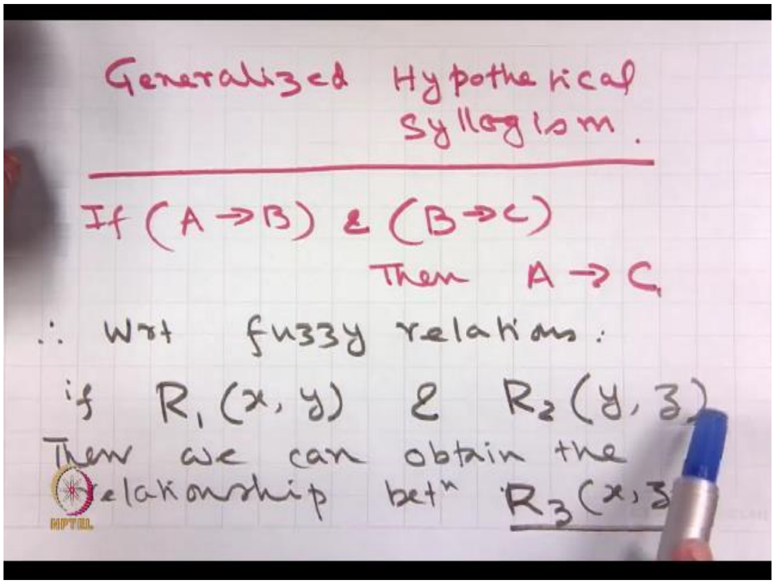

Next let us look at Generalized Hypothetical Syllogism.

It says that with respect to propositional logic:

$$If\ A \to B\ and\ B \to C, then\ A \to C$$

So what is going to happen for fuzzy relations:

If $R_1(x, y)$ is the relationship between x and y and $R_2(y, z)$ is a relationship between y and z then, we can obtain the relationship $R_3(x, z)$ between x and z.

Or, in other words given the two relations $R_1(x, y)$ and $R_2(y, z)$ we get the relation between the variables x and z.

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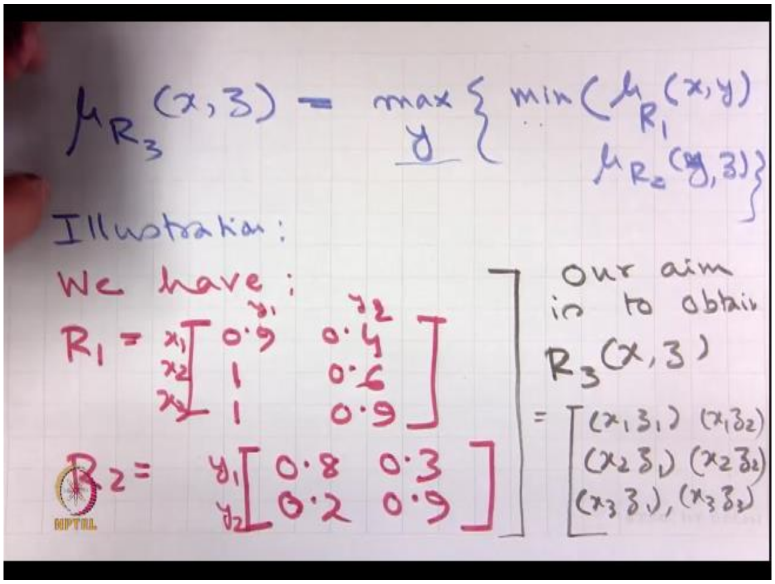

And the formula for this is

$$\mu_{R_3}(x,z) = \max_{y}\left\{\min\left(\mu_{R_1}(x,y), \mu_{R_2}(y,z)\right)\right\}$$

That is we are looking at the strength of relationship between x and y, and y and z.

So for illustration we already had:

$R_1 =$

	y_1	y_2
x_1	0.9	0.4
x_2	1	0.6
x_3	1	0.9

$R_2 =$

	z_1	z_2
y_1	0.8	0.3
y_2	0.2	0.9

So, from these two we want R_3 between x, z which is going to be basically the strength of relationship between $(x_1, z_1), (x_1, z_2), (x_2, z_1), (x_2, z_2)$ and $(x_3, z_1), (x_3, z_2)$

So we do it in a very straightforward way.

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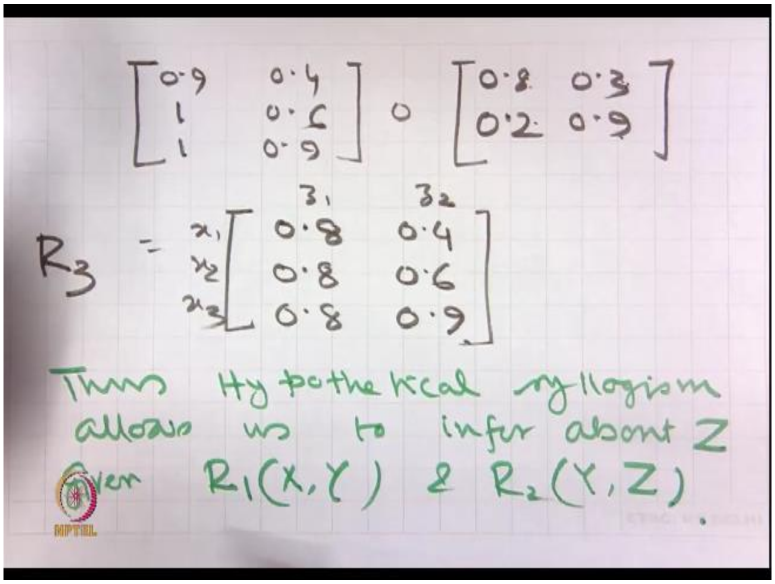

So we take the matrix R_1 and we compose it with R_2 using max-min composition.

$$\begin{bmatrix} 0.9 & 0.4 \\ 1 & 0.6 \\ 1 & 0.9 \end{bmatrix} \circ \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.9 \end{bmatrix}$$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.8 & 0.4 \\ 0.8 & 0.6 \\ 0.8 & 0.9 \end{bmatrix} \end{array}$$

Thus, Hypothetical Syllogism allows us to infer about Z, given R_1 between X and Y and R_2 between Y and Z.

Ok friends I stop here today, but before I stop let me tell you that.

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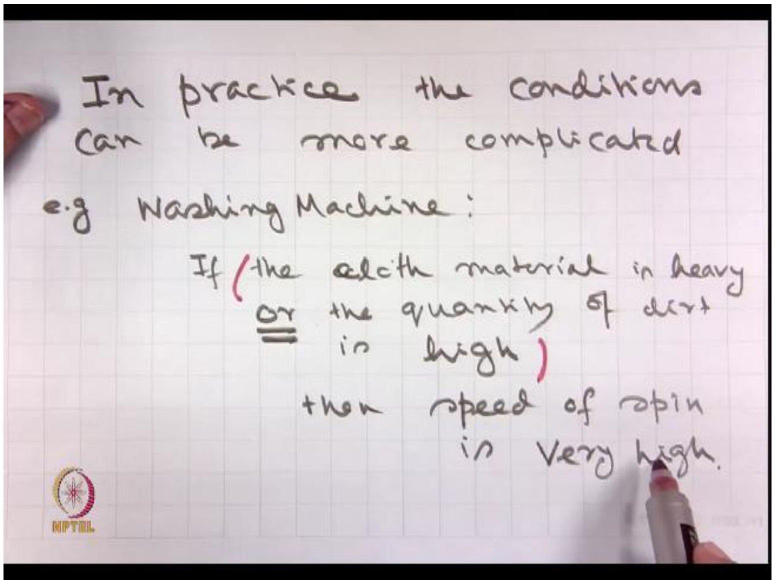

In practice the conditions can be more complicated.

For example, say with respect to washing machine, one can say

If the cloth material is heavy or the quantity of dirt is high, then speed of spin is very high.

Thus we get a disjunction of two classes which imply the value of the variable speed of spin.

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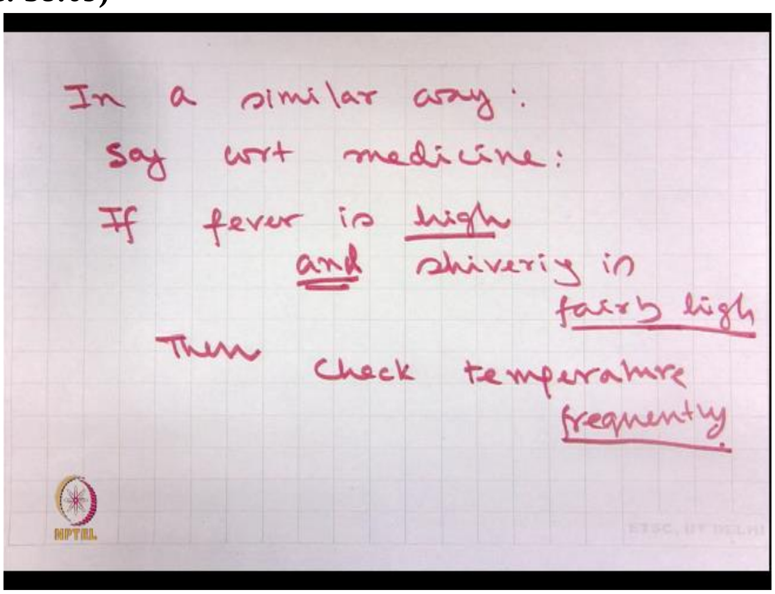

In a similar way, it can be conjunctive, say with respect to medicine,

If fever is high and shivering is a fairly high, then check temperature frequently.

So this is a fuzzy term with a linguistic hedge and we have a conjunction of these two clause which imply this fuzzy term frequently. We know how to handle or/and that is conjunctions and disjunctions and therefore given the background that we have discussed today.

We know how to compute the membership values for this and from there how to get the implication relationship and from there depending upon the problem we can use generalized Modus Ponen or Modus Tollen or Hypothetical Syllogism to infer about some other variables.

Ok students I stop here today. In the next class I shall continue with inferencing with conditional and qualified propositions and after that we shall look into fuzzy quantifiers and how we can infer from fuzzy quantified propositions. Till then thank you all.