Introduction to Fuzzy Sets Arithmetic and Logic
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Lecture minus 22
Fuzzy Sets Arithmetic & Logic

Welcome students to the MOOCs lecture on fuzzy sets arithmetic and logic. This is lecture number 22.

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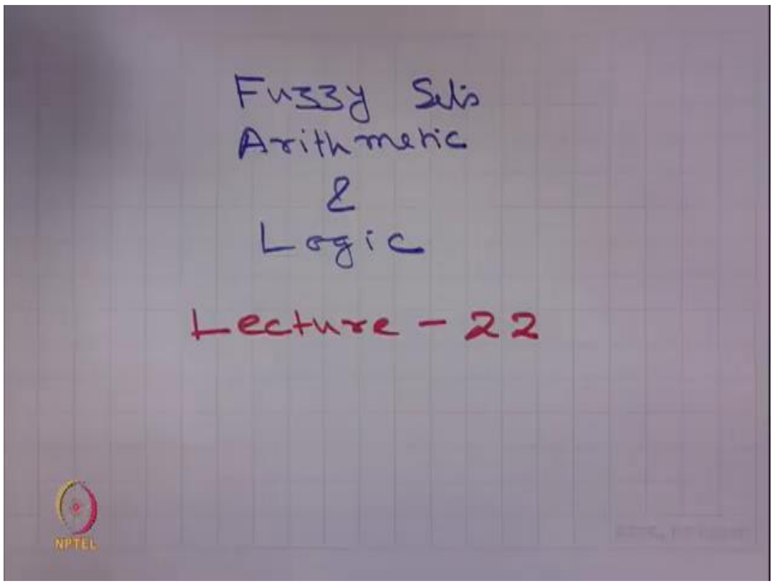

And as you all know in the last class I started fuzzy logic.

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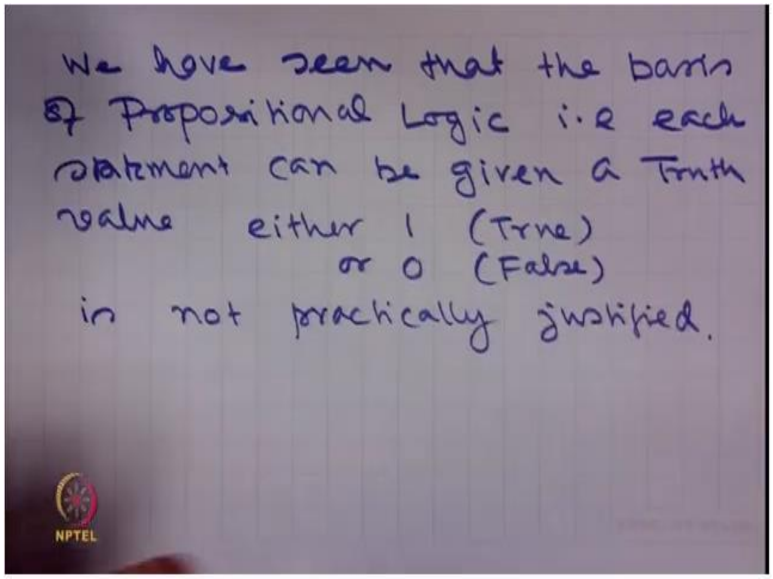

We have seen that the basis of Propositional Logic that is each statement can be given a truth value either 1 that is True or 0 that is False is not practically justified.

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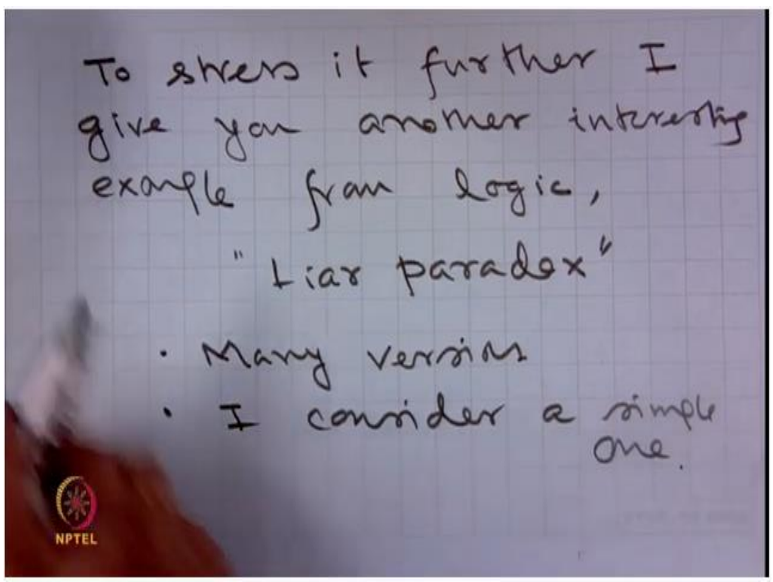

To stress the point further I give you another interesting example from logic known as Liar Paradox.

So there are many versions, I take a simple one.

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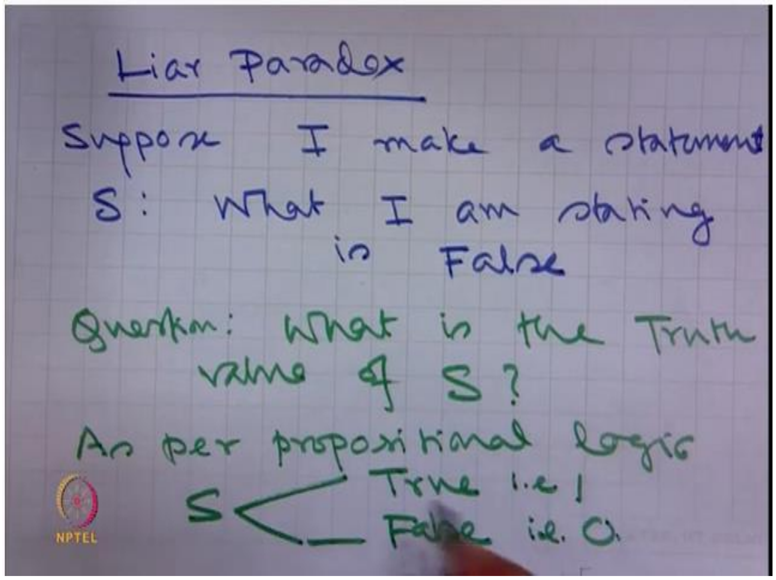

<u>Liar Paradox</u>

Suppose I make a statement

S: What I am stating is False.

Question is what is the truth value of S?

So we know that is per propositional logic

S can have two truth values,

- True that is 1
- False that is 0.

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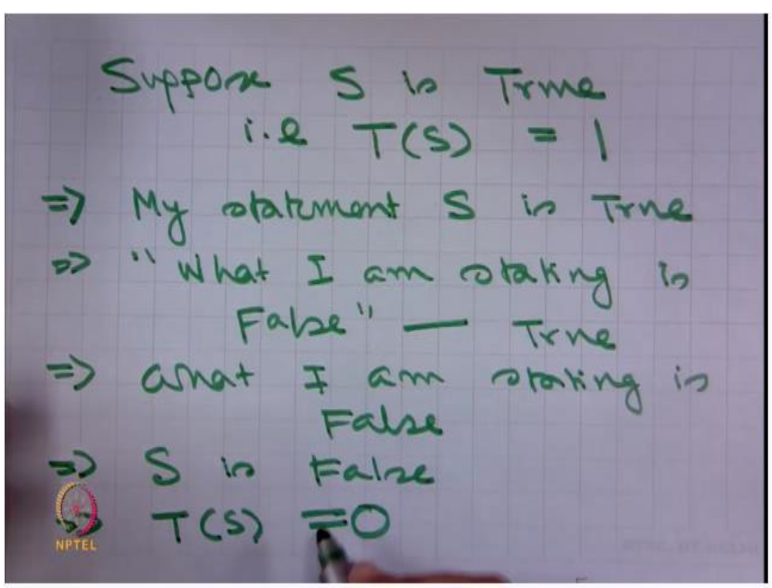

Suppose S is True that is $T(S) = 1$

⇒ My statement S is True

⇒ ‘What I am stating is False’ ---this is True

⇒ What I am stating is False

⇒ S is False

⇒ $T(S) = 0$

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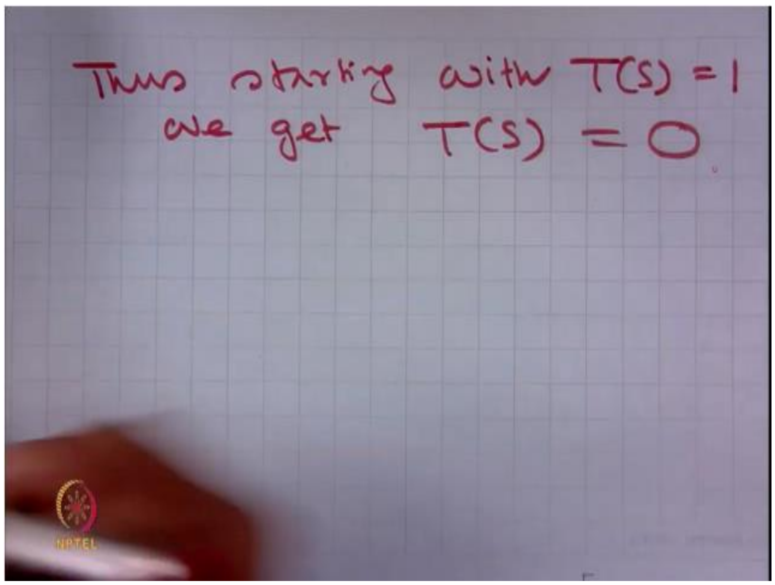

Thus starting with $T(S) = 1$ we get $T(S) = 0$

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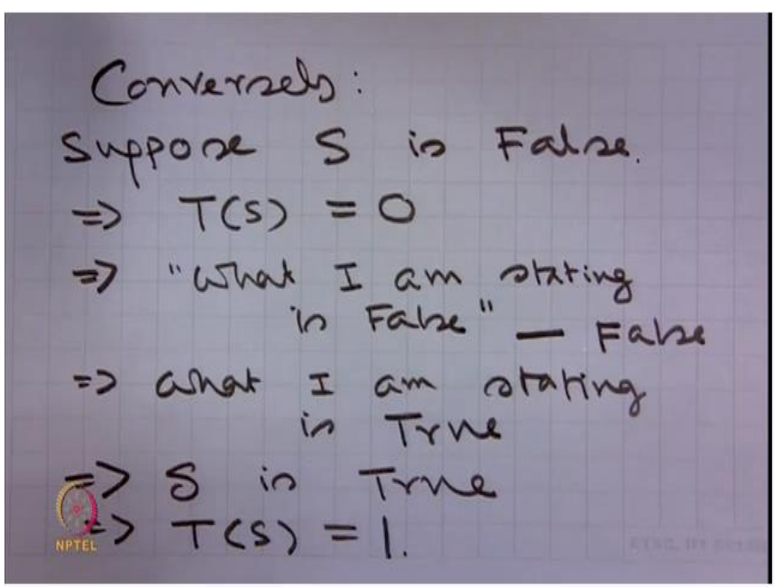

Conversely,

Suppose S is False

$\Rightarrow T(S) = 0$

$\Rightarrow$ 'What I am stating is False', this statement is False

$\Rightarrow$ What I am stating is True

$\Rightarrow S$ is True

$\Rightarrow T(S) = 1$

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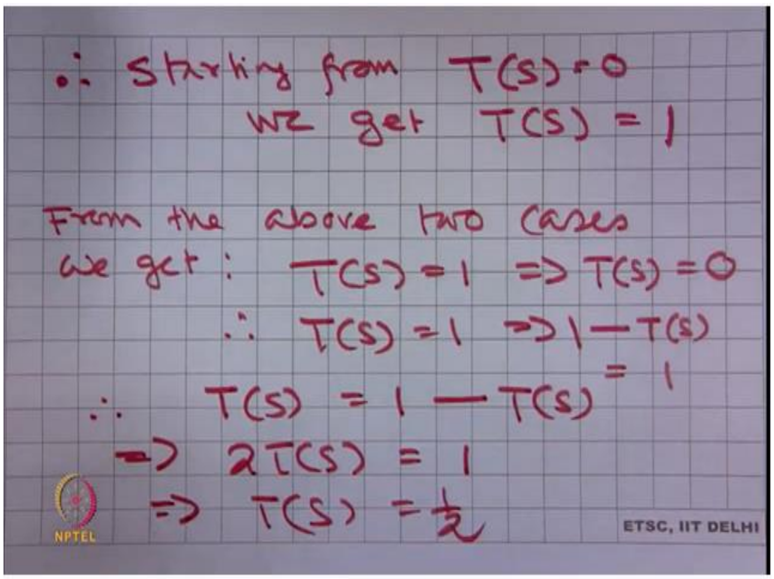

Therefore, starting from $T(S) = 0$ we get $T(S) = 1$

From the above two cases we get:

$T(S) = 1 \Rightarrow T(S) = 0$

$\therefore T(S) = 1 \Rightarrow 1 - T(S) = 1.$

$\therefore T(S) = 1 - T(S)$

$\Rightarrow 2T(S) = 1$

$\Rightarrow T(S) = \frac{1}{2}$

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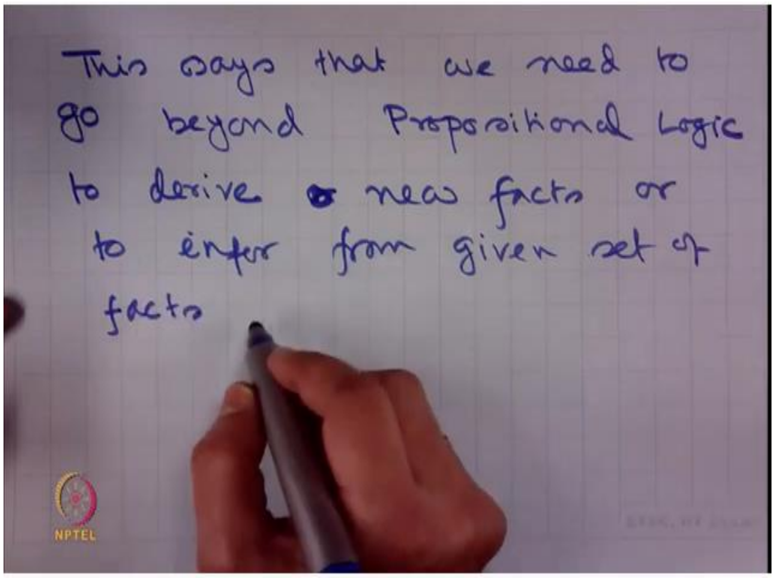

This says that we need to go beyond propositional logic to derive new facts or to infer from given set of facts.

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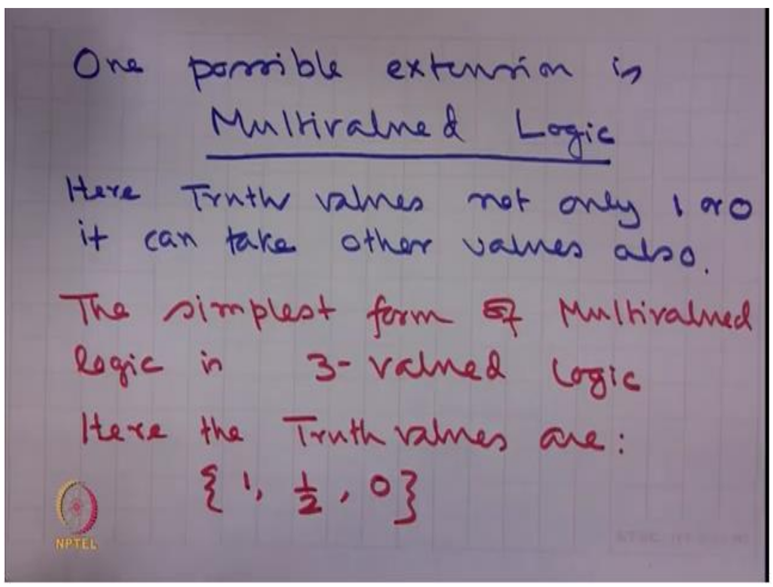

One possible extension is Multivalued Logic.

Here truth values are not only 1 or 0, it can take other values also.

The simplest form of multivalued logic is 3-valued logic. Here the truth values are $\left\{1, \frac{1}{2}, 0\right\}$.

Many researchers have given their interpretation with 3-valued logic.

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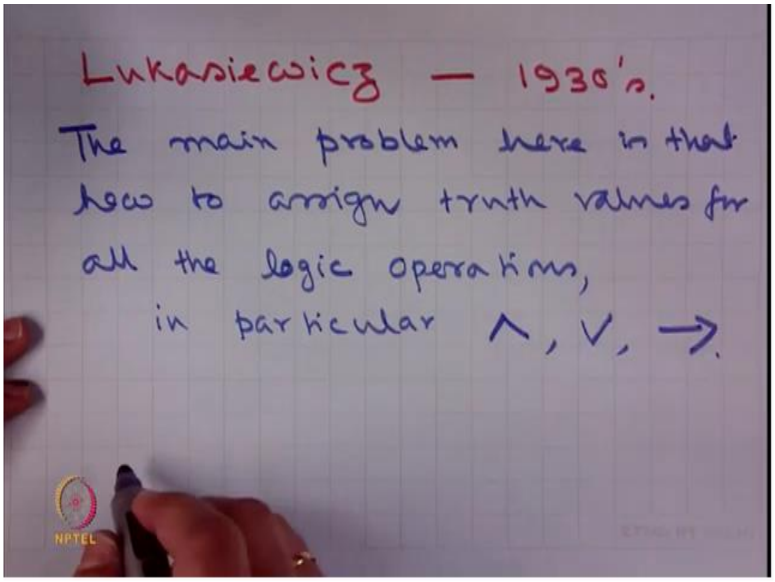

But the most popular one is by Lukasiewicz in the 1930s.

The main problem here is that how to assign truth values for all the logic operations.

In particular, $\wedge, \vee, \rightarrow$

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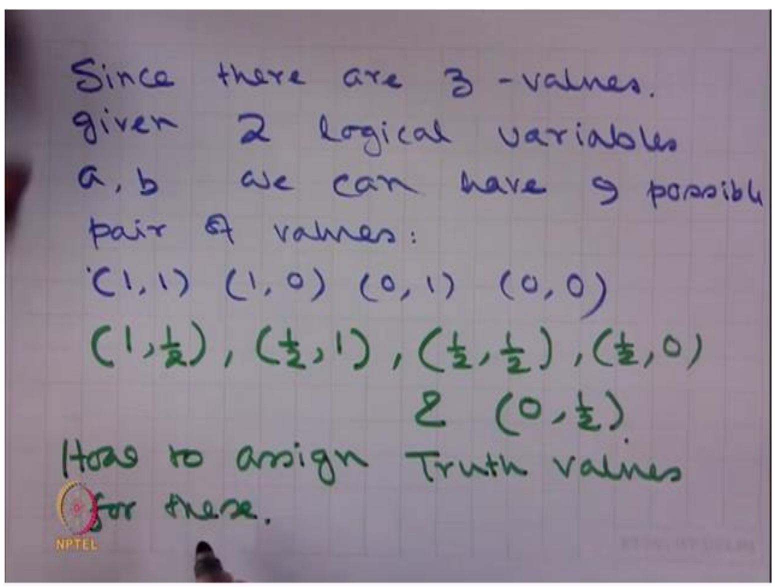

Since there are 3 values given 2 logical variables a, b.

We can have 9 possible pair of values:

- $(1, 1)$
- $(1, 0)$
- $(0, 1)$

- $(0, 0)$
- $\left(1, \frac{1}{2}\right)$
- $\left(\frac{1}{2}, 1\right)$
- $\left(\frac{1}{2}, \frac{1}{2}\right)$
- $\left(\frac{1}{2}, 0\right)$
- $\left(0, \frac{1}{2}\right)$

How to assign truth values for these?

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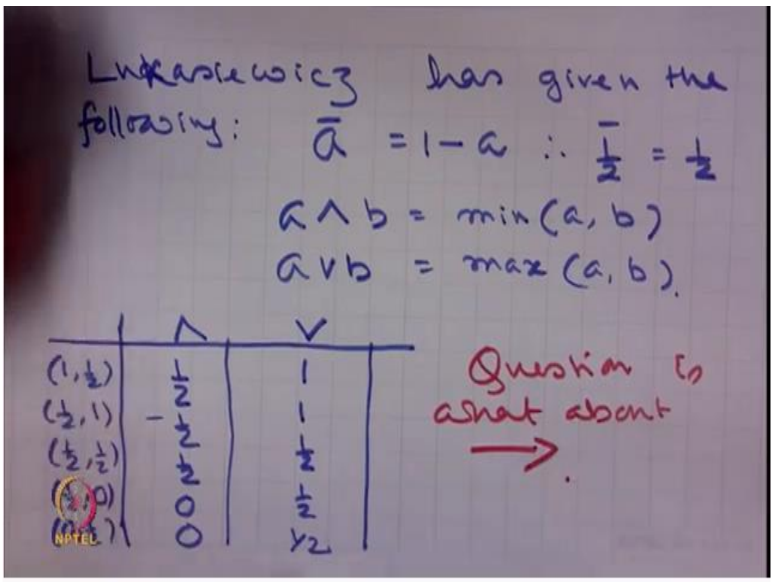

So Lukasiewicz has given the following:

- $\bar{a} = 1 - a \qquad \therefore \bar{\frac{1}{2}} = \frac{1}{2}$
- $a \wedge b = \min(a, b)$
- $a \vee b = \max(a, b)$

	$\wedge$	$\vee$
$\left(1, \frac{1}{2}\right)$	$\frac{1}{2}$	1
$\left(\frac{1}{2}, 1\right)$	$\frac{1}{2}$	1
$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\frac{1}{2}$	$\frac{1}{2}$
$\left(\frac{1}{2}, 0\right)$	0	$\frac{1}{2}$
$\left(0, \frac{1}{2}\right)$	0	$\frac{1}{2}$

The question is what about implication?

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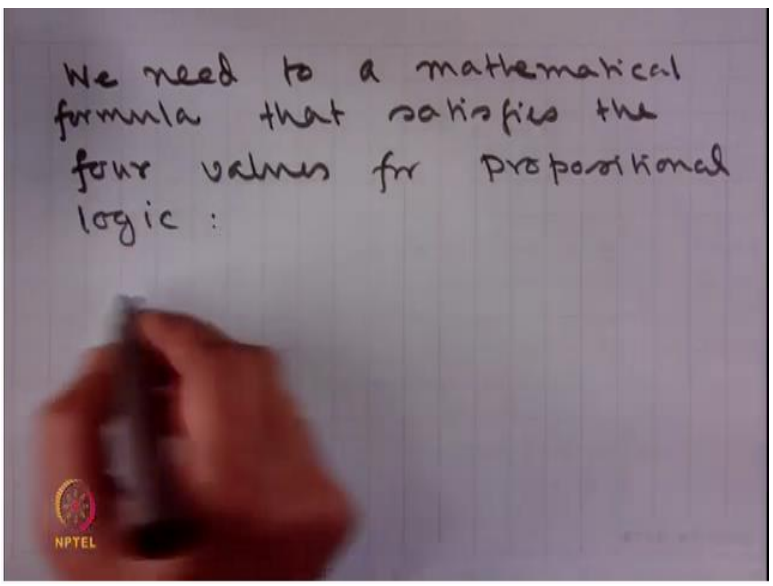

We need to find a mathematical formula that satisfies the four values for Propositional Logic.

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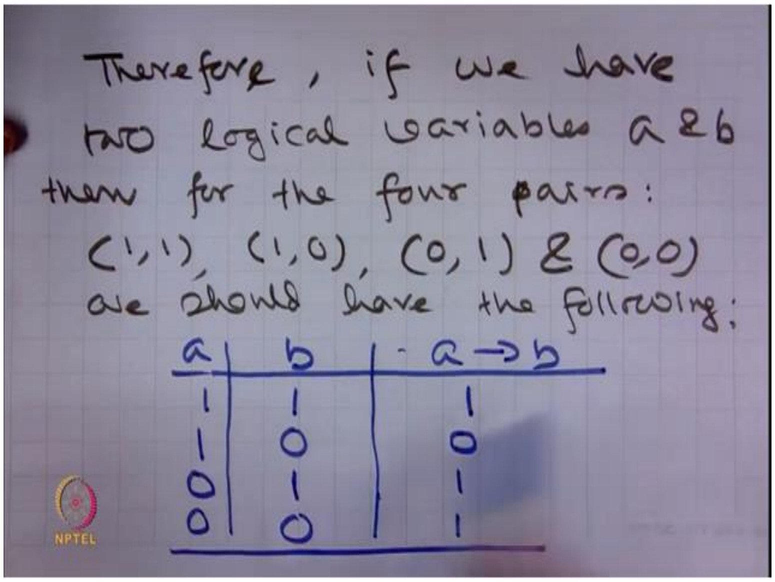

Therefore, if we have two logical variables a and b. Then for the four pairs $(1,1)$, $(1,0)$, $(0,1)$ and $(0,0)$.

We should have the following

a	b	$a \to b$
1	1	1
1	0	0
0	1	1
0	0	1

Because that is the truth value of $a \to b$ when we work with Propositional Logic.

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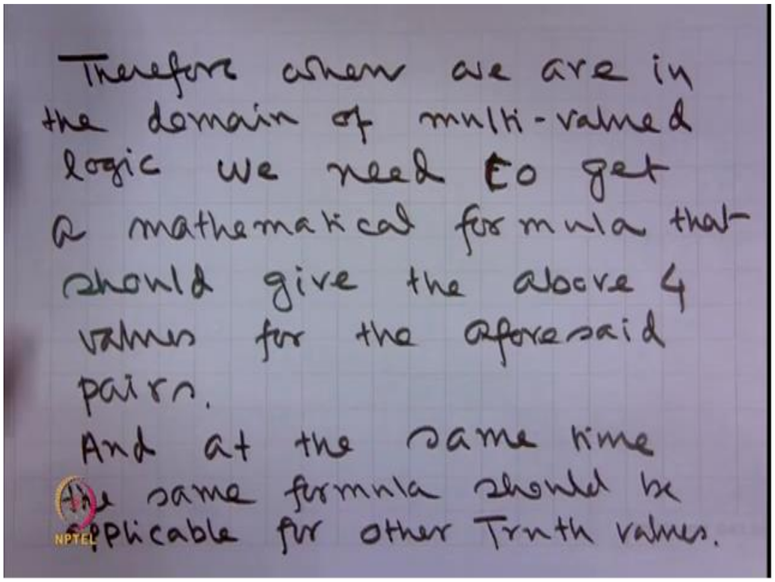

Therefore, when we are in the domain of multi-valued logic, we need to get a mathematical formula that should give the above 4 values for the aforesaid pairs.

And at the same time the same formula should be applicable for other truth values.

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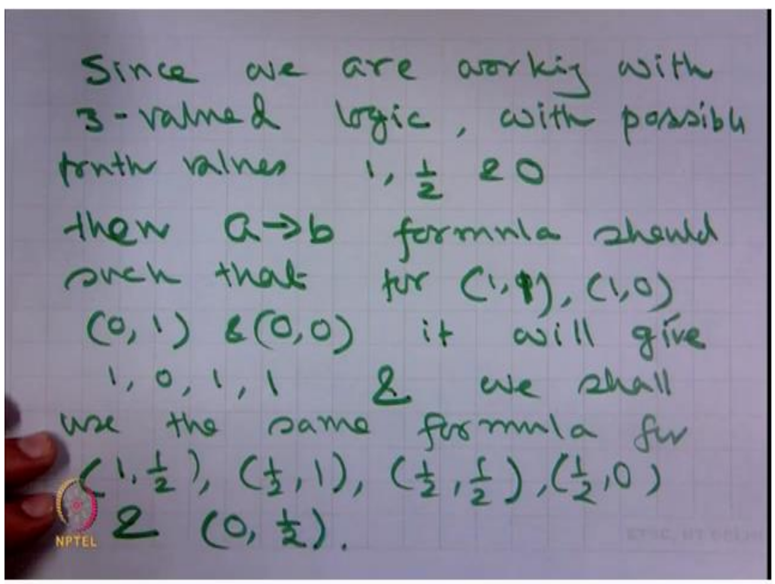

Since we are working with 3-valued logic with possible truth values $1, \frac{1}{2}$ and 0.

Then $a \to b$ formula should be such that for $(1,1)$, $(1,0)$, $(0,1)$ and $(0,0)$ it will give $1, 0, 1$ and 1 and we shall use the same formula for $\left(1,\frac{1}{2}\right)$, $\left(\frac{1}{2},1\right)$, $\left(\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$

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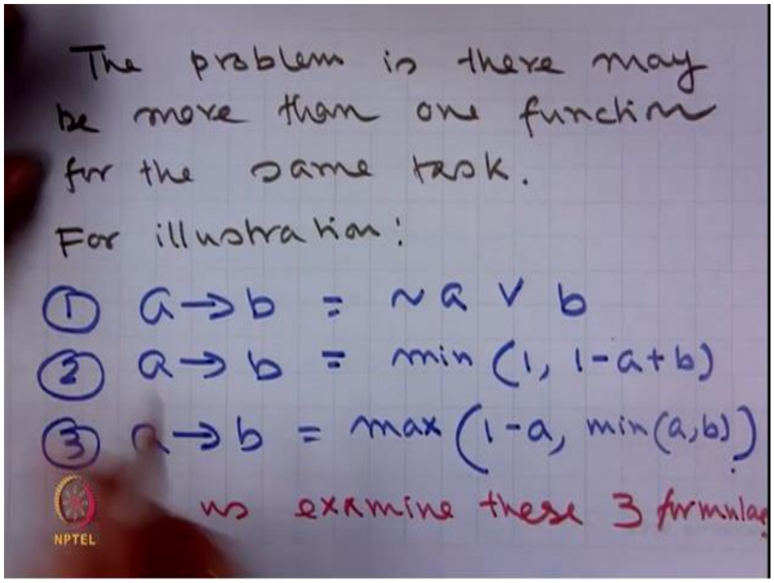

The problem is there may be more than one function for the same task.

For illustration:

1) $a \rightarrow b = {\sim}a \vee b$
2) $a \rightarrow b = \min(1, 1 - a + b)$
3) $a \rightarrow b = \max(1 - a, \min(a, b))$

Let us examine these 3 formulae.

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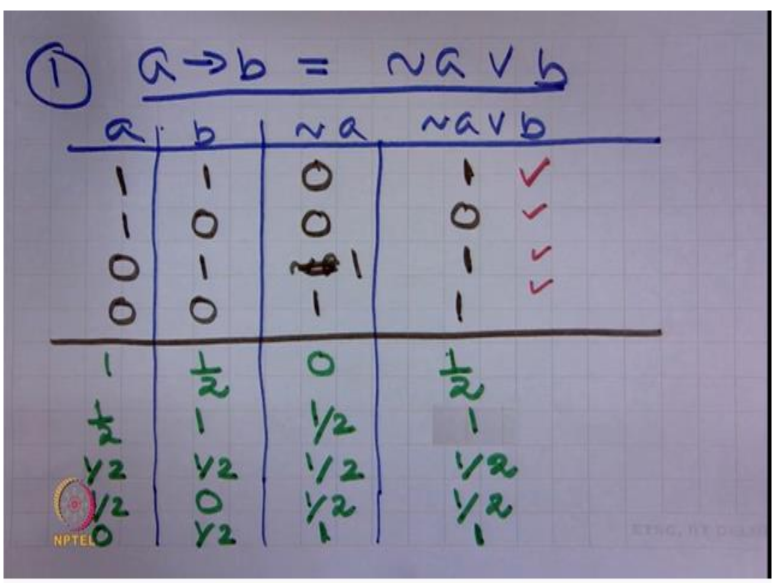

1) $a \to b = \sim a \vee b$

a	b	$\sim a$	$\sim a \vee b$
1	1	0	1
1	0	0	0
0	1	1	1
0	0	1	1
1	1/2	0	1/2
1/2	1	1/2	1
1/2	1/2	1/2	1/2
1/2	0	1/2	1/2
0	1/2	1	1

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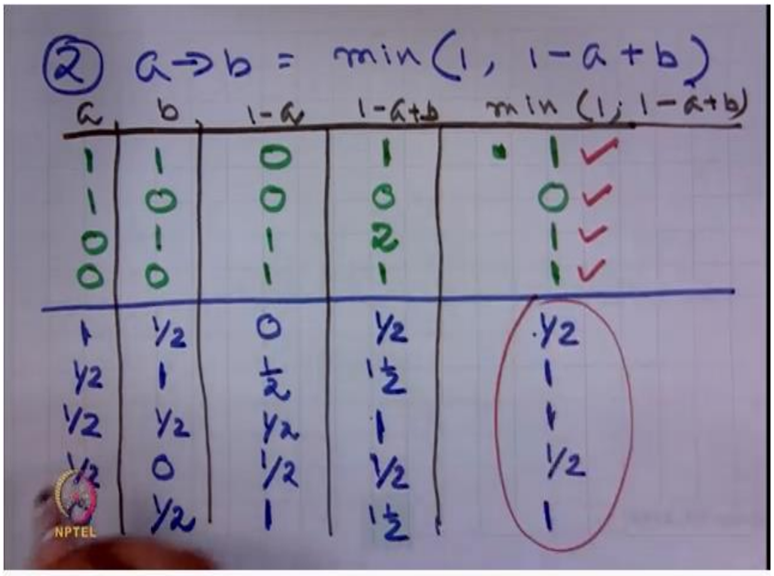

Let us now consider the second formula

2) $a \to b = \min(1, 1 - a + b)$

a	b	$1 - a$	$1 - a + b$	$\min(1, 1 - a + b)$
1	1	0	1	1
1	0	0	0	0
0	1	1	2	1

0	0	1	1	1
1	1/2	0	1/2	1/2
1/2	1	1/2	1/2	1
1/2	1/2	1/2	1	1
1/2	0	1/2	1/2	1/2
0	1/2	1	1/2	1

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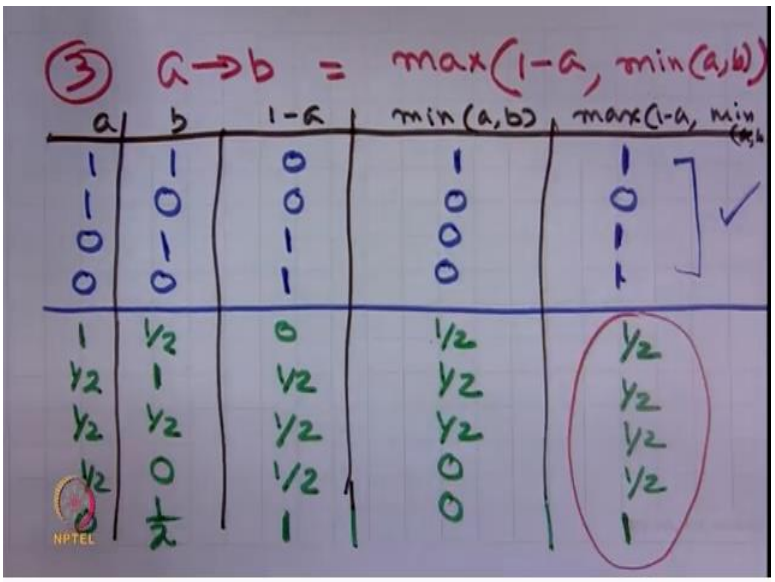

3) $a \to b = \max(1 - a, \min(a, b))$

a	b	$1 - a$	$\min(a, b)$	$\max(1 - a, \min(a, b))$
1	1	0	1	1
1	0	0	0	0
0	1	1	0	1
0	0	1	0	1
1	1/2	0	1/2	1/2
1/2	1	1/2	1/2	1/2
1/2	1/2	1/2	1/2	1/2
1/2	0	1/2	0	1/2
0	1/2	1	0	1

And we see that we get another set of values which are different from the earlier ones.

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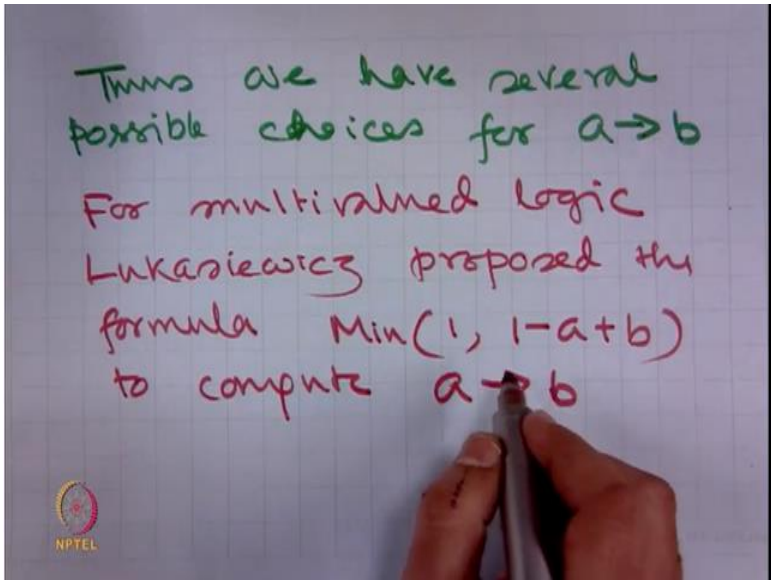

Thus we have several possible choices for $a \to b$.

For multivalued logic Lukasiewicz proposed the formula $\min(1, 1 - a + b)$ to compute $a \to b$

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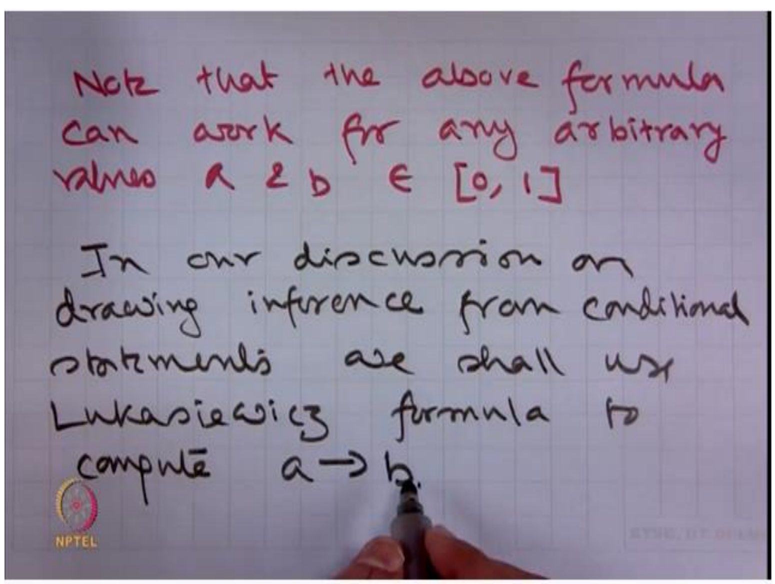

Note that the above formula can work for any arbitrary values a and $b \in [0, 1]$.

In our discussion on drawing inference from conditional statements we shall use Lukasiewicz formula to compute $a \to b$.

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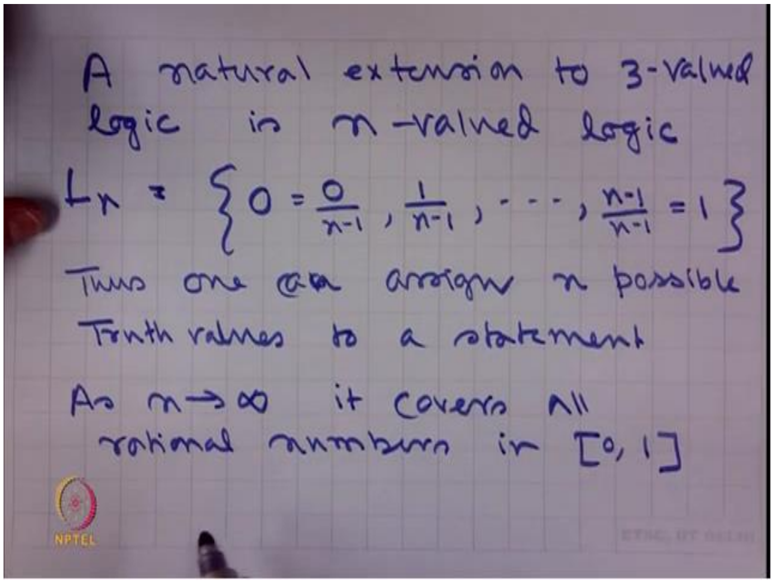

A natural extension to 3-valued logic is n-valued logic which is called

$$L_n = \left\{0 = \frac{0}{n-1}, \frac{1}{n-1}, \dots, \frac{n-1}{n-1} = 1\right\}$$

Thus one can assign n possible truth values to a statement.

As $n \to \infty$ it covers all rational numbers in $[0, 1]$

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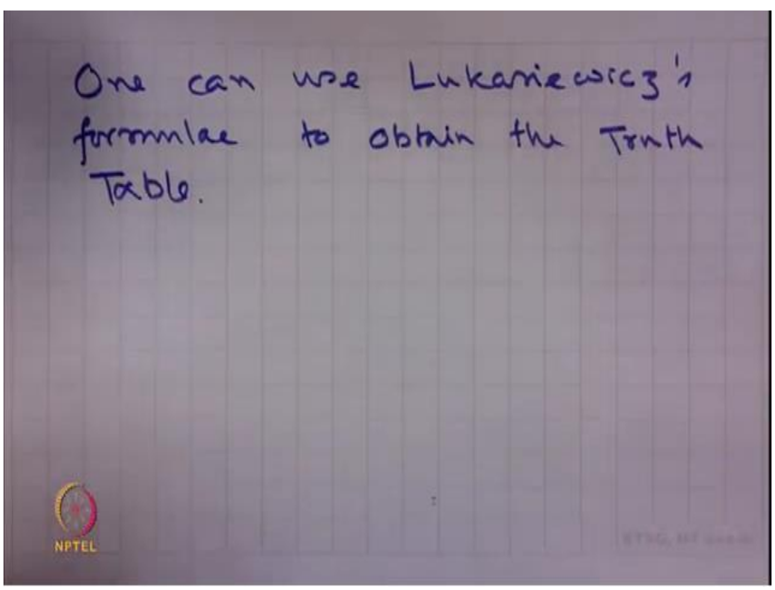

One can use Lukasiewicz's formulae to obtain the truth table.

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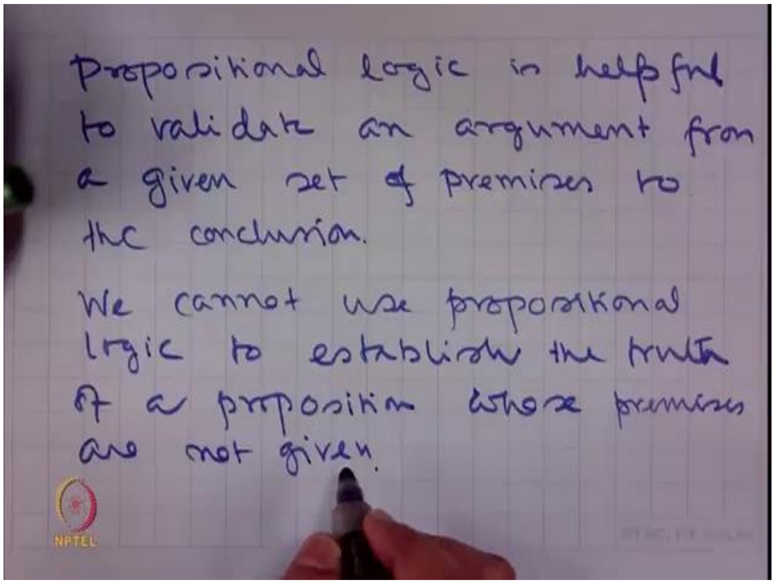

The Propositional Logic is helpful to validate an argument from a given set of premises to the conclusion.

We cannot use propositional logic to establish the truth of a proposition whose premises are not given.

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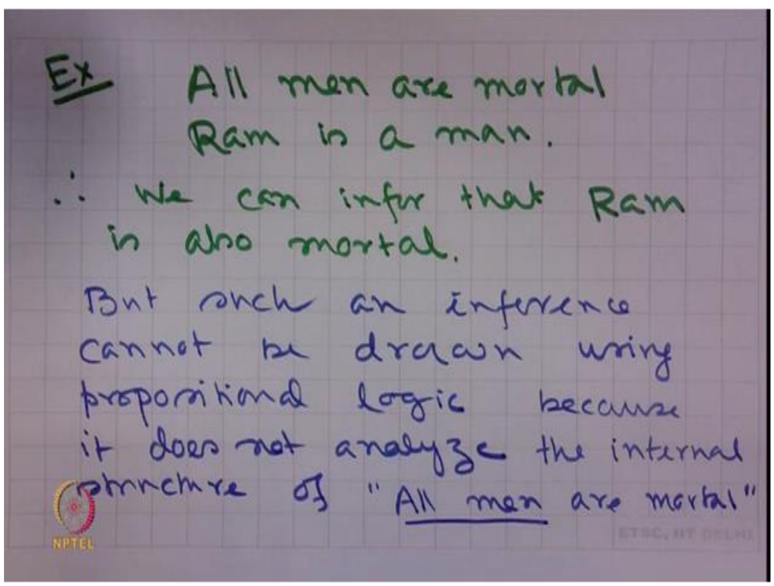

For example:

All men are mortal.

Ram is a man.

Therefore, we can infer that Ram is also mortal.

But such an inference cannot be drawn using propositional logic because it does not analyse the internal structure of ‘All men are mortal’.

So, to propositional logic all men are mortal is one statement which can be True or False. It does not go inside to tell us that we can use this statement to justify that any particular person is also mortal.

And this is because all men covers all the people and therefore, to infer about each individual, we need to use some variable X. So that we can instantiate the value for X with some particular person and therefore from there we can infer that he is also mortal. This internal analysis of the structure is not present in propositional logic.

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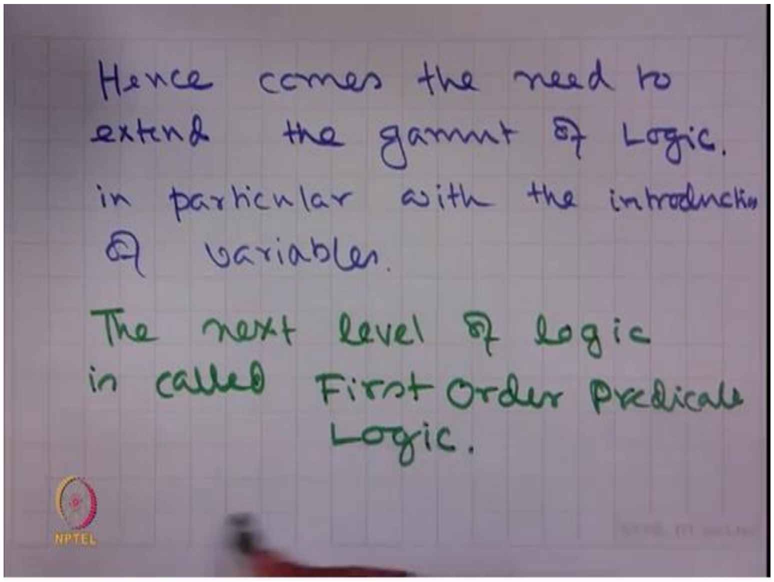

Hence comes the need to extend the gamut of logic in particular with the introduction of variables. The next level of logic is called First Order Predicate Logic.

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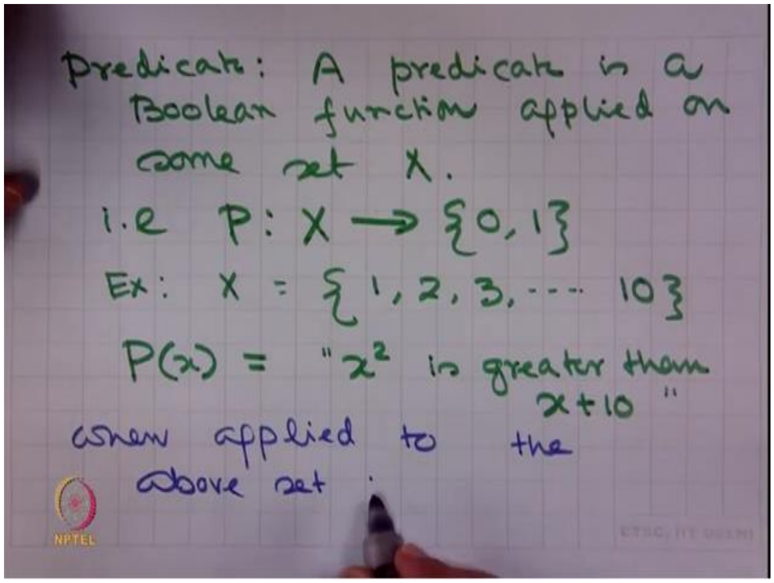

So what is the predicate?

A predicate is a Boolean function applied on some set X that is $P: X \to \{0, 1\}$.

Say for example $X = \{1, 2, 3 \dots 10\}$

And suppose the $P(x) =$ 'x^2 is greater than $x + 10$'. Therefore, when applied to the above set.

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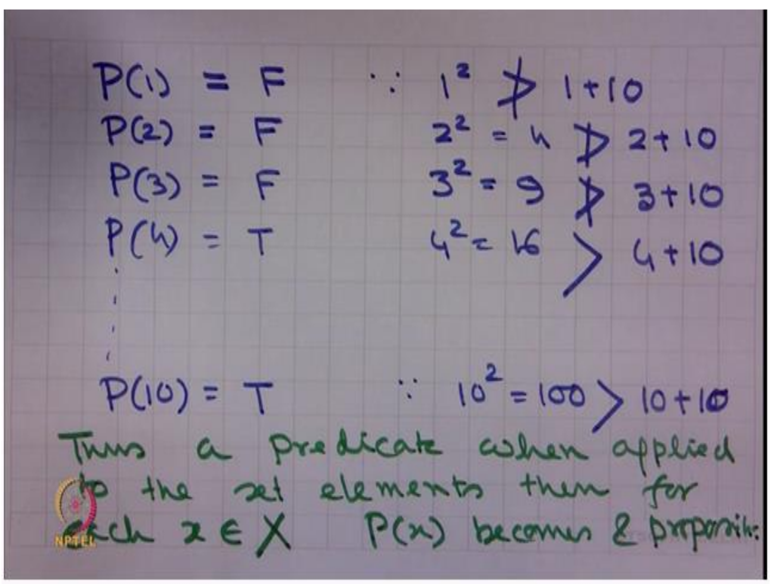

$$
\begin{array}{ll}
P(1) = F & \because 1^2 \ngtr 1 + 10 \\
P(2) = F & \because 2^2 = 4 \ngtr 2 + 10 \\
P(3) = F & \because 3^2 = 9 \ngtr 3 + 10 \\
P(4) = T & \because 4^2 = 16 > 4 + 10 \\
\vdots & \vdots \\
P(10) = T & \because 10^2 = 100 > 10 + 10
\end{array}
$$

Thus a predicate when applied to the set elements then for each $x \in X$, $P(X)$ becomes a proposition. That is, it becomes a statement which says it is True or False.

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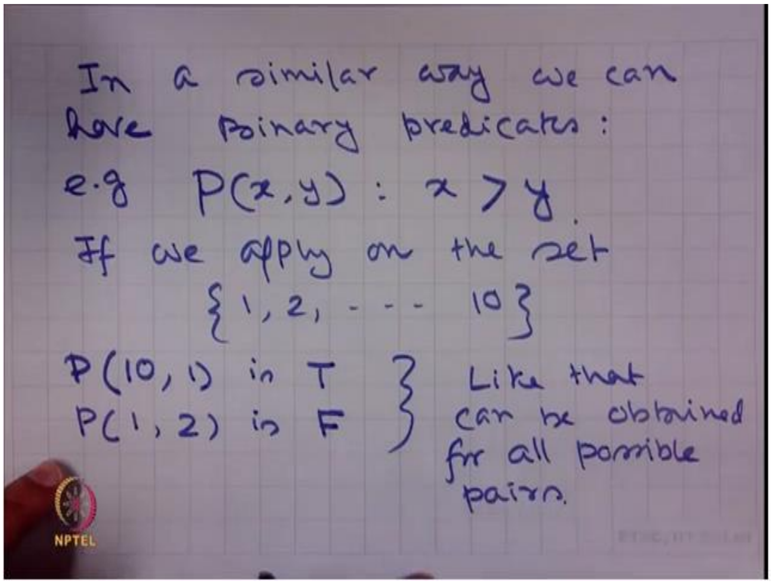

In a similar way we can have binary predicates.

For example

$P\ (x, y)$: $x > y$

If we apply on the same set $\{1, 2, 3 \ldots 10\}$

Then we can see that $P(10,1)$ is True, $P(1, 2)$ is False.

Like that can be obtained for all possible pairs.

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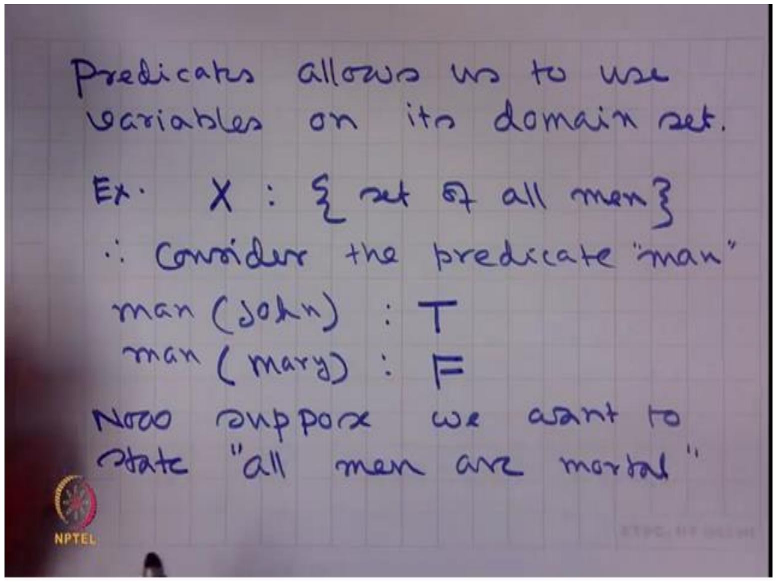

Predicates allows us to use variables on its domain set.

For example

Let X be the domain set, {set of all men}.

Therefore, consider the predicate man.

$$man(John): \quad T$$

$$man(Mary): \quad F$$

Now suppose we want to state 'all men are mortal'.

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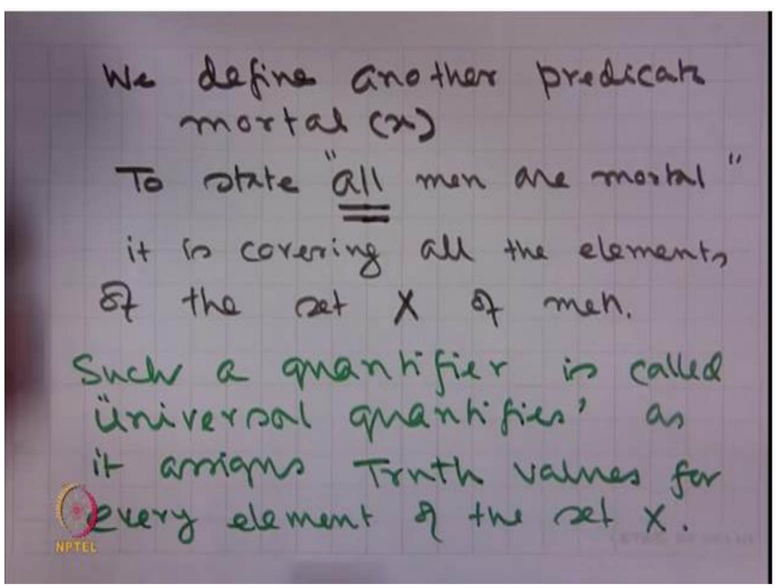

So we define another predicate is $mortal(x)$, which is True if x is mortal, which is False if x is immortal.

Now to state 'all men are mortal', the focus now should go to this quantity all, it is covering all the elements of the set X of men.

Such a quantifier is called universal quantifier as it assigns truth values for every element of the set X.

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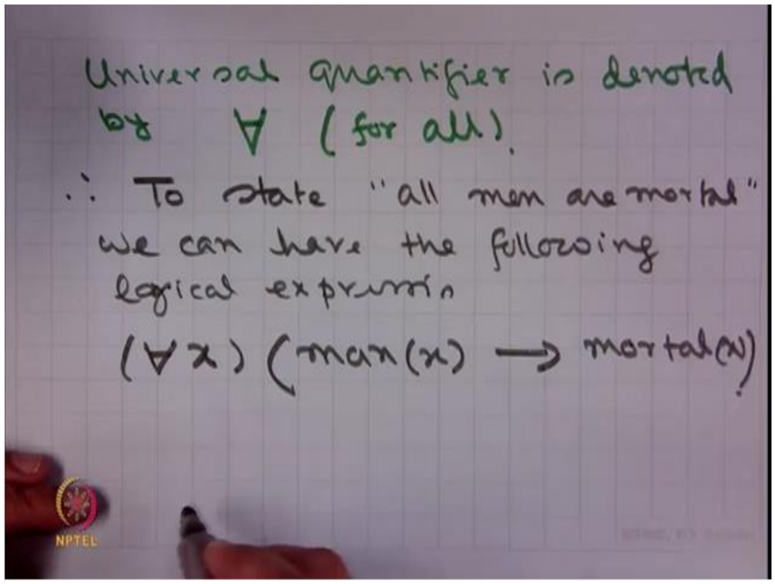

The universal quantifier is denoted by ∀ which means for all.

Therefore, to state all men are mortal we can have the following logical expression

$$\forall x\ \big(man(x) \to mortal(x)\big)$$

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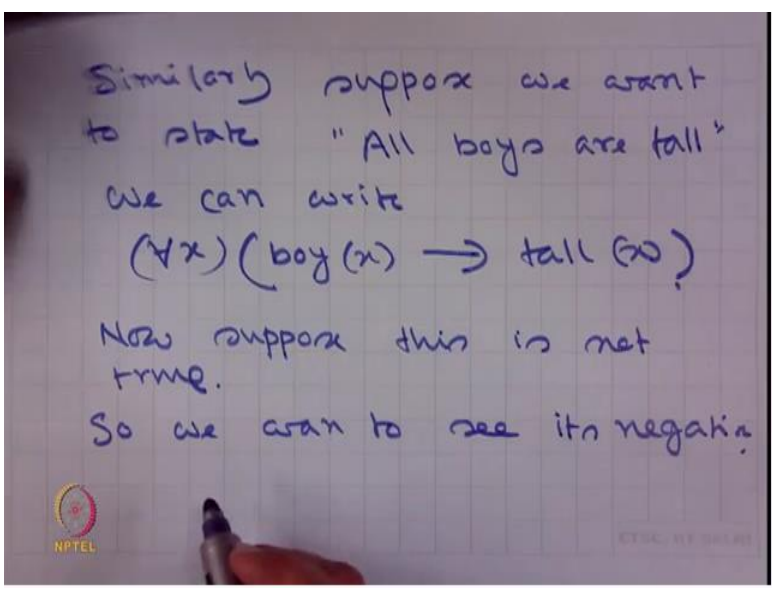

Similarly suppose we want to state

‘All boys are tall’.

We can write

$\forall x\,(boy(x) \rightarrow tall(x))$

Now suppose this is not True. So we want to see its negation.

What can be the negation?

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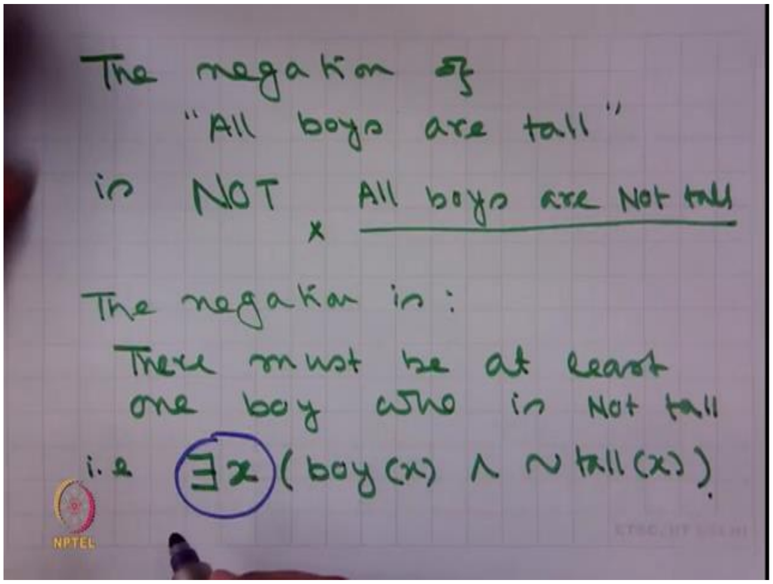

The negation of ‘All the boys are tall’ is NOT ‘All the boys are not tall’.

This is not the negation.

The negation is there must be at least one boy who is not tall.

That is $\exists x(boy(x) \wedge {\sim}tall(x))$

So this gives another quantifier.

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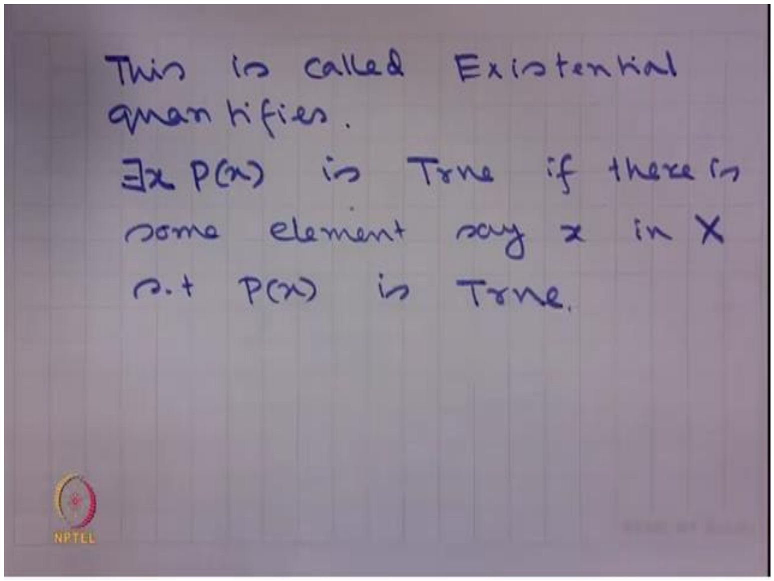

This is called Existential quantifier. $\exists\, x, P(x)$ will be True if there is some element say $x \in X$ such that $P(x)$ is True.

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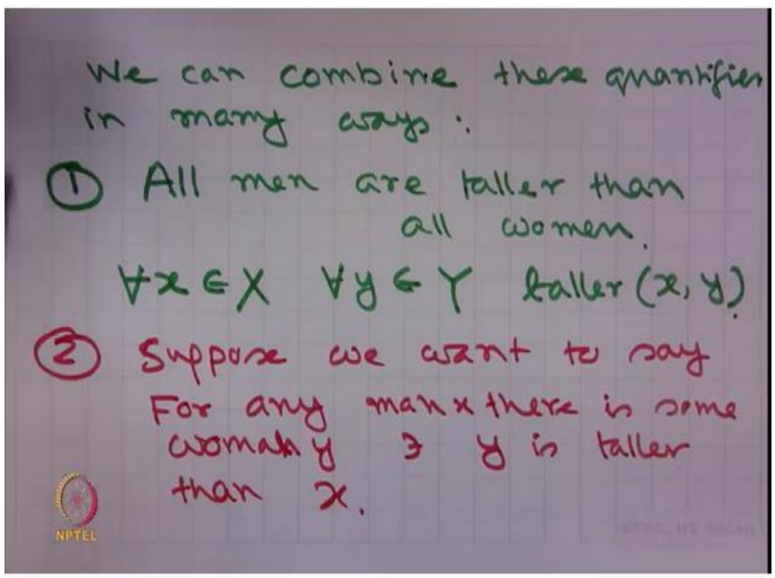

We can combine these quantifiers in many ways.

Suppose we want to say

1) All men are taller than all women.

   So we can write it as $\forall\, x \in X\ \ \forall\, y \in Y\ \ \ taller(x, y)$

On the other hand

2) Suppose we want to say,

   For any man x there is some woman $y \ni$ that y is taller than x.

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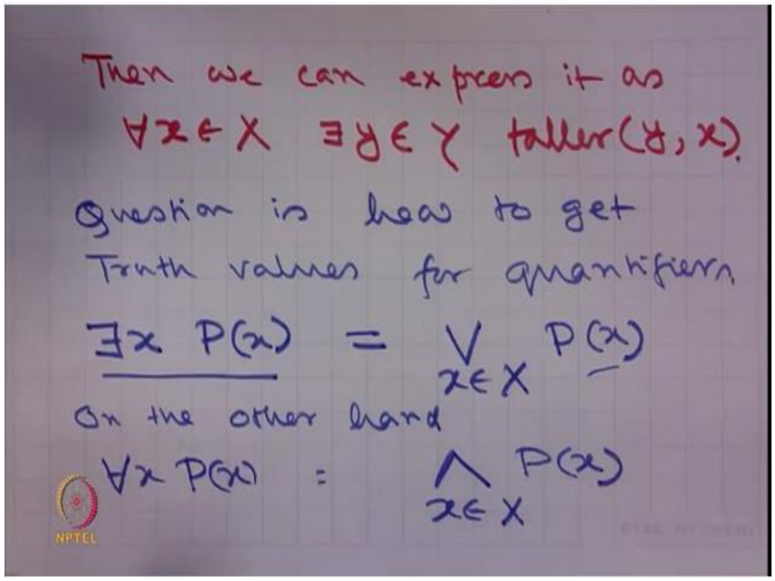

Then we can express it as f$\forall\ x \in X\ \ \exists\ y \in Y\ \ \ taller(x, y)$

Question is how to get truth values for quantifiers?

$$\exists\ x\ P(x) = \bigvee_{x \in X} P(x)$$

That means if for any one of the elements x of X, $P(x)$ is True. Then $\exists\ x\ P(x)$ is going to be True.

On the other hand,

$$\forall x\ \ P(x) = \bigwedge_{x \in X} P(x)$$

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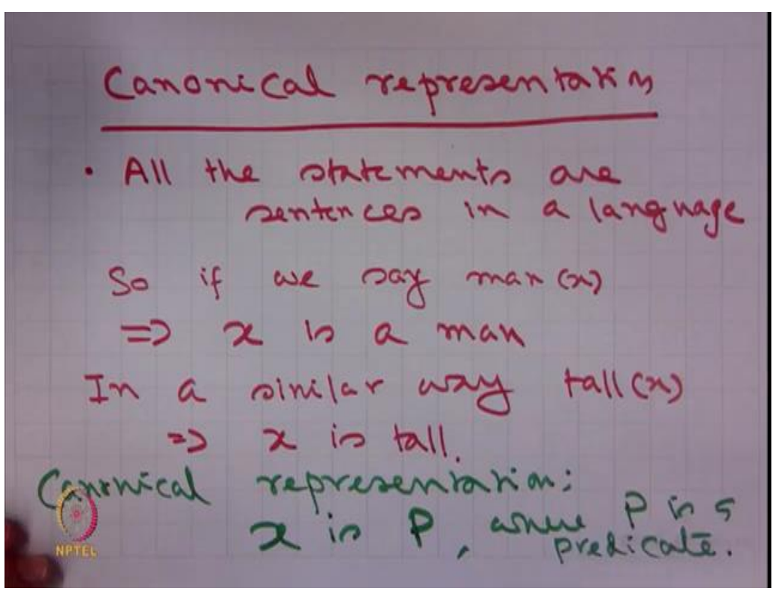

So before we stop I give you a canonical representation.

If we notice that all the statements are sentences in a language.

So if we say $man(x)$

$\Rightarrow x$ is a man.

In a similar way $tall(x)$

$\Rightarrow x$ is tall.

So canonical representation of such statements is:

x is P, where P is a predicate.

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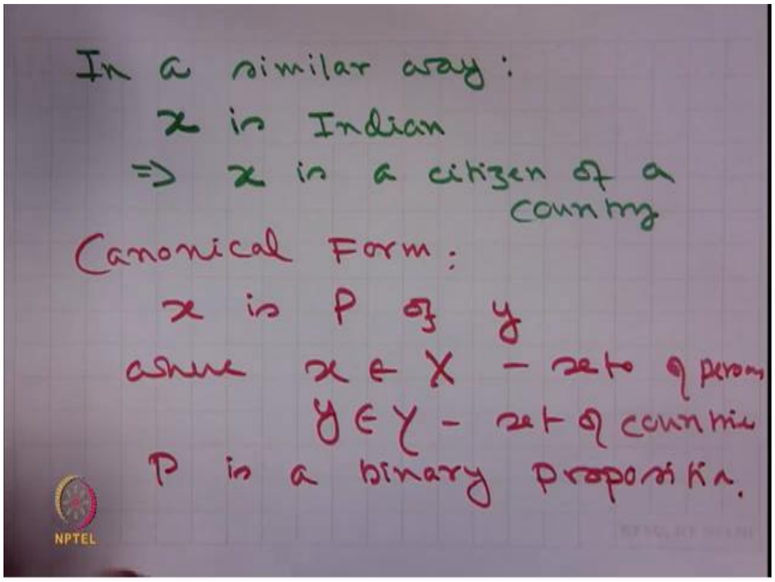

In a similar way, x is Indian is essentially we mean x is a citizen of a country.

Therefore, the canonical form may be

$$x \text{ is } P \text{ of } y$$

, where $x \in X -$ set of persons, $y \in Y -$ set of countries and P is a binary proposition.

Okay students I stop here today. In the next class I shall start with fuzzy proposition. So in these 2 classes I have given you the basics of first order logic and propositional logic.

Of course logic has been extended to much more theoretical details. We have paraconsistent logic, description logic, default logic, defeasible logic there are many different forms of logic that has been developed theoretically for this course we shall look at fuzzy logic and we shall see how we can make inference from fuzzy logical propositions. Thank you.