Introduction to Fuzzy Sets Arithmetic and Logic
Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology - Delhi

Lecture – 21
Fuzzy Sets Arithmetic & Logic

Welcome students to the MOOCs lecture series on introduction to fuzzy sets arithmetic and logic, this is lecture number 21 and as I said towards the end of the last class that in this class I will start fuzzy logic.

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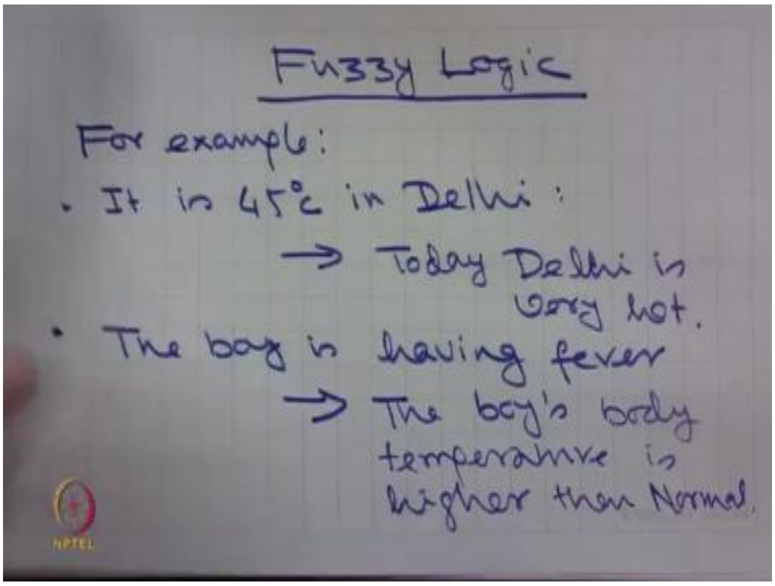

Logic is perhaps the most important human acumen that distinguishes human beings from other living creatures. Logic allows us to derive new facts or new knowledge from a given set of facts.

For example,

- It is 45℃ in Delhi today.

Suppose, this is a fact that is given to a human being, he immediately infers that today Delhi is very hot.

- The boy is having a fever.

One can easily infer from here that the boy’s body temperature is higher than normal.

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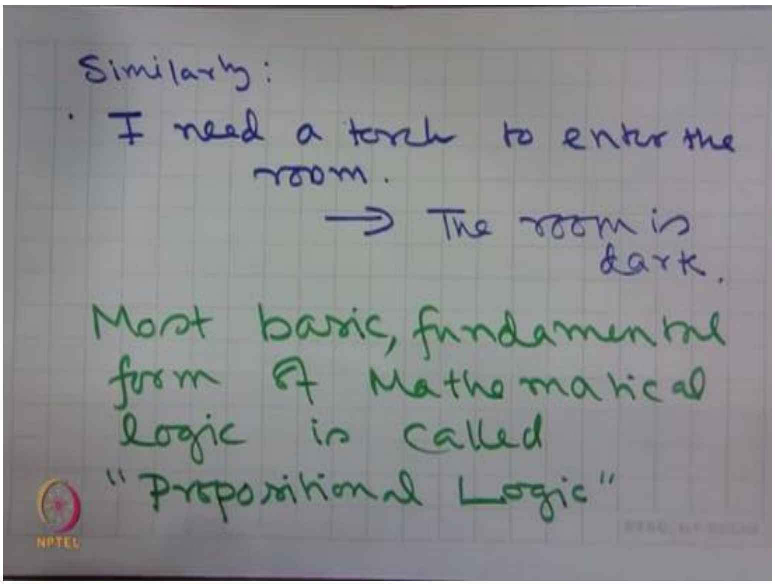

Similarly,

- I need a torch to enter the room.

From here, we can infer that the room is dark.

Like that I can give many examples but perhaps, that is not needed.

I am sure most of you have done some amount of Mathematical logic earlier and all of you remember that the most basic or fundamental form of Mathematical logic is called Propositional Logic.

In this class, I shall quickly recapitulate the basics of Propositional Logic.

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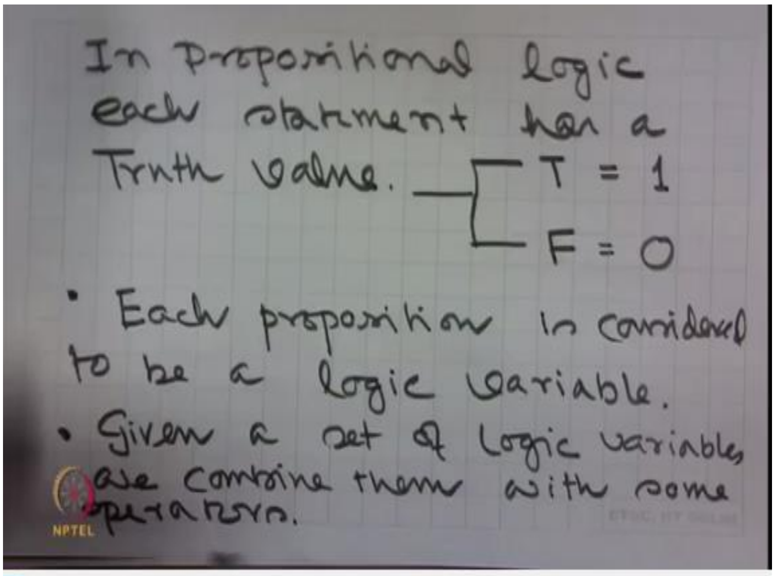

And I will try to give you a small motivation towards fuzzy logic.

- In propositional logic; each statement has a truth value which is either true or false. Mathematically, truth is given the value 1 and false is given the value 0.

- Each proposition is considered to be a logic variable.
- Given a set of logic variables, we combine them with some logical operators.

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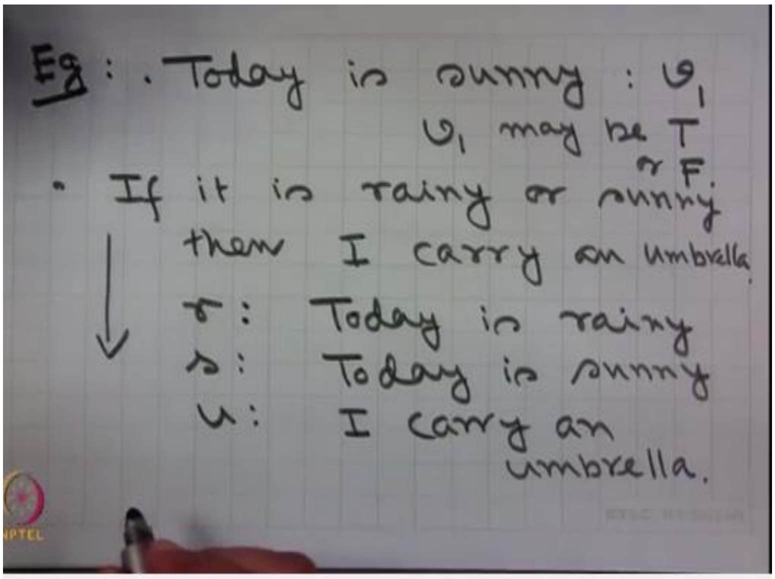

For example,

- Today is sunny : v_1

It is a statement suppose, we give the variable name v_1, now it can be true or false

Suppose, another statement is

- If it is rainy or sunny, then I carry an umbrella.

This is a very complicated sentence to represent mathematically, so I break it into several sentences or statements.

So, suppose

r: Today is rainy

s: Today is sunny.

u: I carry an umbrella.

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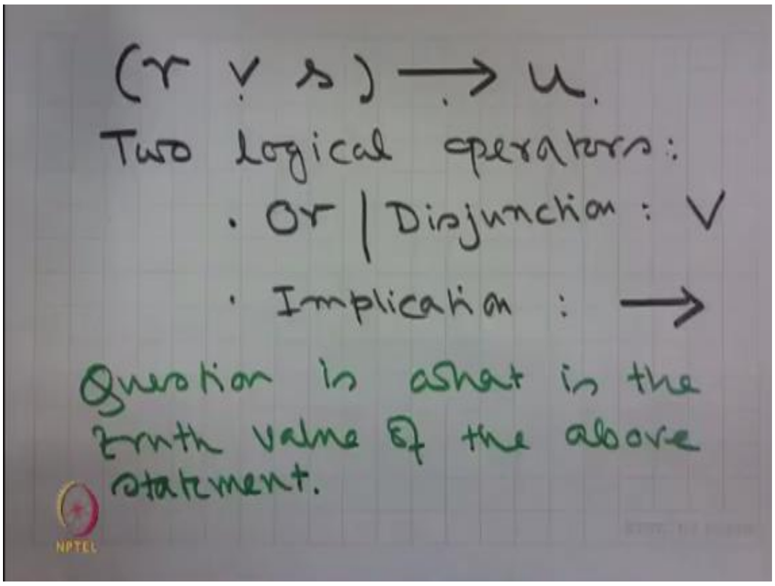

Then I can mathematically put this set of sentences or statements as:

$$(r \vee s) \rightarrow u$$

Here, I have used two operators:

- Or \Disjunction and represented as ∨
- Implication represented as →

Question is what is the truth value of the above statement?

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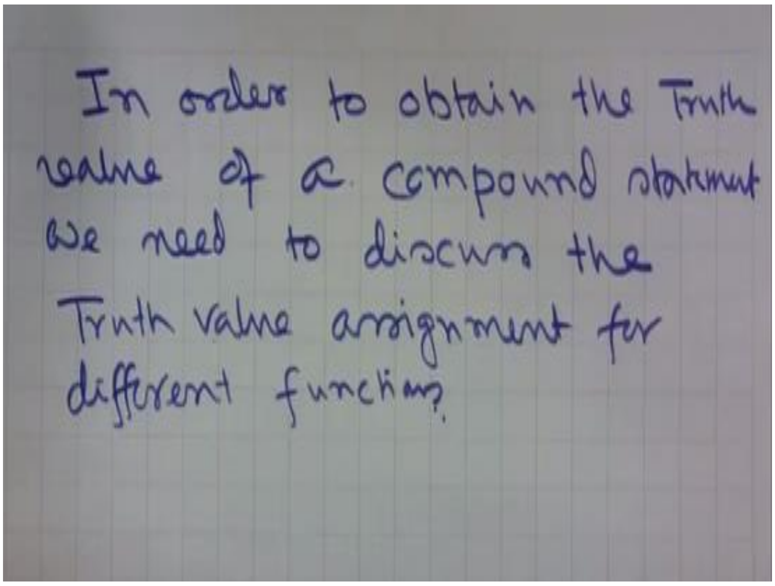

So, in order to obtain the truth value of a compound statement, we need to discuss the truth value assignment for different functions.

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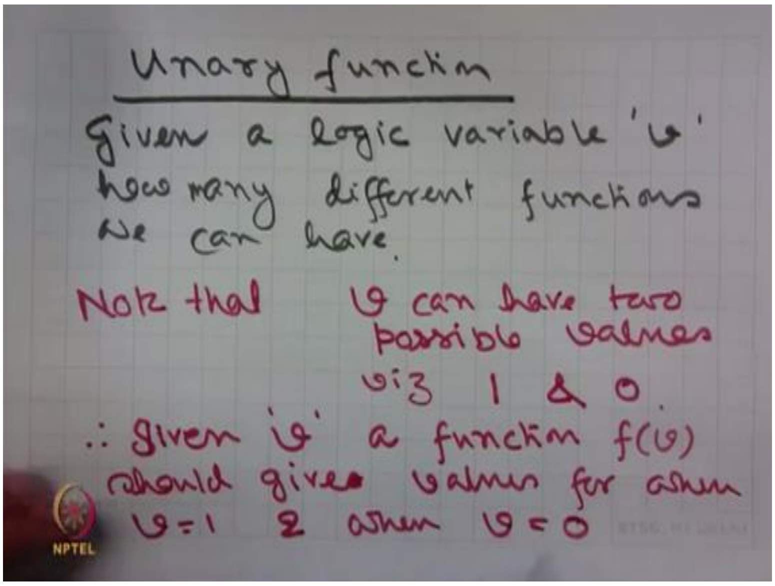

Unary function

That means given a logic variable say v, how many different functions we can have.

Note that v can have 2 possible values namely 1 or true and 0 or false.

So each function will assign some value for 1 and some value for 0.

∴ Given v, a function $f(v)$ should give values for when $v = 1$ and when $v = 0$.

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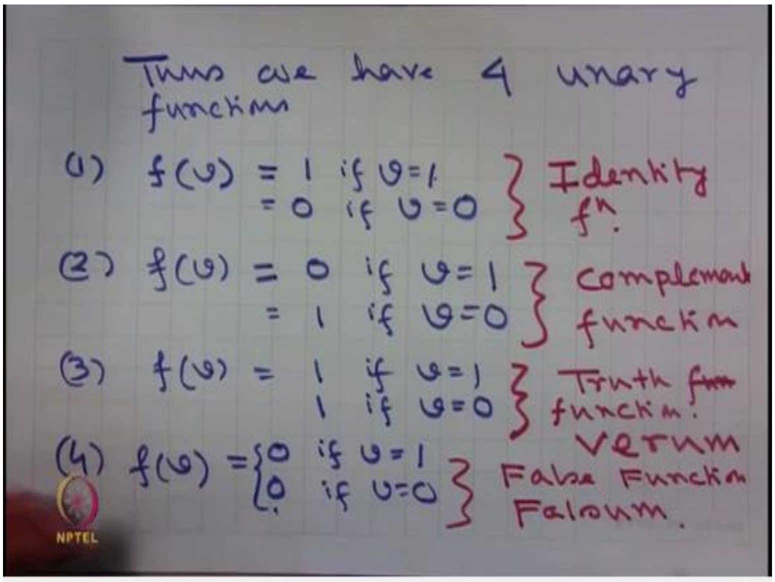

Thus, we have 4 unary functions;

1. $f(v) = \begin{cases} 1 & \text{if } v = 1 \\ 0 & \text{if } v = 0 \end{cases}$ This is the identity function, whatever is the value of v, it is returning the same value.
2. $f(v) = \begin{cases} 0 & \text{if } v = 1 \\ 1 & \text{if } v = 0 \end{cases}$ This is our very familiar complement function.

3. $f(v) = \begin{cases} 1 & \text{if } v = 1 \\ 1 & \text{if } v = 0 \end{cases}$ This is called truth function and its Latin name is Verum.

That means, whatever is the value of v, the function is returning the value 1

4. $f(v) = \begin{cases} 0 & \text{if } v = 1 \\ 0 & \text{if } v = 0 \end{cases}$ It is called false function or in Latin it is called Falsum.

So, you understand how given a logical variable, how many different functions can be constructed.

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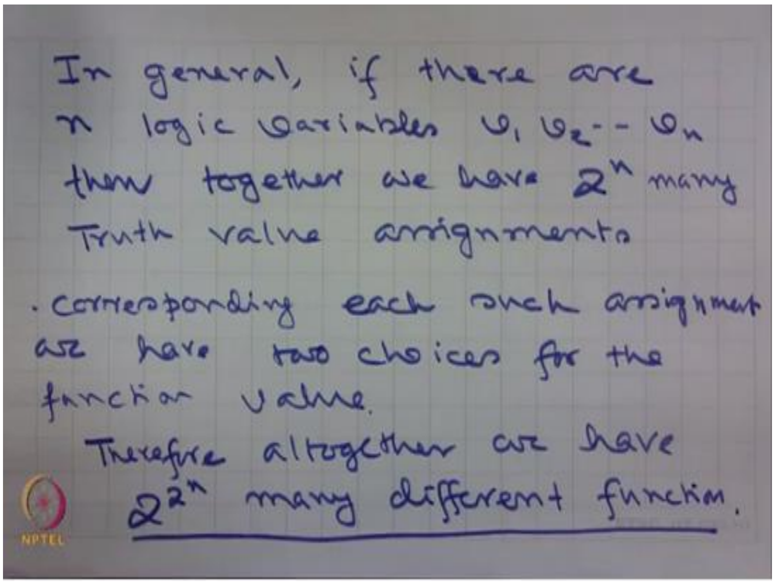

In general, if there are n logic variables say, $v_1, v_2, \ldots\ v_n$ then, together we have 2^n many truth value assignments.

Corresponding to each such assignment, we have two choices for the function value that is 0 or 1. Therefore, altogether we have 2^{2^n}many different functions.

That means, if there are n variables then we can have 2^{2^n}many different logic functions.

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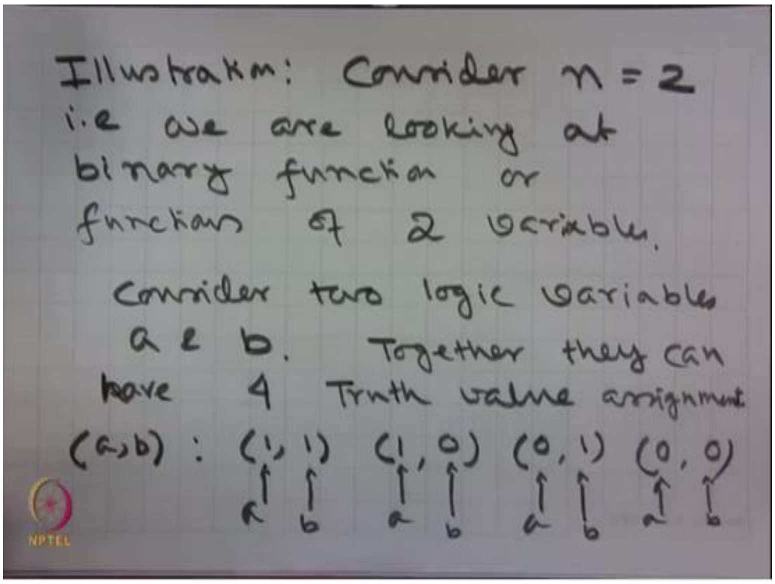

For illustration:

Consider $n = 2$ that is we are looking at binary functions or functions of 2 variables, so let us first illustrate how there can be 16 functions.

So, consider two logic variables a and b. Together they can have 4 truth value assignments, so (a, b) can be $(1, 1), (1, 0), (0, 1)$ and $(0, 0)$

Let me now define the 16 binary functions on logic variables.

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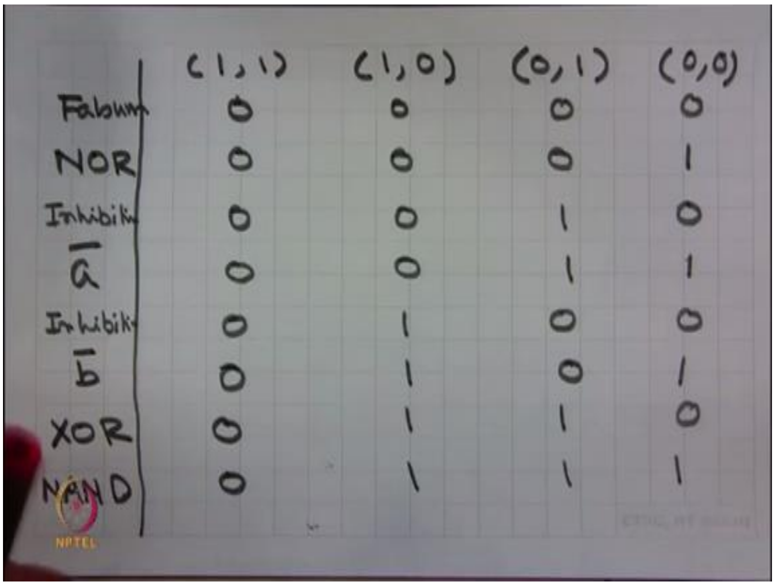

	(1,1)	(1,0)	(0,1)	(0,0)
Falsum	0	0	0	0
NOR	0	0	0	1
Inhibition	0	0	1	0
$\bar{a}$	0	0	1	1
Inhibition	0	1	0	0
$\bar{b}$	0	1	0	1
XOR	0	1	1	0
NAND	0	1	1	1

Now, if you pay attention to how I am constructing the functions actually, I started with $0, 0, 0, 0$ and then I am incrementing in a very systematic way, I am changing the values from 0 to 1 and 1 to 0. And like that we have covered all possible assignments for the last 3 of them keeping $1, 1$ at the value 0. In a similar way now, we will get other 8 functions.

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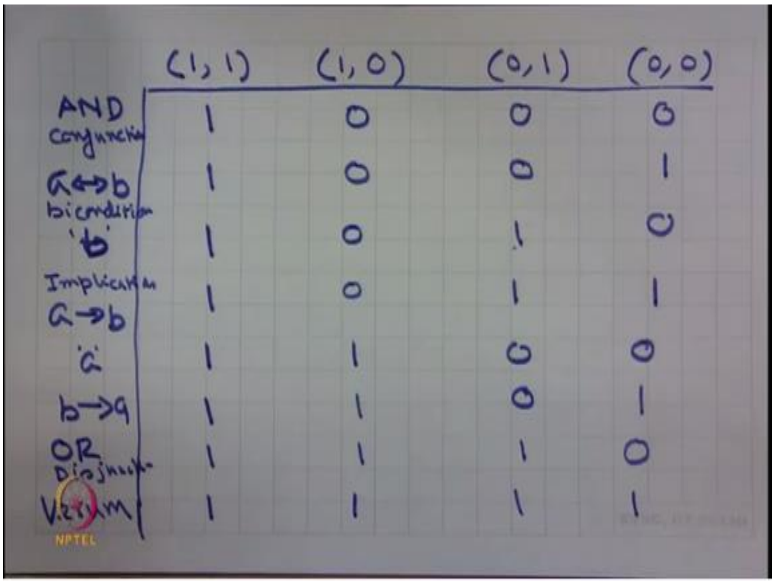

	(1,1)	(1,0)	(0,1)	(0,0)
AND (Conjunction)	1	0	0	0
$a \leftrightarrow b$ (Bi-condition)	1	0	0	1
b	1	0	1	0
$a \rightarrow b$ (Implication)	1	0	1	1

a	1	1	0	0
$b \rightarrow a$ (Implication)	1	1	0	1
OR (Disjunction)	1	1	1	0
Verum	1	1	1	1

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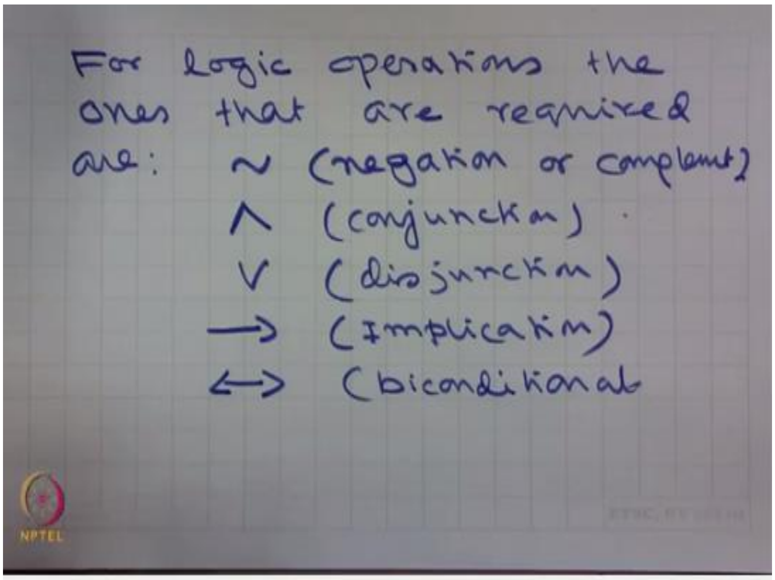

Obviously, it is very difficult to remember all the 16 possibilities, for logic operations, the ones that are required are:

- $\sim$ (negation or complement)
- $\wedge$ (conjunction)
- $\vee$ (disjunction)
- $\rightarrow$ (Implication)
- $\leftrightarrow$ (biconditional)

The first 4 are the most important and corresponding truth values should be on our fingertips.

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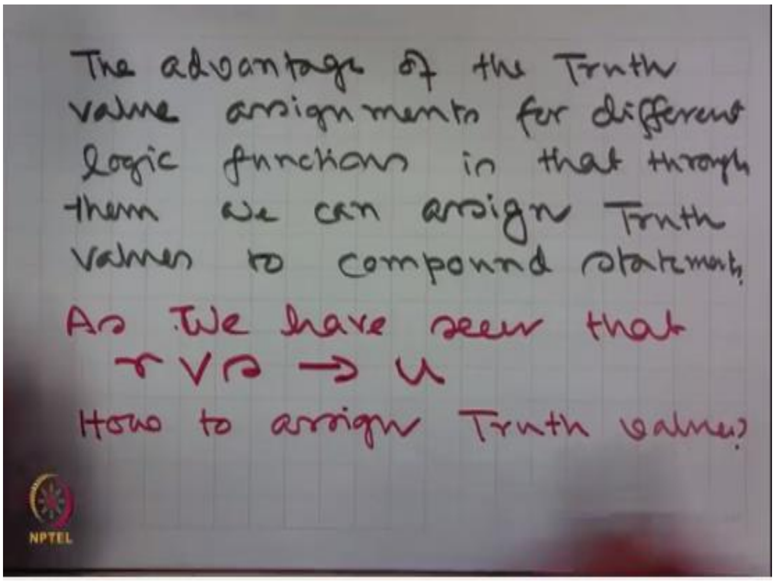

The advantage of the truth value assignments for different logic functions is that through them we can assign truth values to compound statements.

As we have seen that

$$(r \vee s) \rightarrow u$$

That means, If it is rainy or if it is a sunny day that means, I will take an umbrella.

How to assign the truth values?

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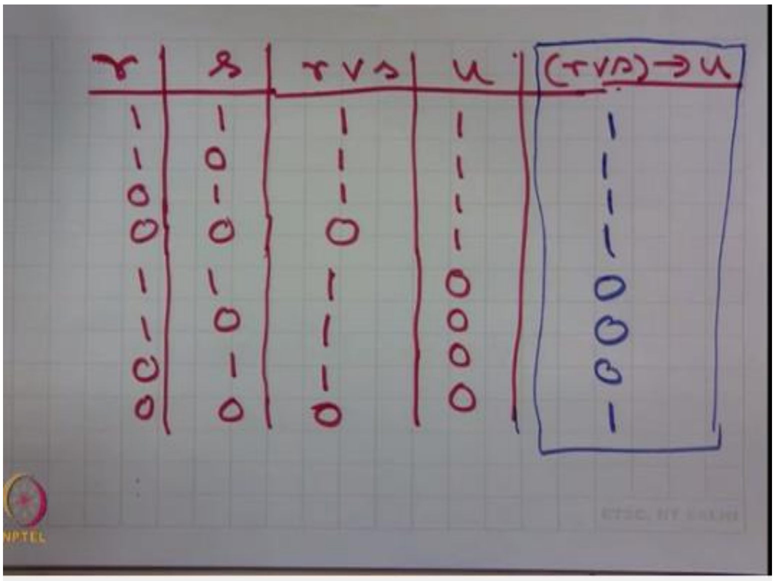

r	s	$r \vee s$	u	$(r \vee s) \rightarrow u$
1	1	1	1	1

1	0	1	1	1
0	1	1	1	1
0	0	0	1	1
1	1	1	0	0
1	0	1	0	0
0	1	1	0	0
0	0	0	0	0

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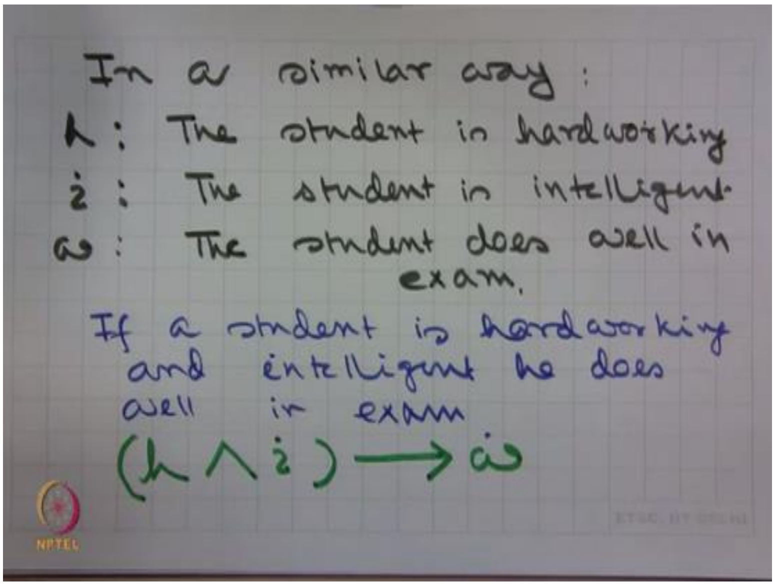

In a similar way, suppose our statements are:

h: The student is hardworking

i: The student is intelligent

w: The student does well in exam

Suppose I want to state the fact that

If a student is hardworking and intelligent, he does well in exam.

So, we can write this statement as

$$(h \wedge i) \rightarrow w$$

Therefore, we have 3 variables as in the previous case, we had 3 variables; r, s and u here we have h, i and w

Therefore, we can have 2^3 that is 8 different possibilities of truth values for h, i and w and from there we need to compute the truth value of $(h \wedge i) \rightarrow w$.

I leave it as an exercise for you to try this.

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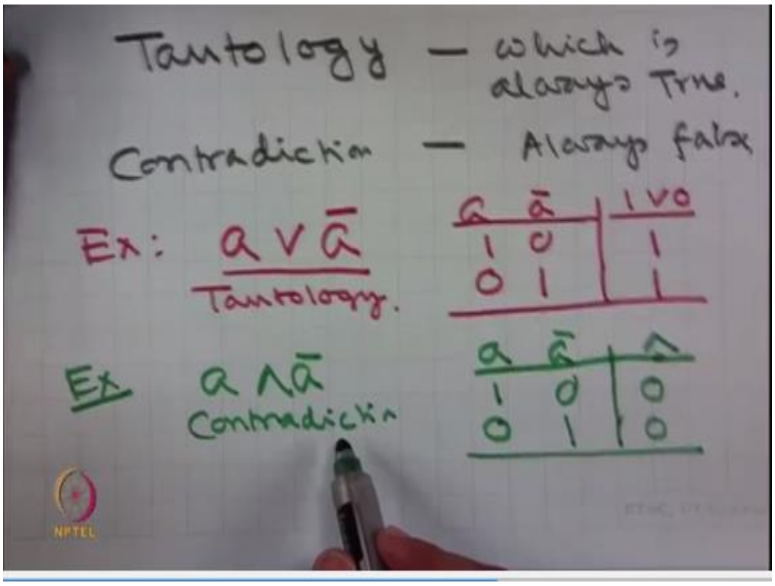

A special type of formula is called tautology which is always true and another one is contradiction that is always false.

For example,

$a \vee \bar{a}$ is a Tautology

a	$\bar{a}$	$a \vee \bar{a}$
1	0	1
0	1	1

On the other hand, $a \wedge \bar{a}$ is a Contradiction.

a	$\bar{a}$	$a \wedge \bar{a}$
1	0	0
0	1	0

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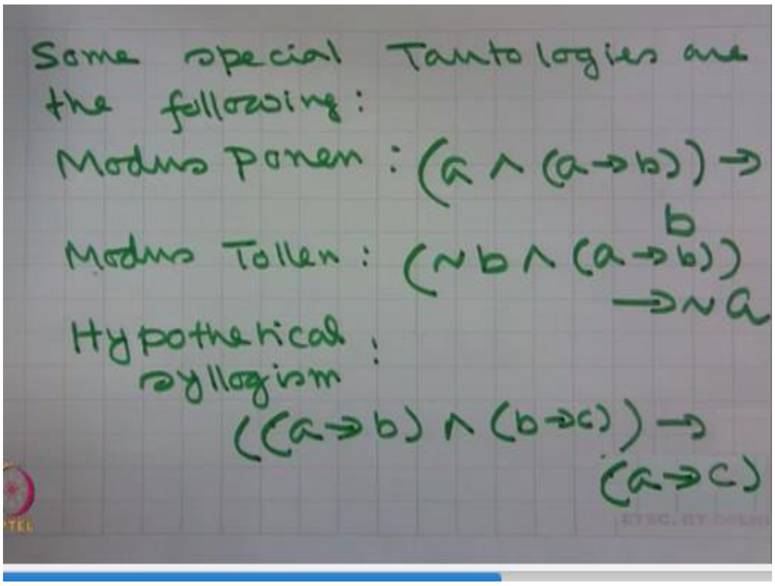

Some special Tautologies are the following;

- Modus Ponen: $\left(a \wedge (a \rightarrow b)\right) \rightarrow b$
- Modus Tollen: $\left(\sim b \wedge (a \rightarrow b)\right) \rightarrow \sim a$
- Hypothetical syllogism: $\left((a \rightarrow b) \wedge (b \rightarrow c)\right) \rightarrow (a \rightarrow c)$

These tautologies allow us to infer new facts from given set of facts.

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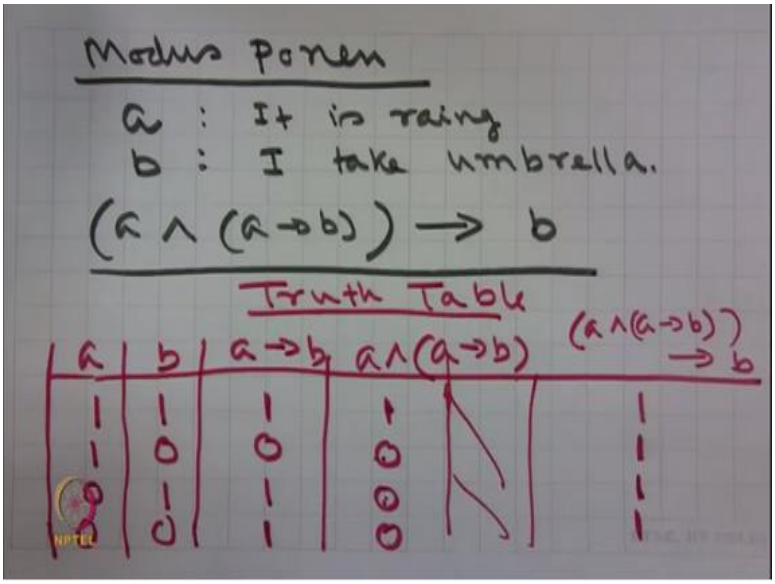

Consider for example,

Modus Ponen;

a; It is raining,

b; I take umbrella,

Then what does it say?

$$\left(a \land (a \to b)\right) \to b$$

It means that, if it is raining, it is true and if it rains, I will take umbrella that is true, so both of them are true that implies that I am going to take umbrella

This allows us to infer about b from the given two set of facts.

Question is; is it a tautology?

So, let us look into the truth table

a	b	$a \to b$	$a \land (a \to b)$	$\left(a \land (a \to b)\right) \to b$
1	1	1	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	1

Thus, we find that $\left(a \land (a \to b)\right) \to b$ is a tautology.

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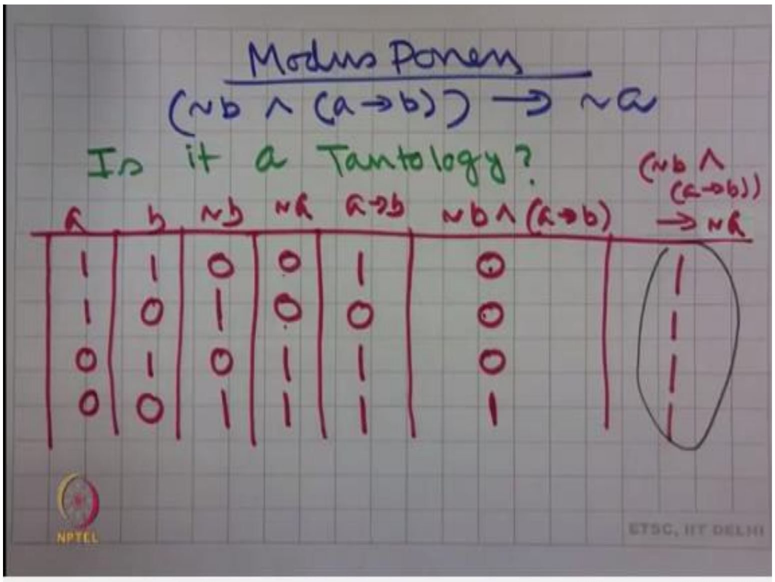

So, Modus Ponen that is $\left(\sim b \land (a \to b)\right) \to \sim a$,

Now, let us see whether it is a tautology or not, so let us fill in the truth table

a	b	$\sim b$	$\sim a$	$a \to b$	$\sim b \land (a \to b)$	$\left(\sim b \land (a \to b)\right) \to \sim a$
1	1	0	0	1	0	1
1	0	1	0	0	0	1
0	1	0	1	1	0	1
0	0	1	1	1	1	1

Thus, we find this is also a tautology because whatever may be the combination of values for a and b, we are going to get 1.

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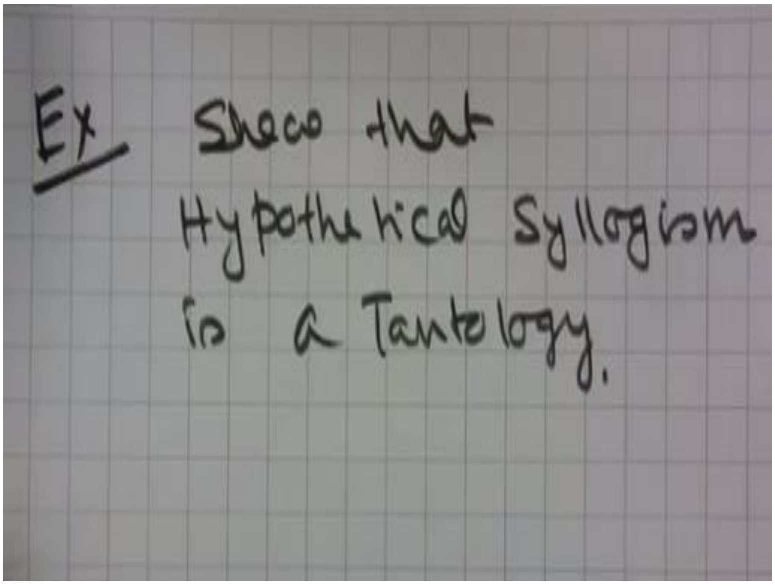

In a similar way, I give you an exercise show that hypothetical syllogism is a tautology, I leave it for you as an exercise.

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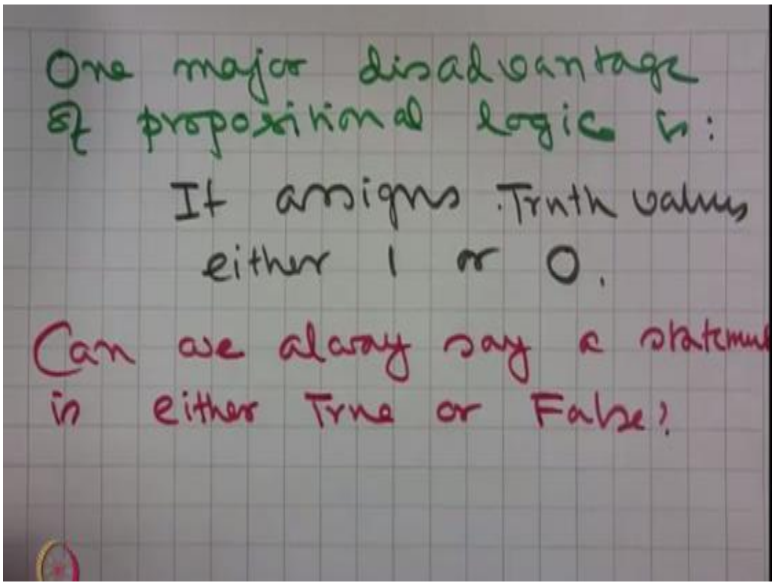

One major disadvantage of propositional logic is:

It assigns truth values either 1 or 0 but can we always say a statement is either True or False?

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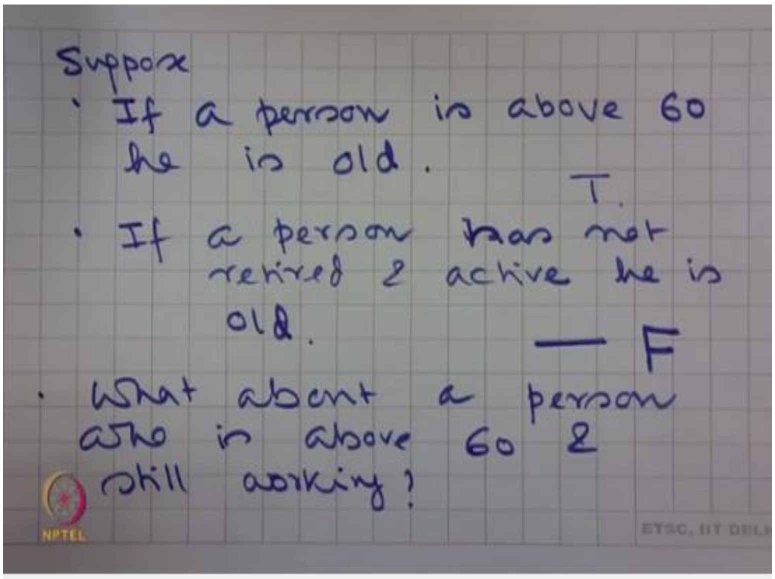

Suppose we say

- If a person is above 60, he is old

  One can think it is true because he is a senior citizen,

On the other hand suppose, we say

- If a person is say has not retired and active, he is old.

  Then we know that it is false because such a person is not old.

Then question comes what about a person who is above 60 and still working?

We can easily understand that we cannot clearly assign a truth value; true or false to this statement.

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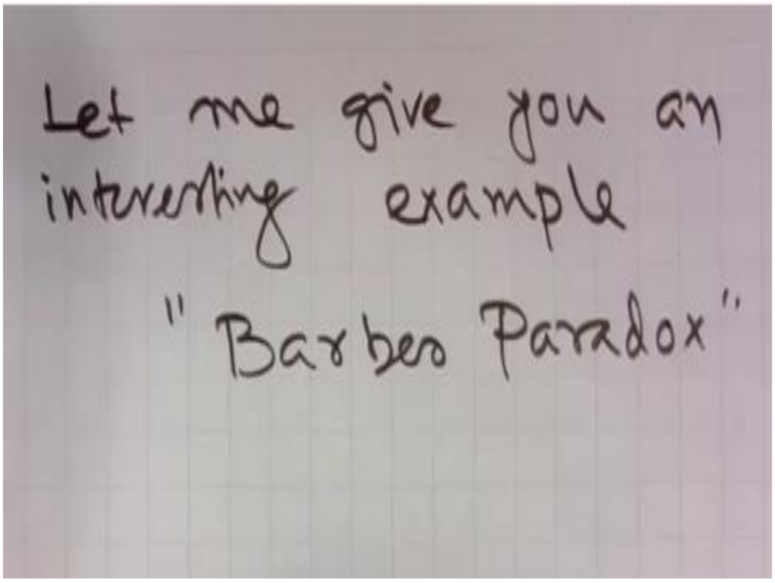

To stress the point further, let me give you an interesting example which is known as Barber Paradox.

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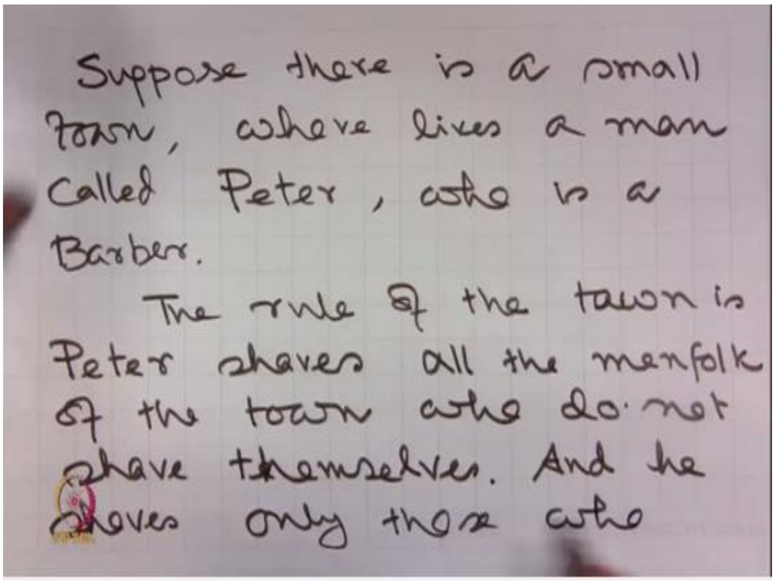

Suppose, there is a small town where lives a man called Peter who is a barber.

The rule of the town is Peter shaves all the men folk of the town who do not shave themselves.

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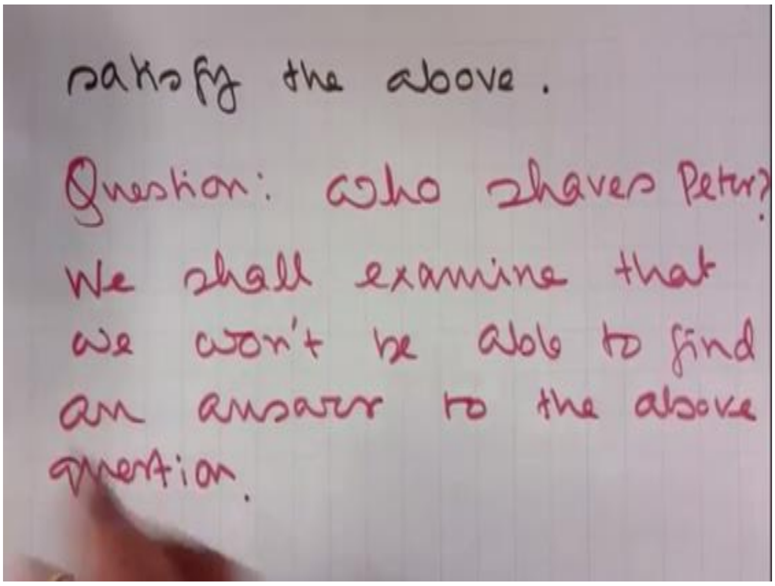

And he shaves only those who satisfy the above.

Question is who shaves Peter?

We shall examine that we would not be able to find an answers to the above question.

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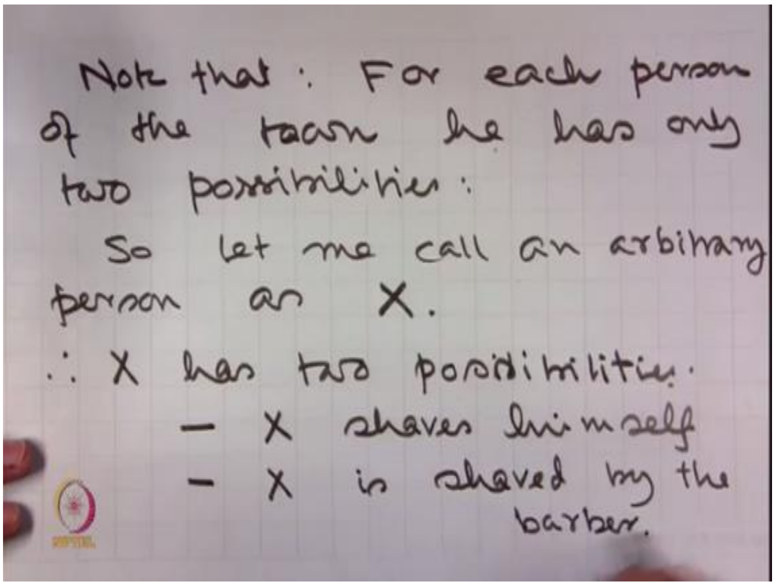

So, note that for each person of the town, he has that means the person has only two possibilities.

So let me call an arbitrary person as X, therefore X has 2 possibilities;

- X shaves himself,
- X is shaved by the barber.

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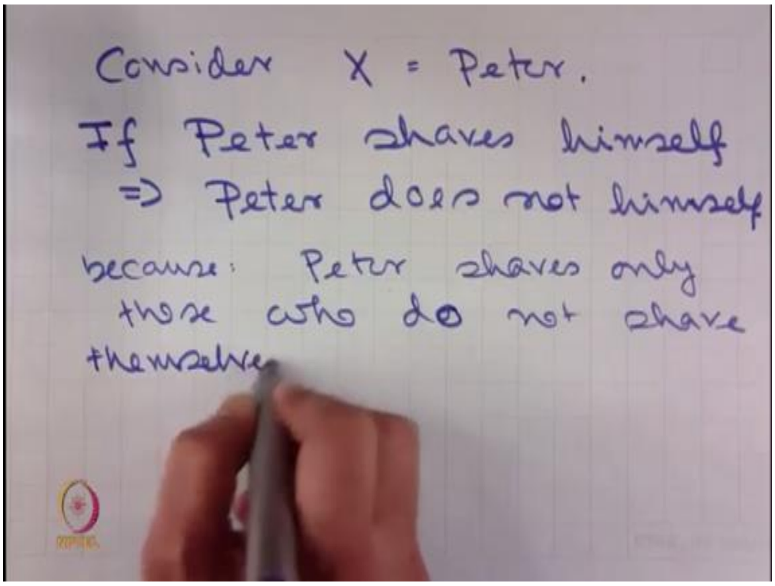

Consider $X =$ Peter, so if Peter shaves himself implies Peter does not shave himself because Peter shaves only those who do not shave themselves.

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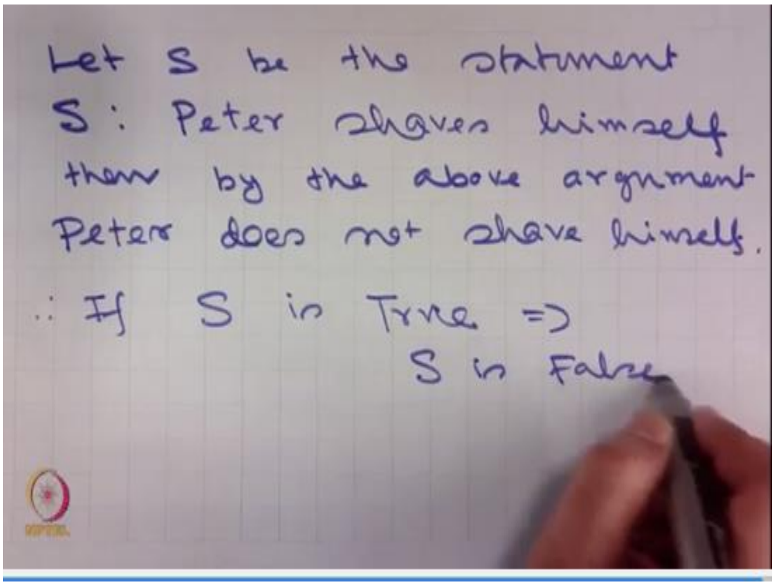

So, let S be the statement, Peter shaves himself, then by the above argument Peter does not shave himself.

∴ If S is True ⇒ S is False.

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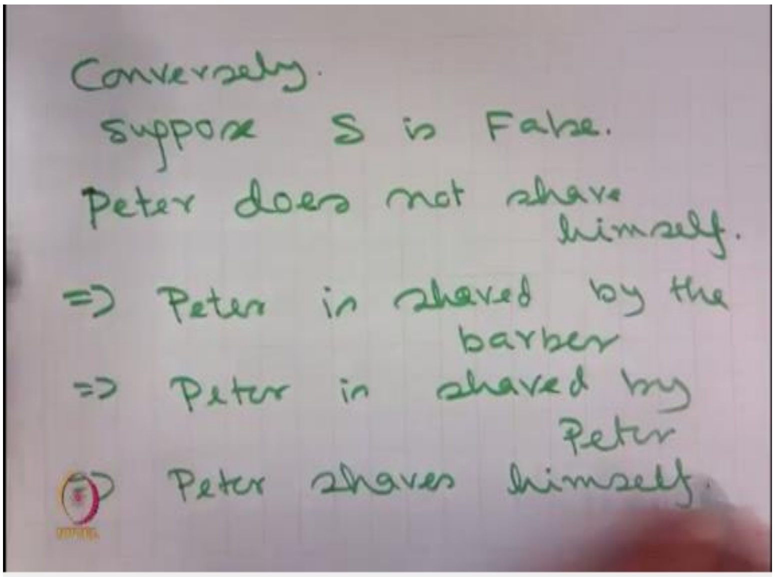

Conversely, suppose S is false, that is Peter does not shave himself.

⇒ Peter is shaved by the barber,

⇒ Peter is shaved by Peter,

⇒ Peter shaves himself.

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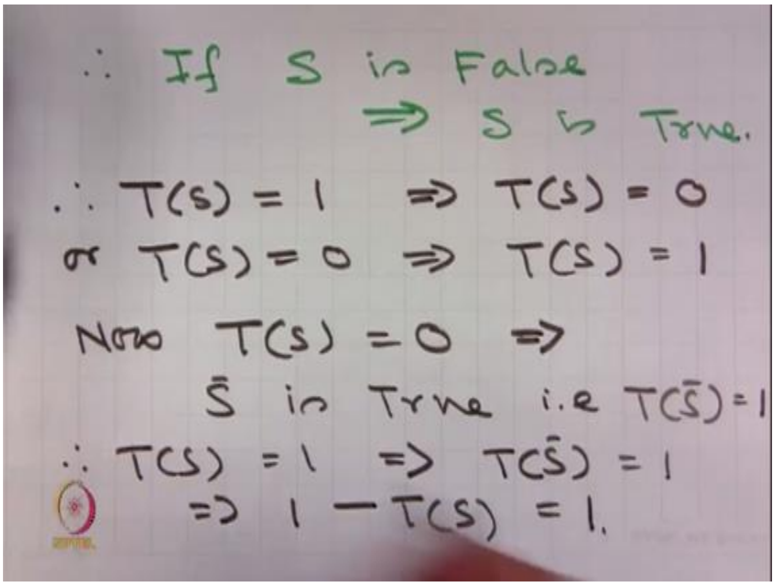

∴ If S is false ⇒ S is True

∴ $T(S) = 1 \Rightarrow T(S) = 0$

Or $T(S) = 0 \Rightarrow T(S) = 1$

Now, $T(S) = 0 \Rightarrow \bar{S}$ is True i.e. $T(\bar{S}) = 1$

∴ $T(S) = 1 \Rightarrow T(\bar{S}) = 1 \Rightarrow 1 - T(S) = 1$

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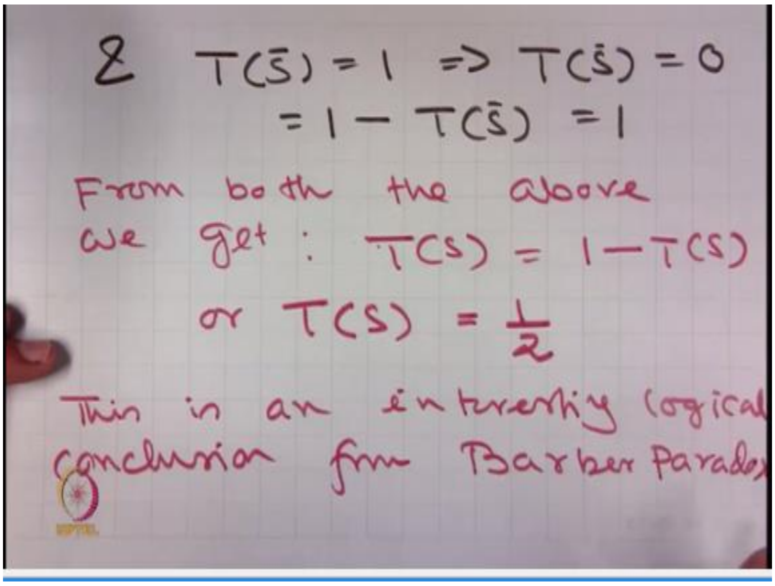

And $T(\bar{S}) = 1 \Rightarrow T(\bar{S}) = 0 \Rightarrow 1 - T(\bar{S}) = 1$

From both the above we get $T(S) = 1 - T(S)$ or $T(S) = \frac{1}{2}$

This is an interesting logical conclusion from Barber Paradox.

This shows the limitation of propositional logic as there is a case when we can see that only 0 and 1, this binary value for truth value of a logic variable is not always correct.

Hence, we need to look into possibilities of multi values or multiple values for truth values, this gives us what is called a multivalued logic, so I stop here today, in the next class I shall start with multivalued logic, thank you.