Introduction to Fuzzy Sets Arithmetic and Logic
Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology – Delhi

Lecture - 20
Fuzzy Sets Arithmetic and Logic

Welcome students to the MOOCs course on fuzzy sets arithmetic and logic, this is lecture number 20.

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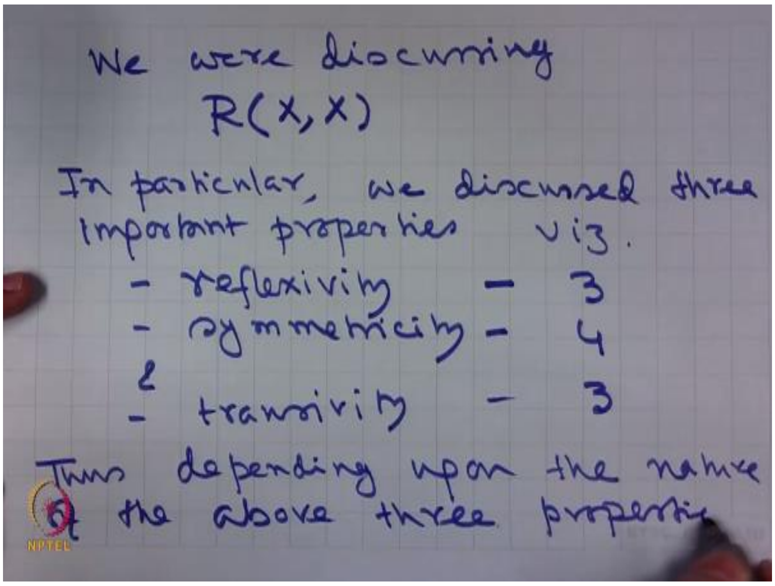

In the last class we were discussing binary relations between the same set that is $R(X,X)$.
In particular, we discussed three important properties namely

- reflexivity
- symmetricity
- transitivity

Reflexivity has 3 variations, Symmetricity has 4 variations and transitivity also has 3 variations.

Thus, depending upon the nature of the above three properties the characteristic of the relation is defined.

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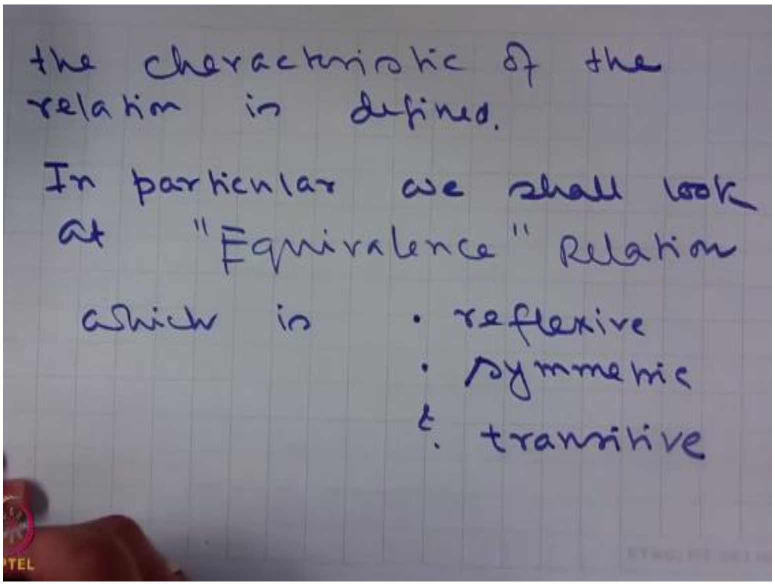

In particular, we look at equivalence relation which is reflexive, symmetric and transitive.

For some other relations such as partially ordered set or strictly order set you can proceed in a very similar way and see how those relationships can be defined with respect to crisp sets and also fuzzy sets.

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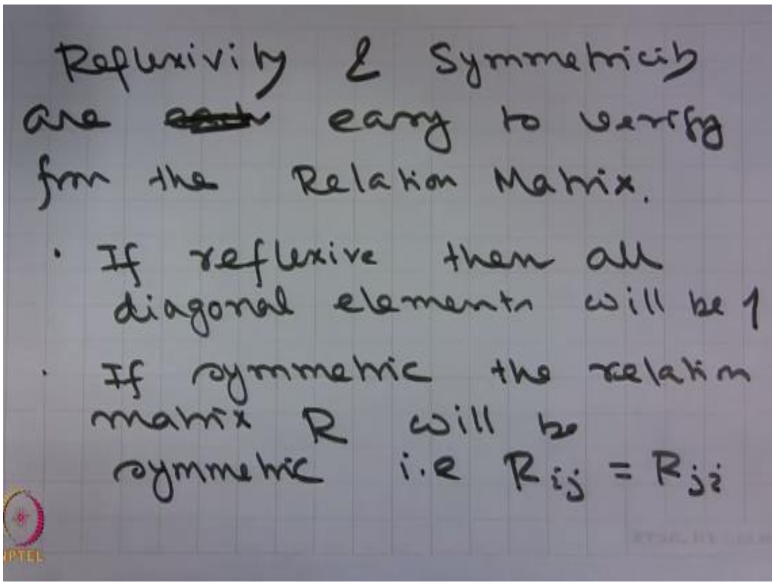

Now reflexivity and symmetricity are easy to verify from the relation matrix.

- If it is reflexive, then all diagonal elements will be 1 and
- If symmetric the relation matrix R will be symmetric, that is $R_{ij} = R_{ji}$ where R_{ij} is the element at the i^{th} row and j^{th} column and R_{ji} is the element in the j^{th} row, i^{th} column.

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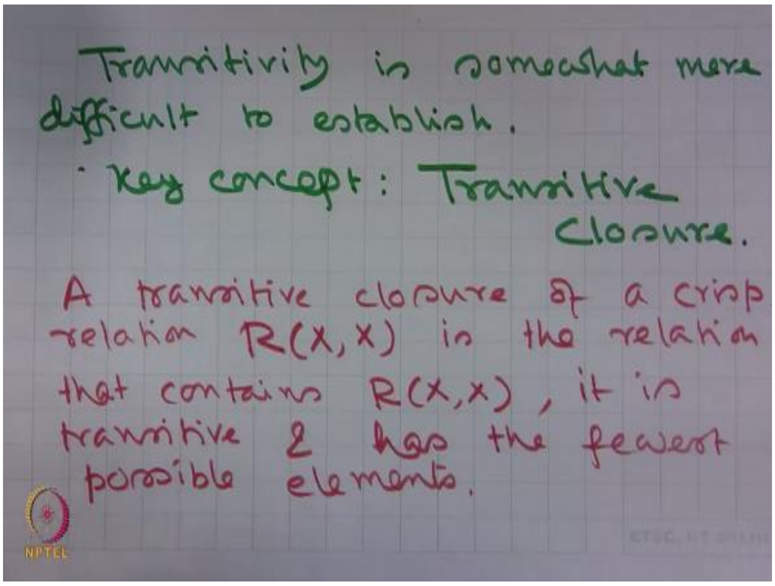

However, transitivity is somewhat more difficult to establish.

The key concept here is transitive closure.

A transitive closure of a crisp relation $R(X, X)$ is the relation that contains $R(X, X)$, it is transitive and has the fewest possible elements,

That is given a relation R from X to X, the transitive closure is the smallest set that contains R and also that is transitive.

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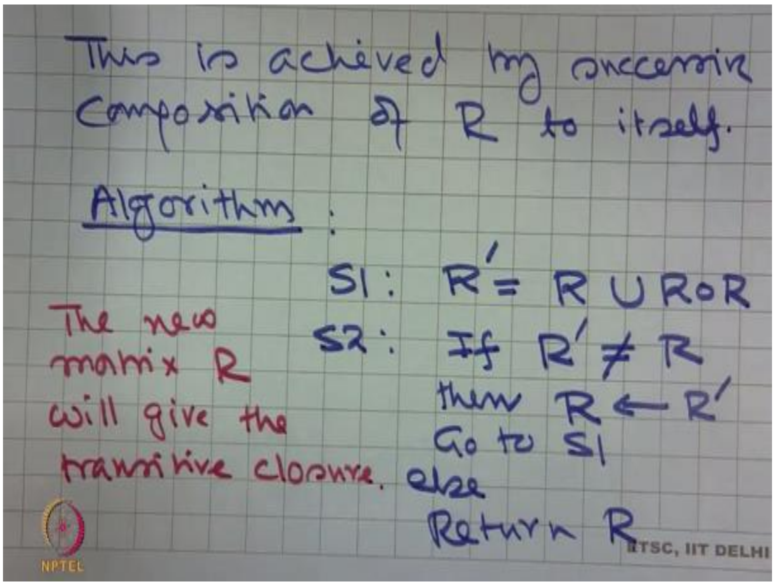

This is achieved by successive composition of R to itself.

So there is a simple algorithm for that one for achieving transitive closure.

Step 1. $R' = R \cup (R \circ R)$

Step 2. If $R' \neq R$ then, $R \leftarrow R'$ and go to Step 1,

else return R.

The new matrix R will give the transitive closure.

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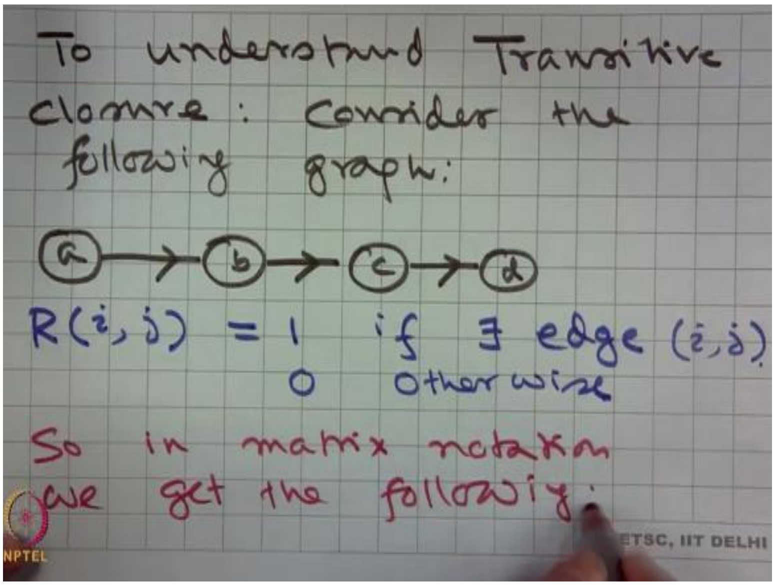
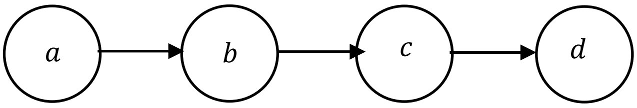


To understand the meaning of transitive closure consider the following graph.



There are four nodes a, b, c and d. From a we can reach b, from b we can reach c and from c we can reach d. So,

$$R(i,j) = \begin{cases} 1 & \text{if } \exists\, edge(i,j) \\ 0 & \text{otherwise} \end{cases}$$

So in matrix notation we get the following.

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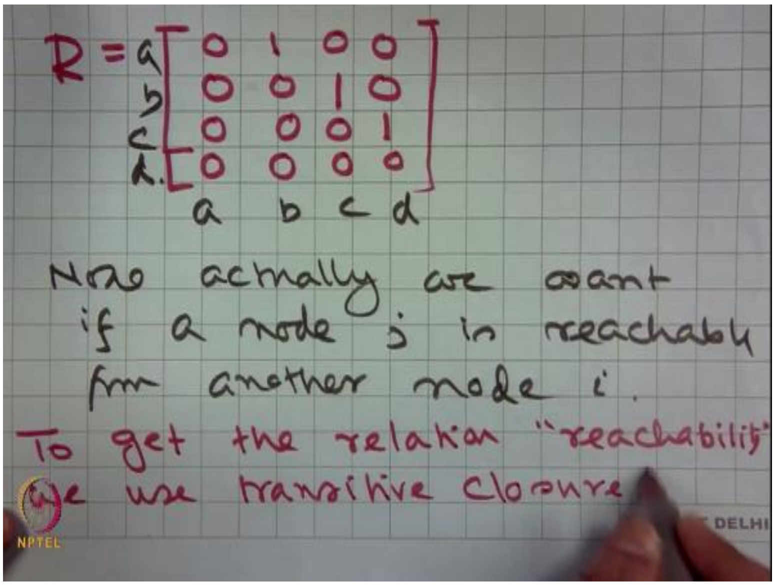

$$R = \begin{array}{c} a \\ b \\ c \\ d \\ \\ \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{array}{cccc} a & b & c & d \end{array}$$

This shows that there is an edge from a to b, there is an edge from b to c and there is an edge from c to d.

Now, actually we want if a node j is reachable from another node i.

And if we go back to the graph we see that c is reachable from a via b, d is also reachable from a via b and c and d is also reachable from b.

Therefore, to get the relation reachability we use transitive closure.

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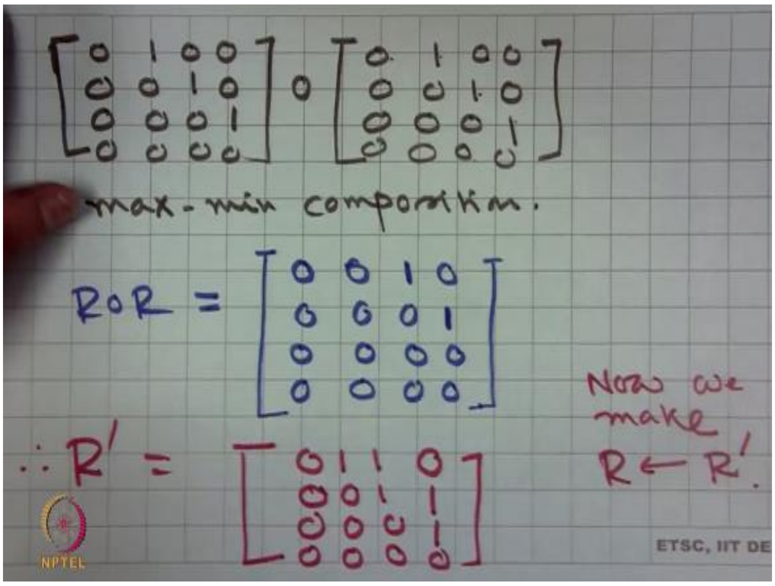

So we compose R with itself and we use max-min composition. So we get the following matrix.

$$R \circ R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore R' = R \cup (R \circ R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it is different from original R now we make $R \leftarrow R'$

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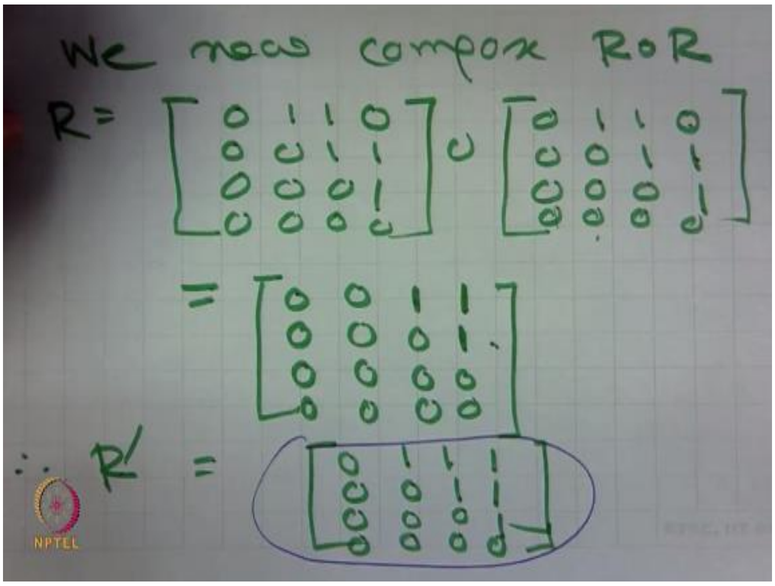

We now compose this new R with itself

$$R \circ R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, new R' after the second iteration is

$$\therefore R' = R \cup (R \circ R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it is different from these R, now this matrix is going to be the new R and I make a composition with itself.

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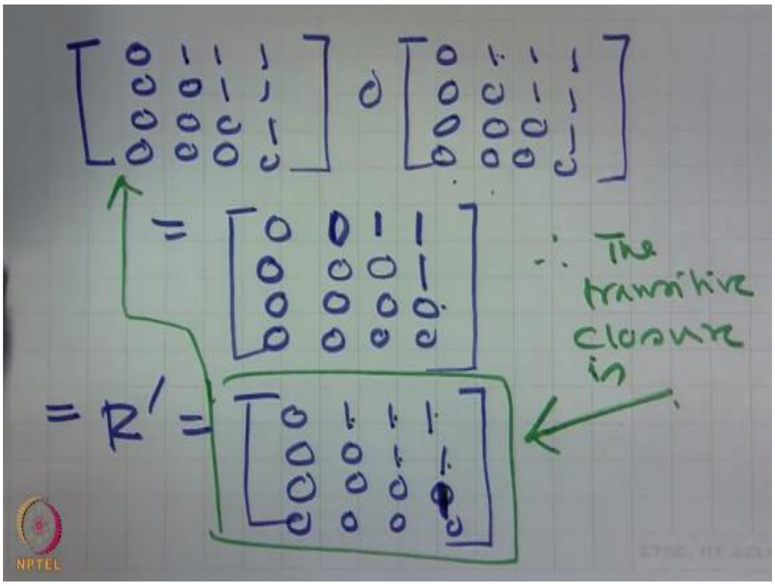

Therefore,

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Again if I take the union new R' is going to be

$$R' = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we get this R' is same as this R.

Therefore, the transitive closure is this matrix and it shows that

- b is reachable from a,
- c is reachable from a,
- d is reachable from a,
- c is reachable from b,
- d is reachable from b and
- d is reachable from c.

So that gives the idea of transitive closure.

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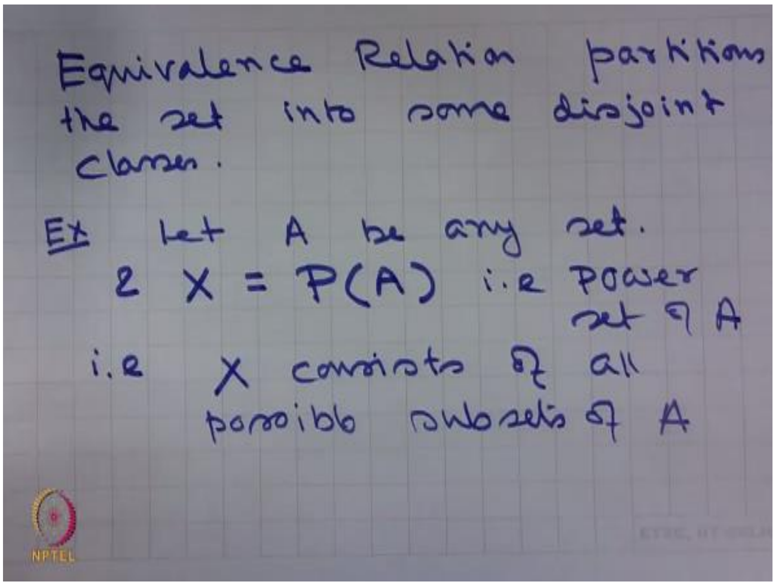

Equivalence relation partitions the set into some disjoint classes.

Let me give you an example.

Let A be any set and $X = P(A)$ that is the power set of A, that means X consists of all possible subsets of A.

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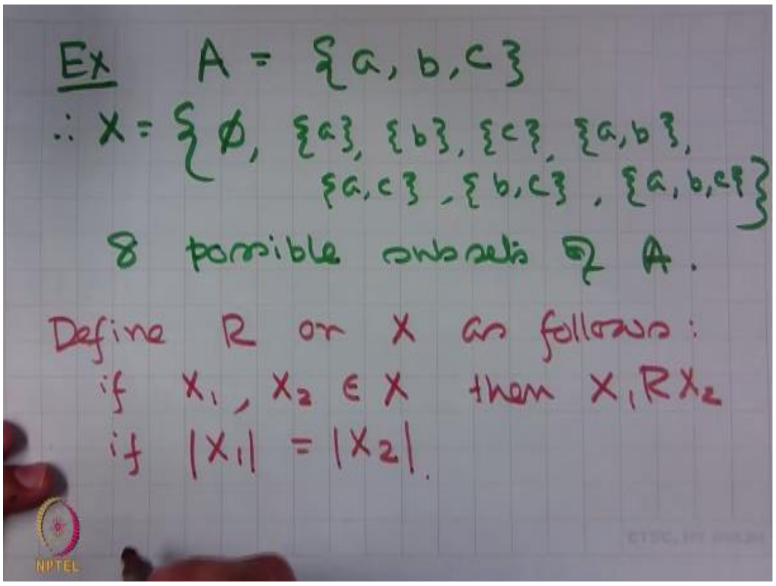

Example

Let $A = \{a, b, c\}$

Therefore, $X = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

So, these are the 8 possible subsets of A.

We define R on X as follows.

For $X_1, X_2 \in X$, $X_1 \; R \; X_2$ if $|X_1| = |X_2|$

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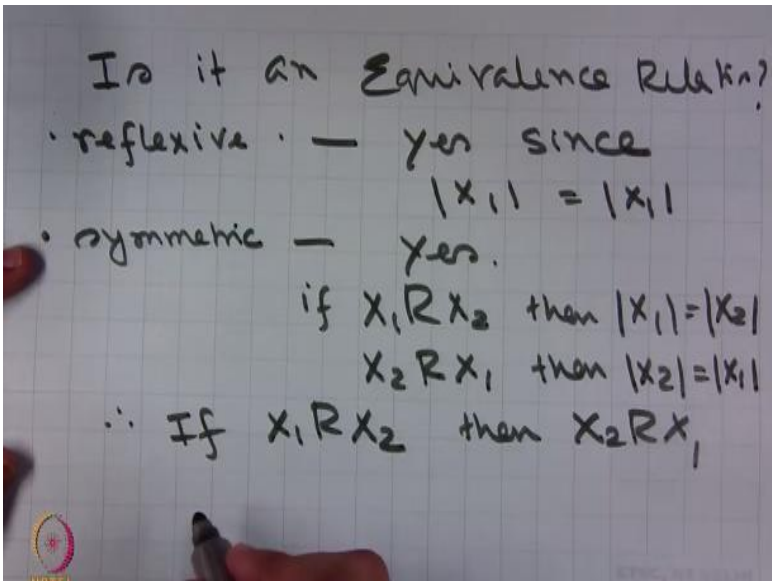

Question is, is it an Equivalence Relation?

So let us check is it

- Reflexive, yes,

  Since, $|X_1| = |X_1|$. Therefore, X_1 is related to itself.

- Symmetric, yes,

  Because if $X_1\ R\ X_2$ then, $|X_1| = |X_2|$ and $X_2\ R\ X_1$ then $|X_2| = |X_1|$

  Therefore, if $X_1\ R\ X_2$ then $X_2\ R\ X_1$

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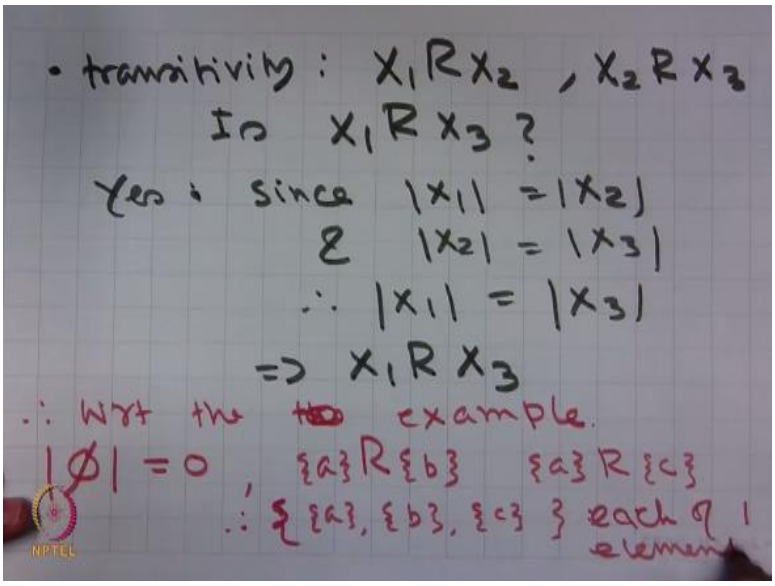

- Transitivity, suppose $X_1\ R\ X_2$ and $X_2\ R\ X_3$.

  Question is, is $X_1\ R\ X_3$?

  Yes, since $|X_1| = |X_2|$ and $|X_2| = |X_3|$. Therefore, $|X_1| = |X_3|$

  $\Rightarrow X_1\ R\ X_3$

∴ With respect to the example we get that ϕ is not related to any other set since $|\phi| = 0$.

$$\{a\}R\{b\}, \{a\}R\{c\}$$

∴ one class is comprising of the 3 sets $\{a\}, \{b\}$ and $\{c\}$, because each has cardinality 1.

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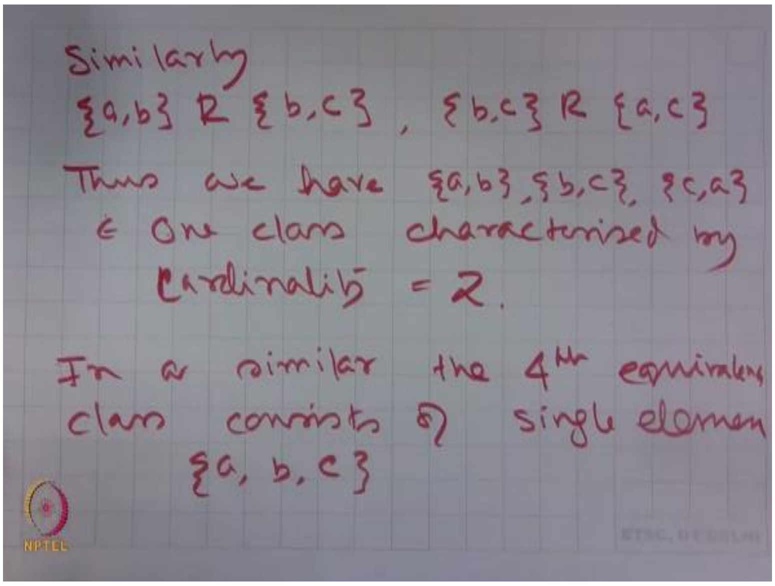

Similarly, $\{a, b\}R\{b, c\}$ and $\{b, c\}\ R\ \{a, c\}$

Thus, we have $\{a, b\}, \{a, c\}$ and $\{b, c\}$ belong to one class characterized by cardinality is equal to 2 and in a similar way the fourth equivalence class consists of single element $\{a, b, c\}$, which has 3 elements.

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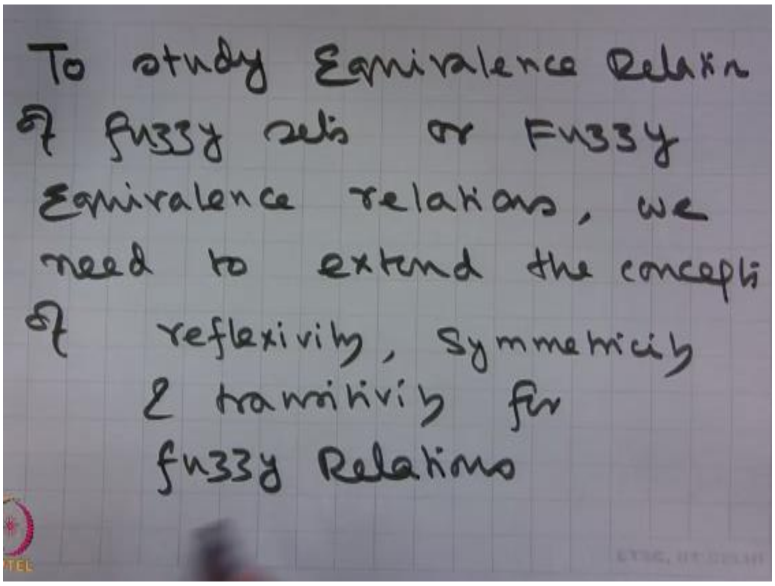

Now, to study Equivalence Relation on fuzzy sets or Fuzzy Equivalence relations, we need to extend the concepts of the 3 properties reflexivity, symmetricity, and transitivity for fuzzy relations.

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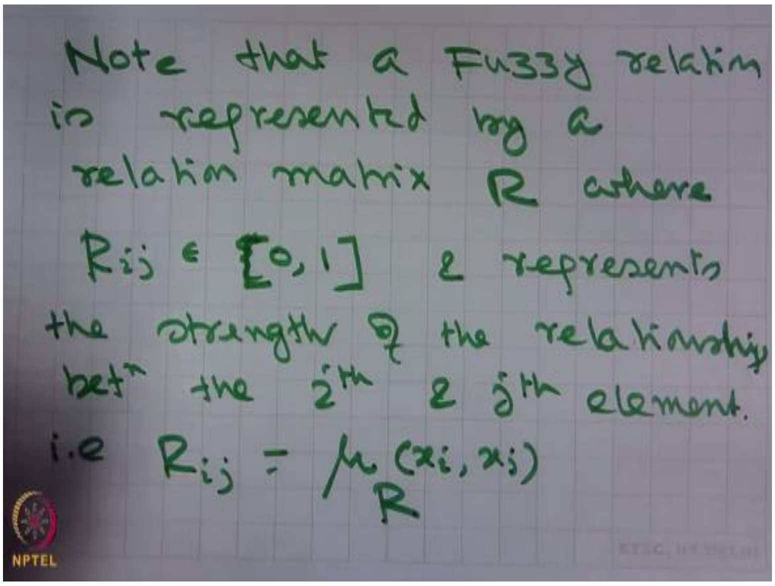

Note that a fuzzy relation is represented by a relation matrix R where $R_{ij} \in [0, 1]$ and represents the strength of the relationship between the i^{th} and j^{th} element that is

$$R_{ij} = \mu_R(x_i, x_j)$$

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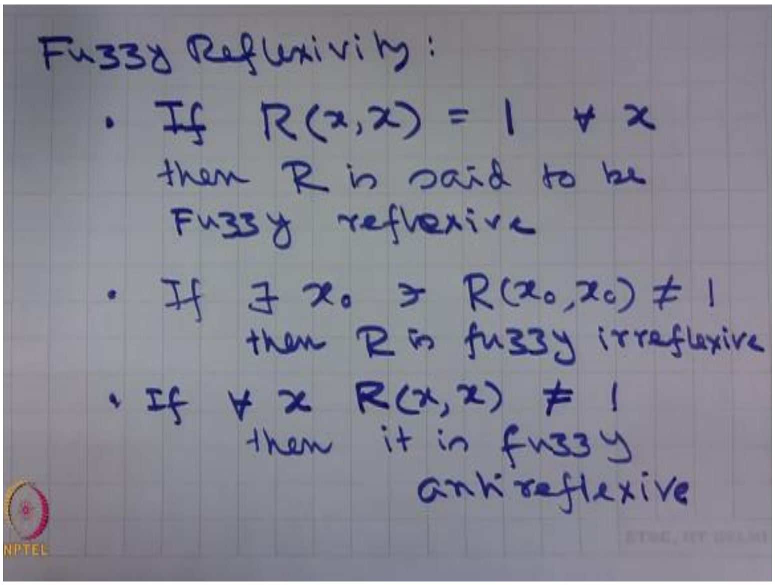

With that background let us now define Fuzzy Reflexivity.

- If $R(x, x) = 1 \;\; \forall \, x$ then R is said to be fuzzy reflexive.
- If $\exists \, x_0$ such that $R(x_0, x_0) \neq 1$ then R is fuzzy irreflexive.
- If $\forall \, x, \;\; R(\, x, x) \neq 1$, then it is fuzzy antireflexive.

So these 3 are similar to what we define with respect to crisp relations.

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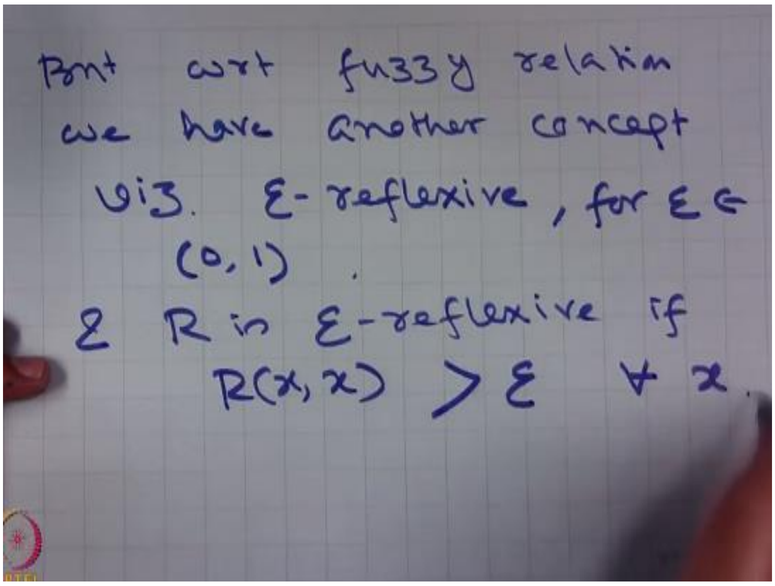

But with respect to fuzzy relation we have another concept namely ϵ-reflexive for $\epsilon \in [0, 1]$ and R is ϵ-reflexive if $R(x, x) > \epsilon \;\; \forall\, x$

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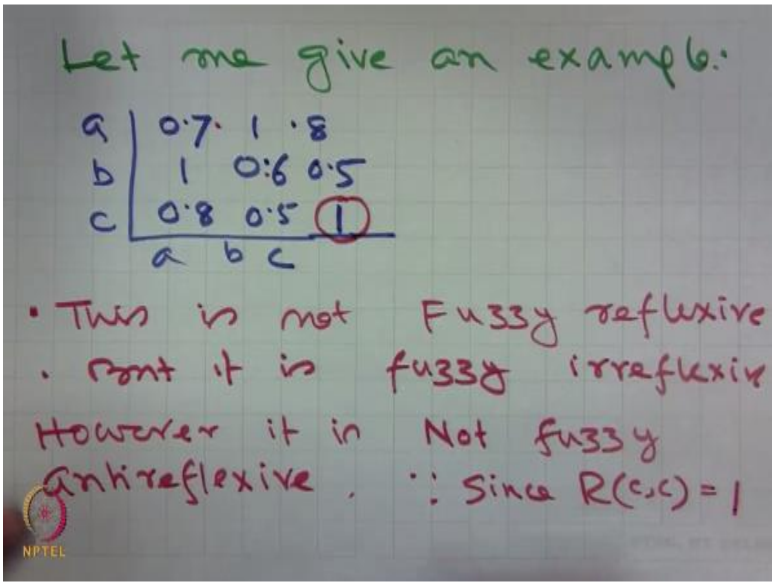

So let me give an example:

	a	b	c
a	0.7	1	0.8
b	1	0.6	0.5
c	0.8	0.5	1

- This is not fuzzy reflexive because some diagonal elements are not 1,
- But it is fuzzy irreflexive; however, it is not fuzzy anti-reflexive since $R(c, c) = 1$

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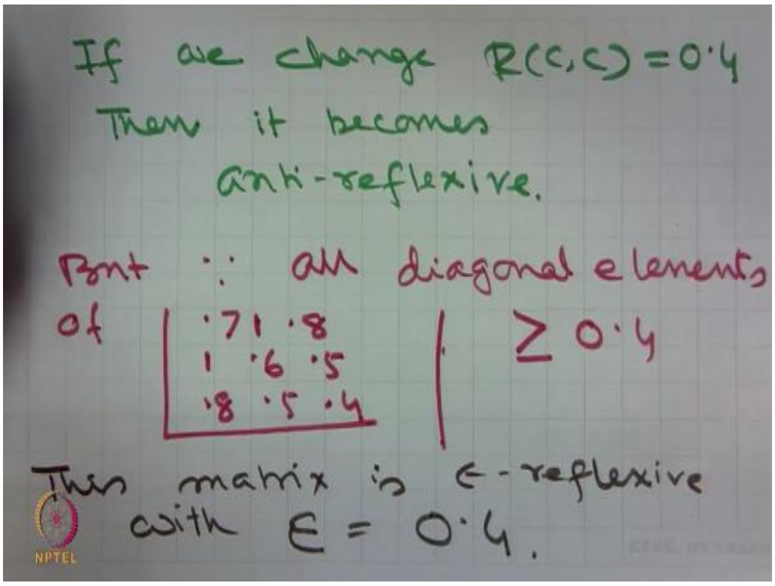

If we change $R(c,c) = 0.4$ then it becomes anti-reflexive.

But since all diagonal elements of

a	0.7	1	0.8
b	1	0.6	0.5
c	0.8	0.5	0.4
	a	b	c

are greater than equal to 0.4.

This matrix is ϵ-reflexive with $\epsilon = 0.4$

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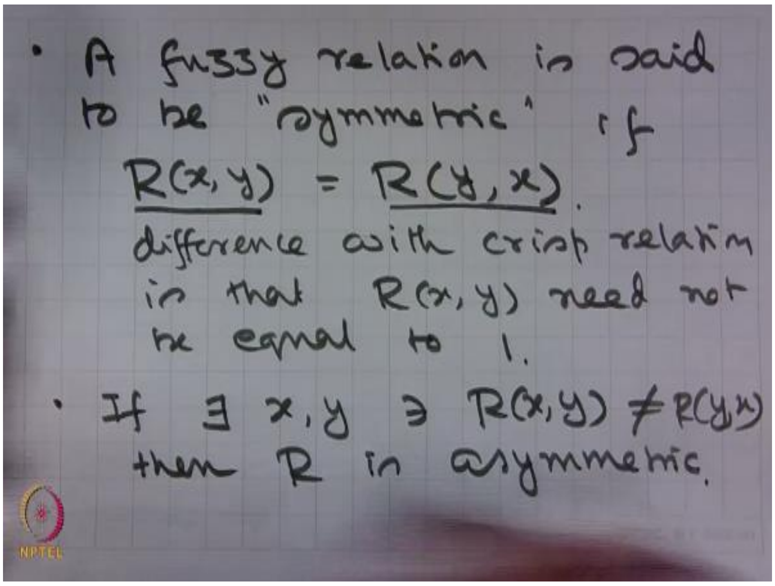

- A fuzzy relation is said to be symmetric if $R(x,y) = R(y,x)$. With respect to crisp relation also we have seen this, but the difference is that $R(x,y)$ need not be equal to

1. That is if the strength of relationship between x and y is same as the strength of the relationship between y and x, whatever value it is we call it symmetric.

- Similarly, if there exists x, y such that $R(x, y) \neq R(y, x)$ then R is asymmetric.

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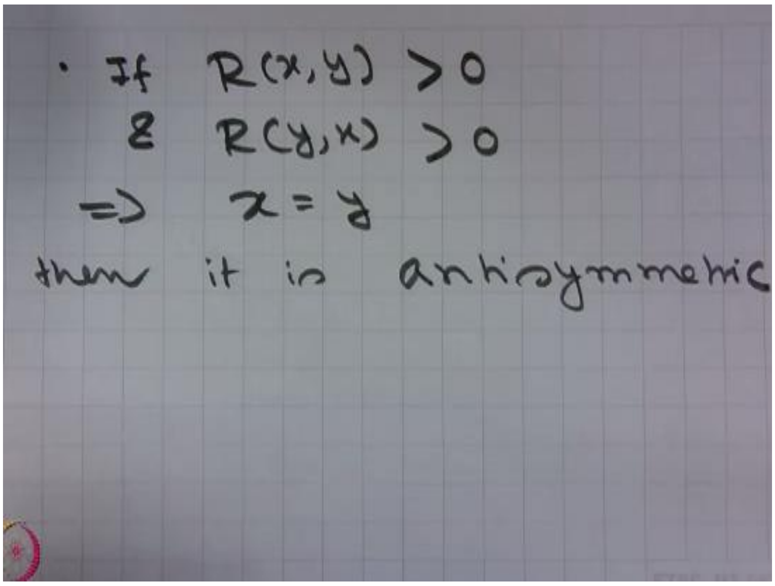

- If $R(x, y) > 0$ and $R(y, x) > 0$ implies $x = y$, then it is anti-symmetric.

Thus we find a subtle difference with respect to anti-symmetricity between crisp relation and a fuzzy relation.

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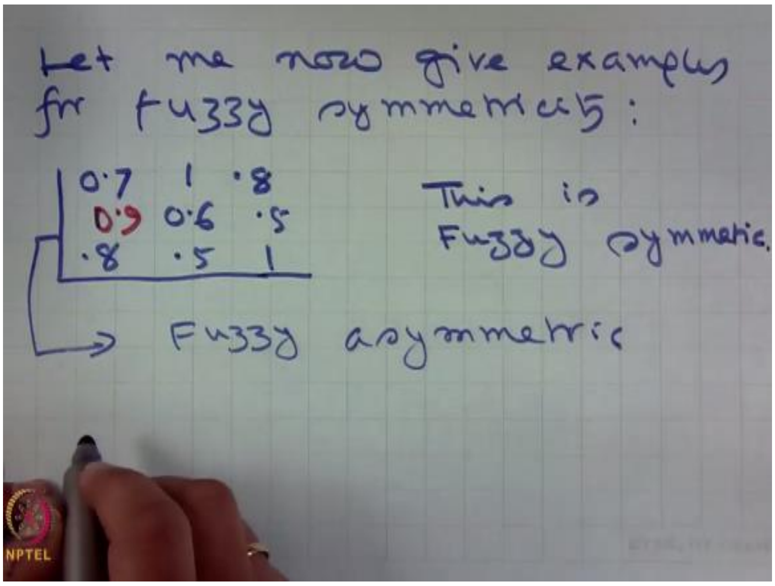

Let me now give examples for fuzzy symmetricity.

Consider

a	0.7	1	0.8
b	1	0.6	0.5

c	0.8	0.5	1
	a	b	c

This is fuzzy symmetric.

If we change $R(b,a)$ to 0.9 then it becomes fuzzy asymmetric.

a	0.7	1	0.8
b	0.9	0.6	0.5
c	0.8	0.5	1
	a	b	c

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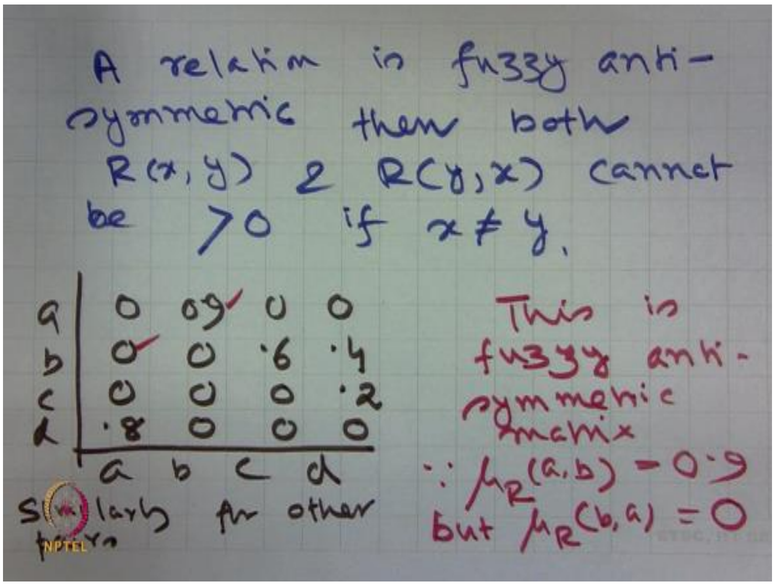

A relation is fuzzy anti-symmetric then both $R(x,y)$ and $R(y,x)$ cannot be > 0, if $x \neq y$

Therefore, consider this matrix

a	0	0.9	0	0
b	0	0	0.6	0.4
c	0	0	0	0.2
d	0.8	0	0	0
	a	b	c	d

This is a fuzzy anti-symmetric matrix.

Since $\mu_R(a,b) = 0.9$ but $\mu_R(b,a) = 0$. Similarly, for other pairs. Please verify that it is fuzzy anti-symmetric.

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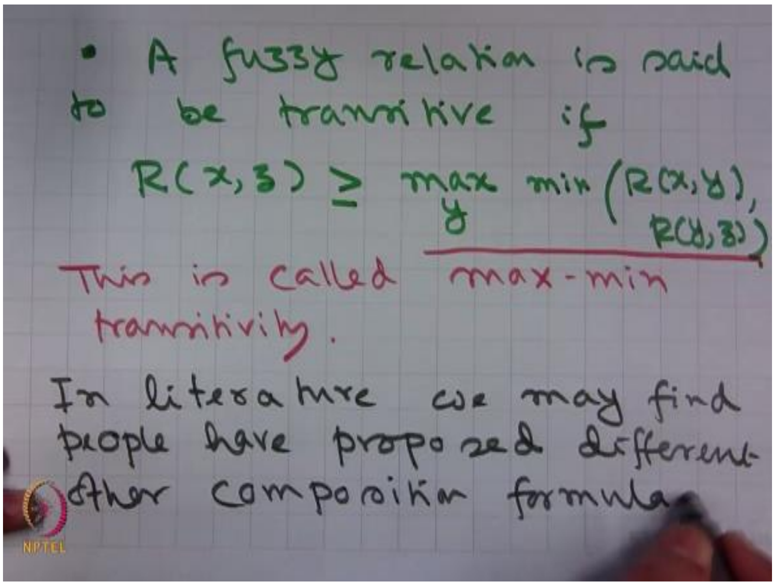

- A fuzzy relation is said to be transitive if
  $R(x,z) \geq \max_{y}\{\min(R(x,y), R(y,z))\}$

So this is called max-min transitivity because in the composition of $R(x,y)$ and $R(y,z)$ we are using the max-min formula. Min is actually the standard intersection and max gives standard union.

In literature we may find people have proposed different other composition formula.

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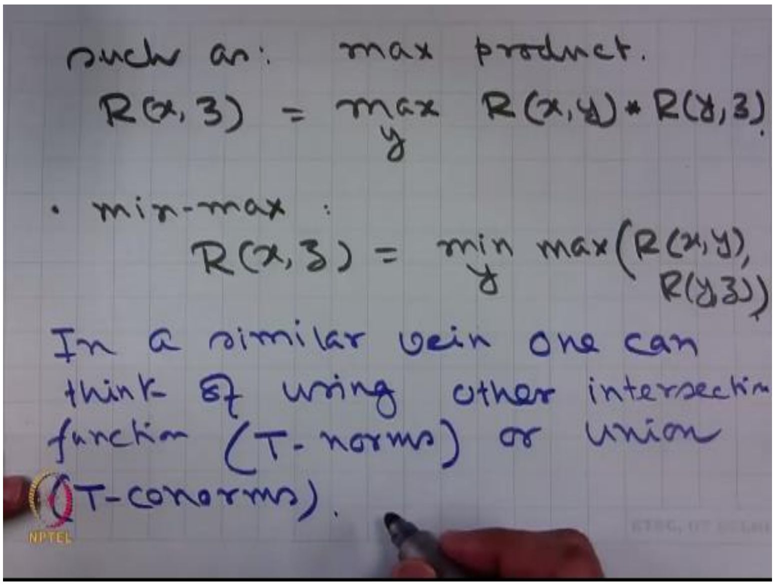

Such as max-product that is

$$R(x,z) = \max_{y}\{R(x,y) \cdot R(y,z)\}$$

there can be min-max that is

$$R(x,z) = \max_{y}\{\min(R(x,y), R(y,z))\}$$

So in different applications people have tried different way of relation composition, but max min is the most commonly practiced one. In a similar vein one can think of using other intersection function that is t-norms or union that is t-conorms, but in this course we are not going into much details with respect to these different formulae.

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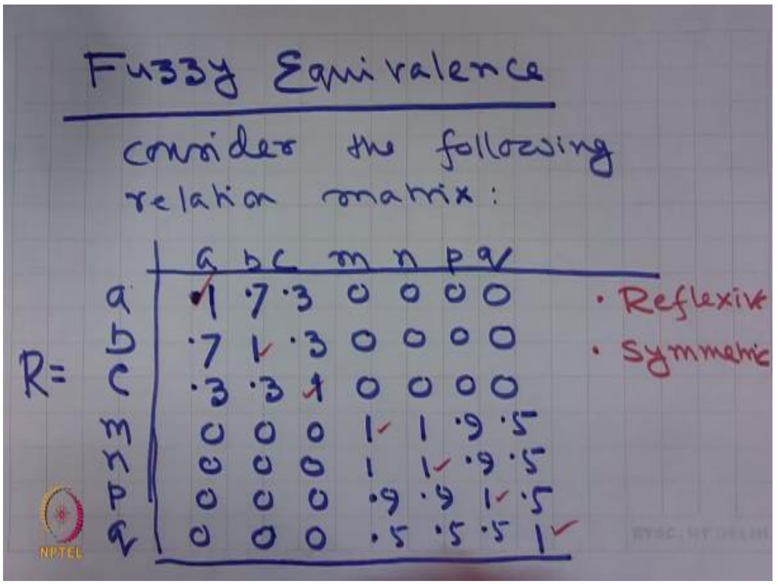

With that background let us now explain Fuzzy Equivalence.

Consider the following relation matrix.

	a	b	c	m	n	p	q
a	1	0.7	0.3	0	0	0	0
b	0.7	1	0.3	0	0	0	0
c	0.3	0.3	1	0	0	0	0
m	0	0	0	1	1	0.9	0.5
n	0	0	0	1	1	0.9	0.5
p	0	0	0	0.9	0.9	1	0.5
q	0	0	0	0.5	0.5	0.5	1

So let us call it R.

If we examine the matrix carefully, we see that all the diagonal elements are 1. Therefore, it is reflexive. If you recall the definition that a fuzzy relation is reflexive if $R(x, x) = 1$ for all x.

Also note that it is symmetric.

Now is it transitive?

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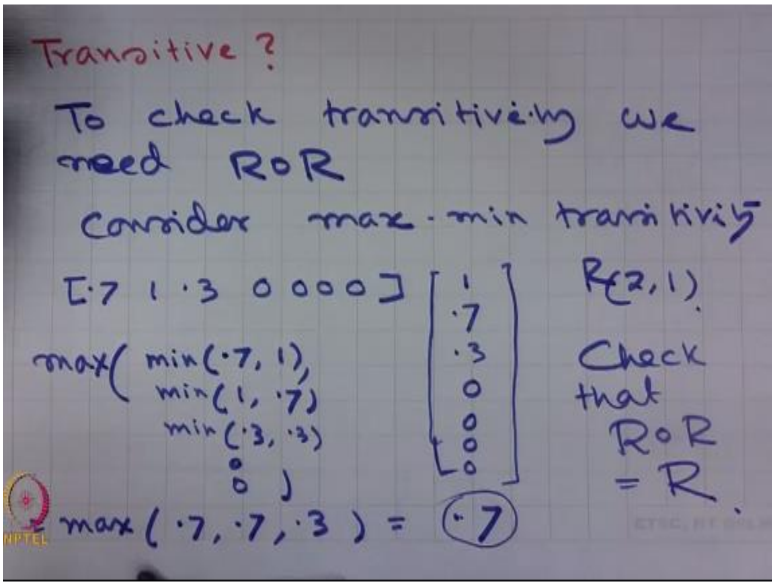

To check transitivity, we need to compose R with itself that is, $R \circ R$ and suppose we work on max-min transitivity.

Let us consider any arbitrary element say ij and if we compose what we shall get. So let us look at the say second row first column element.

The second row is $[0.7 \quad 1 \quad 0.3 \quad 0 \quad 0 \quad 0 \quad 0]$ and the first column is $\begin{bmatrix} 1 \\ 0.7 \\ 0.3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. What is the max-min?

$$R \circ R(2,1) = \max \left\{ \begin{matrix} \min(0.7,1) \\ \min(1,0.7) \\ \min(0.3,0.3) \\ \min(0,0) \\ \min(0,0) \\ \min(0,0) \\ \min(0,0) \end{matrix} \right\} = \max\{0.7,0.3,0\} = 0.7$$

In the original matrix the $2,1$ element was 0.7 and after composition also it remains 0.7.

I request you to verify for all the 49 elements and see that $R \circ R = R$

That is the transitive closure of R if we apply that algorithm that I have given some time back after the first iteration itself we find that R' is same as R and therefore, this relation is transitive. Also if we look at the matrix very carefully you can see that the first 3 elements are related with each other and the last 4 elements are related with each other, these have been kept 0. Therefore, clear distinction between the elements which are related with each other.

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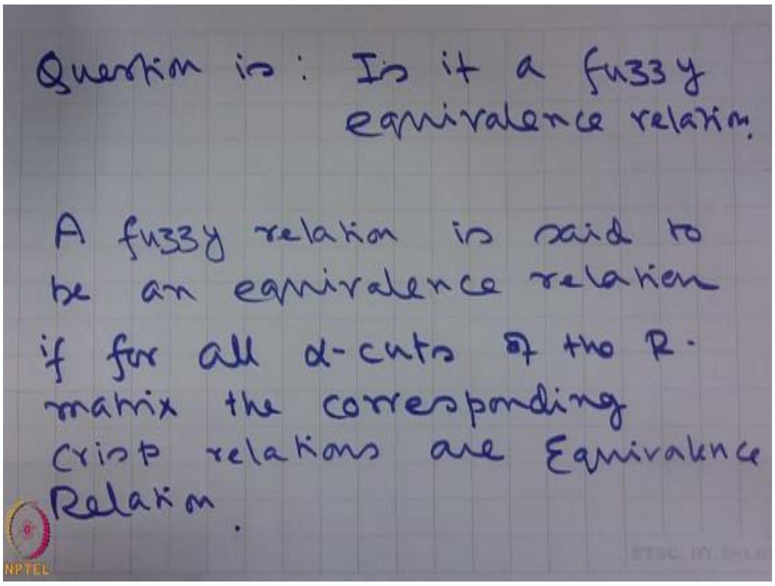

Therefore, the question is, is it a fuzzy equivalence relation?

A fuzzy relation is said to be an equivalence relation if for all α-cuts of the R matrix, the corresponding crisp relations are equivalence relation.

Therefore, if we have been given a fuzzy relation matrix then corresponding to all α-cuts we need to verify if we are getting an equivalence relation among the elements of the underlying set.

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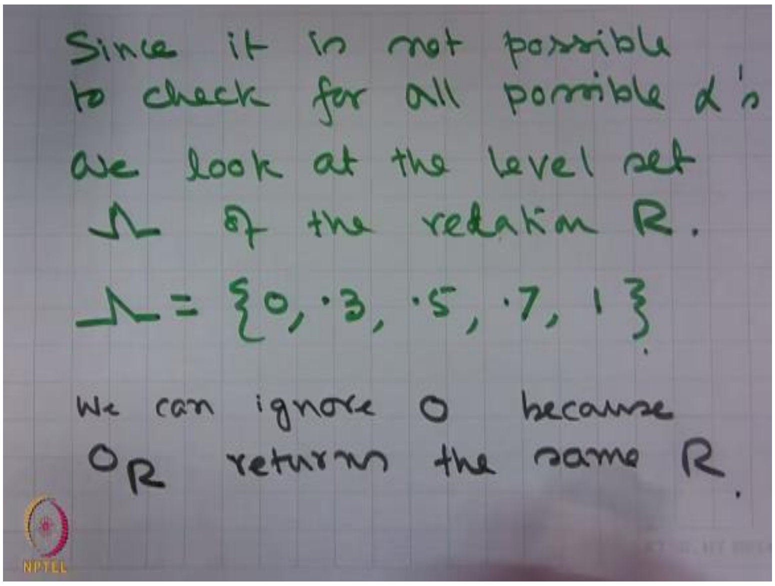

Since it is not possible to check for all possible α's.

We look at the level set (Λ) of the relation R. Now if we look at the matrix very carefully then we get that different α's belonging to the level set $\Lambda = \{0, 0.3, 0.5, 0.7, 1\}$. Therefore, we need to check for different alphas and see what happens. We can ignore 0, because ${}^{0}R$ returns the same R.

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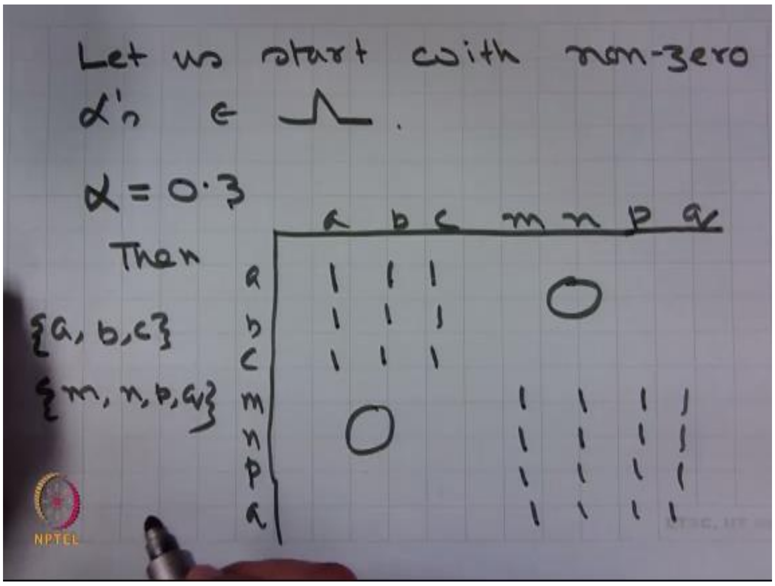

So let us start with non-zero α's belonging to the level set Λ.

Consider $\alpha = 0.3$ then, the matrix that we will get is

	a	b	c	m	n	p	q
a	1	1	1	0	0	0	0
b	1	1	1	0	0	0	0
c	1	1	1	0	0	0	0
m	0	0	0	1	1	1	1
n	0	0	0	1	1	1	1
p	0	0	0	1	1	1	1
q	0	0	0	1	1	1	1

It is very clear that these are 2 different equivalence classes.

Therefore, we get that 2 classes to be $\{a, b, c\}$ and $\{m, n, p, q\}$.

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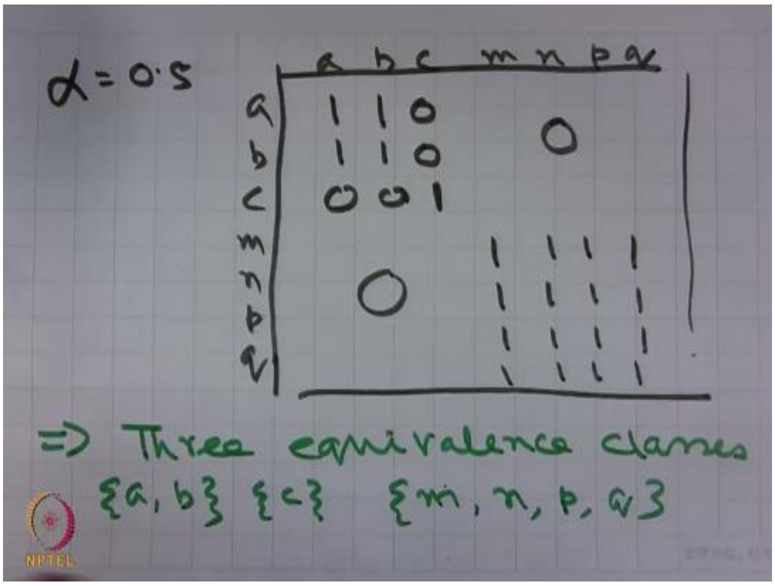

Let us now consider $\alpha = 0.5$, therefore, the matrix that we get is,

	a	b	c	m	n	p	q
a	1	1	0	0	0	0	0
b	1	1	0	0	0	0	0
c	0	0	1	0	0	0	0
m	0	0	0	1	1	1	1
n	0	0	0	1	1	1	1
p	0	0	0	1	1	1	1
q	0	0	0	1	1	1	1

So that gives us 3 equivalence classes namely $\{a, b\}, \{c\}$ and $\{m, n, p, q\}$.

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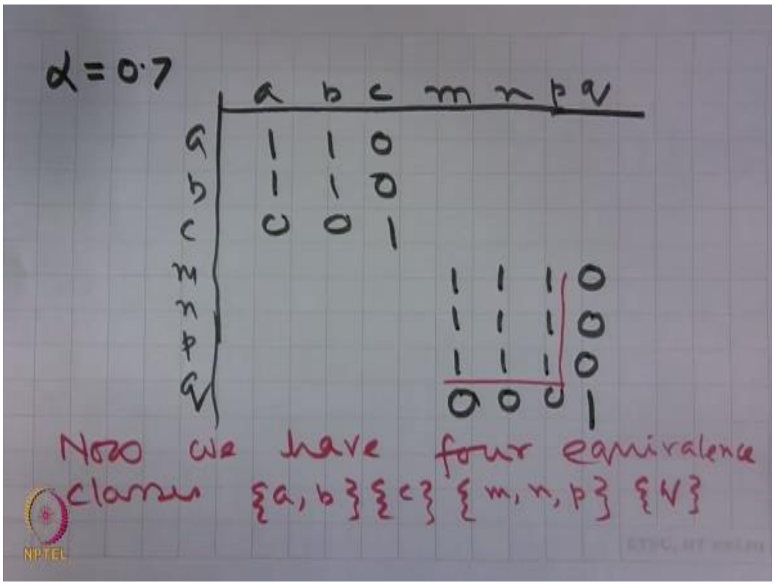

When we take $\alpha = 0.7$,

	a	b	c	m	n	p	q
a	1	1	0	0	0	0	0
b	1	1	0	0	0	0	0
c	0	0	1	0	0	0	0
m	0	0	0	1	1	1	0
n	0	0	0	1	1	1	0
p	0	0	0	1	1	1	0
q	0	0	0	0	0	0	1

Thus, with $\alpha = 0.7$, now we have 4 equivalence classes namely $\{a, b\}, \{c\}, \{m, n, p\}$ and $\{q\}$.

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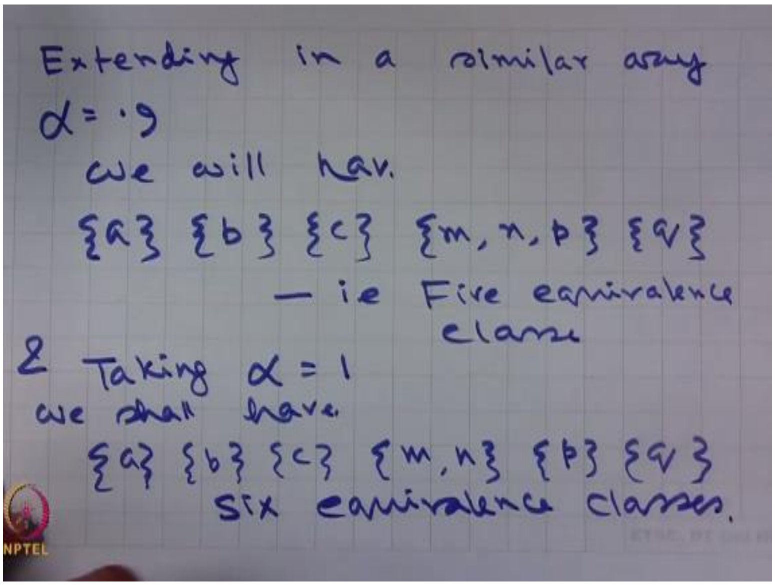

Extending in a similar way for $\alpha = 0.9$, we will have $\{a\}, \{b\}, \{c\}, \{m, n, p\}$ and $\{q\}$ that is, 5 equivalence classes and taking $\alpha = 1$ we shall have $\{a\}, \{b\}, \{c\}, \{m, n\}, \{ p\}$ and $\{q\}$ that is 6 equivalence classes.

Thus what we find that for different α-cuts we get the corresponding partitions and each one of them represent equivalence classes.

Therefore, we can say that this fuzzy relation is actually a fuzzy equivalence relation.

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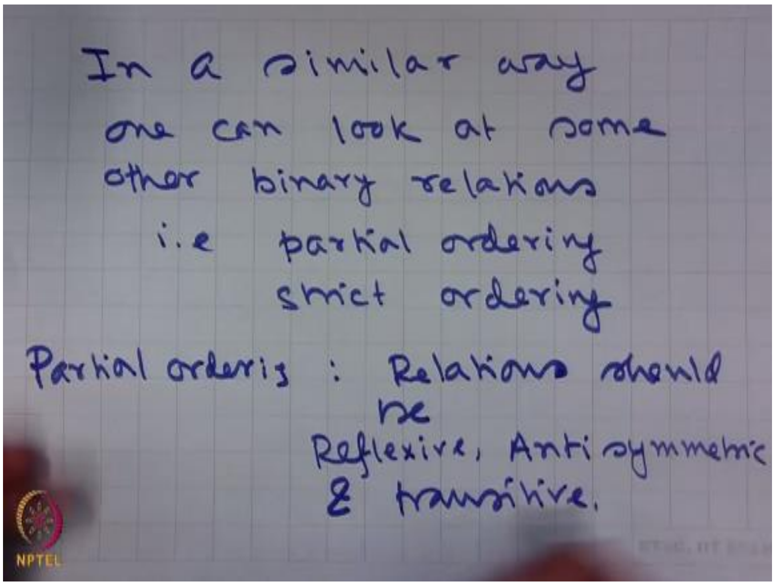

In a similar way one can look at some other binary relations namely partial ordering, strict ordering.

For partial ordering we have seen the relation should be reflexive anti-symmetric and transitive.

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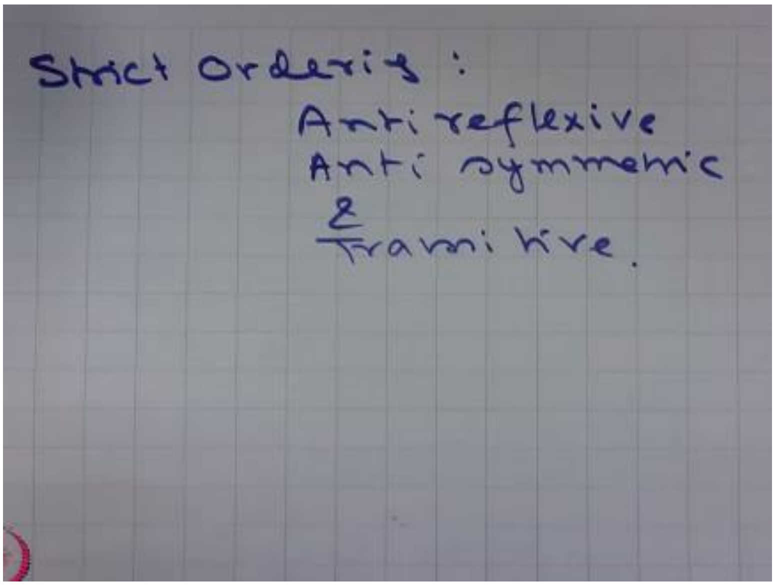

For strict ordering that it should be anti-reflexive, anti-symmetric and transitive.

We have already extended the definitions of these properties with respect to fuzzy relations. So one can apply them for a given relation matrix and examine their properties. Okay students I stop here today and with this I complete my chapter on relations, from the next class onwards I shall start fuzzy logic which is an important concept with respect to decision making and artificial intelligence. Thank you.