**Introduction to Fuzzy Sets Arithmetic and Logic**

**Prof. Niladri Chatterjee**
**Department of Mathematics**
**Indian Institute of Technology – Delhi**

**Lecture - 19**
**Fuzzy Sets Arithmetic and Logic**

Welcome students to the MOOCs course on Fuzzy sets, Arithmetic and logic. This is lecture number 19.

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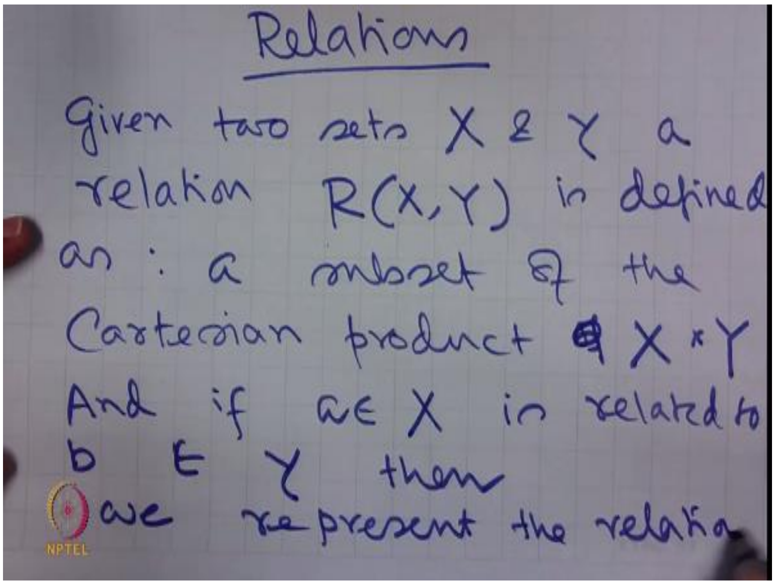


In the last class I have introduced you to the concept of relations.

Given two sets $X$ and $Y$, a relation $R(X,Y)$ is defined as a subset of the Cartesian product $X \times Y$

And if $a \in X$ is related to $b \in Y$ then we represent the relation in the following ways:

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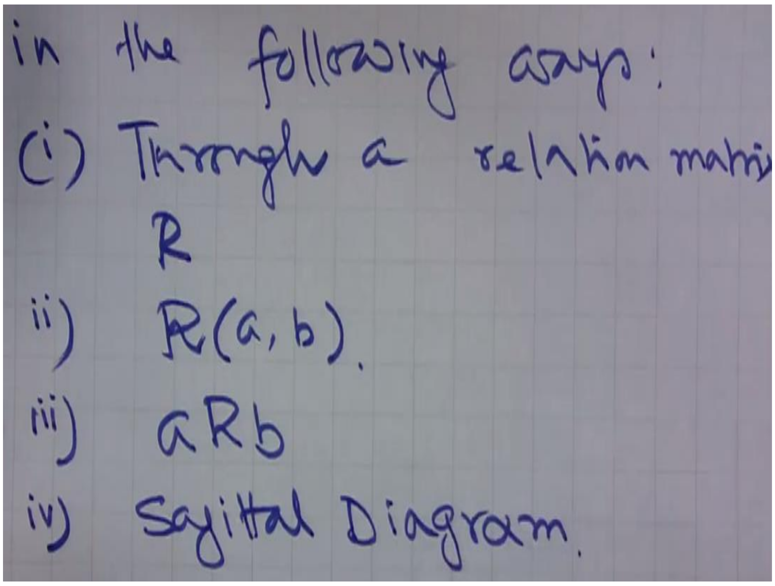

i. Through a relation matrix $R$

ii. Also we can write it as $R(a, b)$

iii. Also we can write it as $a\ R\ b$

So these are the different notations that you may find in different books or rather reading materials.

iv. Through *Sagittal Diagram*.

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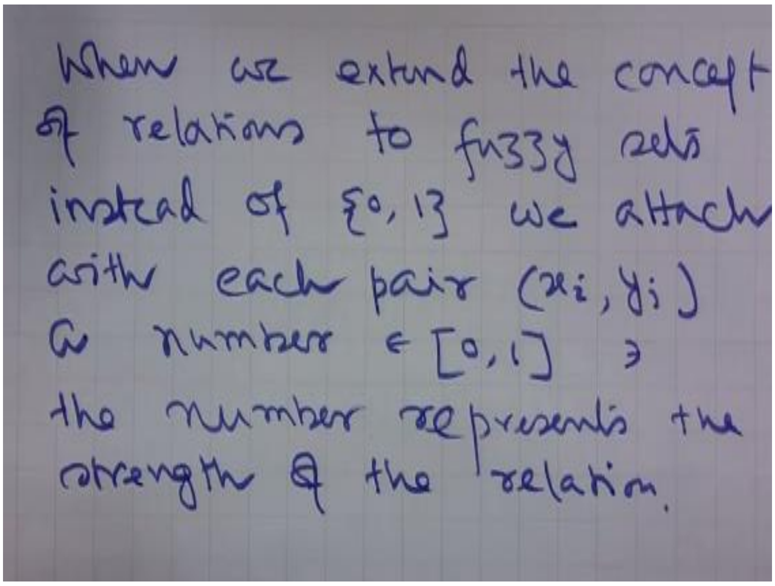


When we extend the concept of relation to fuzzy sets instead of $\{0, 1\}$ we attach with each pair $(x_i, y_j)$ a number belonging to the interval $[0, 1]$ such that the number represents the strength of the relation.

This I have explained to you.

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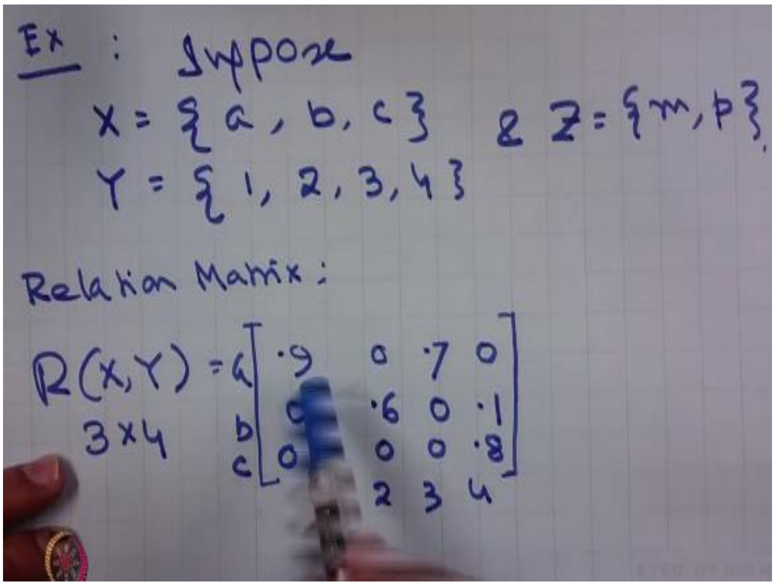

Let me give you some example.

Suppose $X = \{a, b, c\}, Y = \{1, 2,\ 3,\ 4\}$ and $Z = \{m, p\}$

So consider the relation matrix $R(X, Y)$

Since $X$ has 3 elements and $Y$ has 4 elements, it is going to be a $3 \times 4$ matrix

For example, suppose this matrix is

$R(X, Y) =$

$$\begin{array}{cc} \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 0.9 & 0 & 0.7 & 0 \\ 0 & 0.6 & 0 & 0.1 \\ 0 & 0 & 0 & 0.8 \end{bmatrix} \\ & \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 \end{matrix} \end{array}$$

This effectively says that $a$ is not related with 2, but the strength of relation between $a$ and 1 is 0.9. Similarly, for all other elements.

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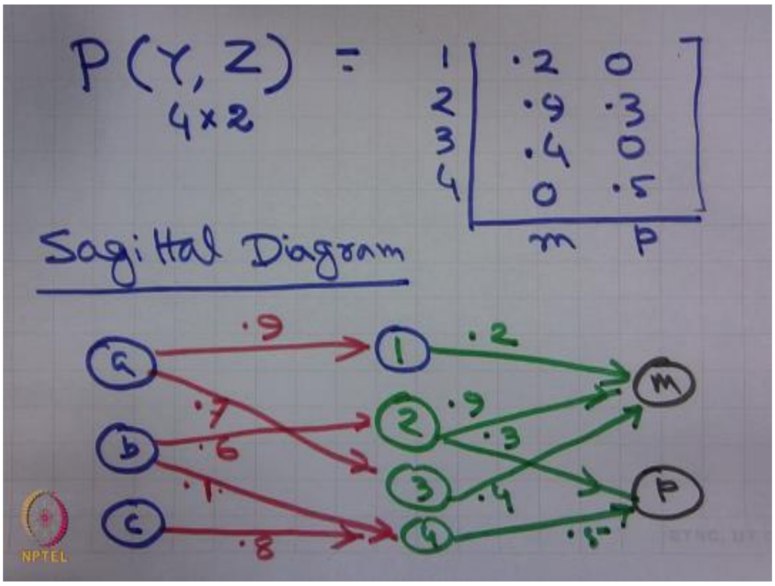


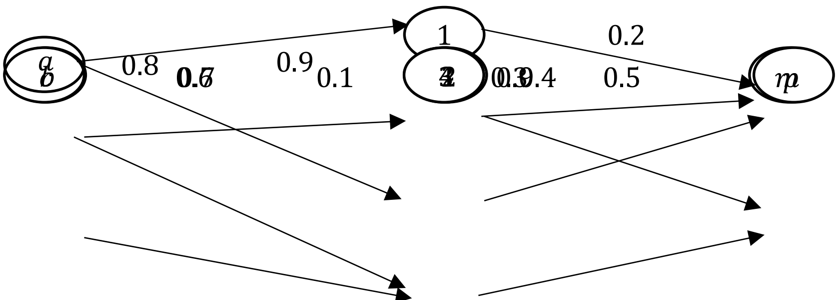

And then we can have another relation say $P(\, Y, Z)$ and let us define it as

$P(Y, Z)_{4\times 2} =$

$$\begin{array}{cc} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0.2 & 0 \\ 0.9 & 0.3 \\ 0.4 & 0 \\ 0 & 0.5 \end{bmatrix} \\ & \begin{matrix} m & \quad p \end{matrix} \end{array}$$

So this is the matrix that gives us the fuzzy relationship between $Y$ and $Z$.

Another way of representation is through *Sagittal Diagram*:



So this gives us the *Sagittal Diagram* representing both the relationships $R(X, Y)$ and $P(Y, Z)$.

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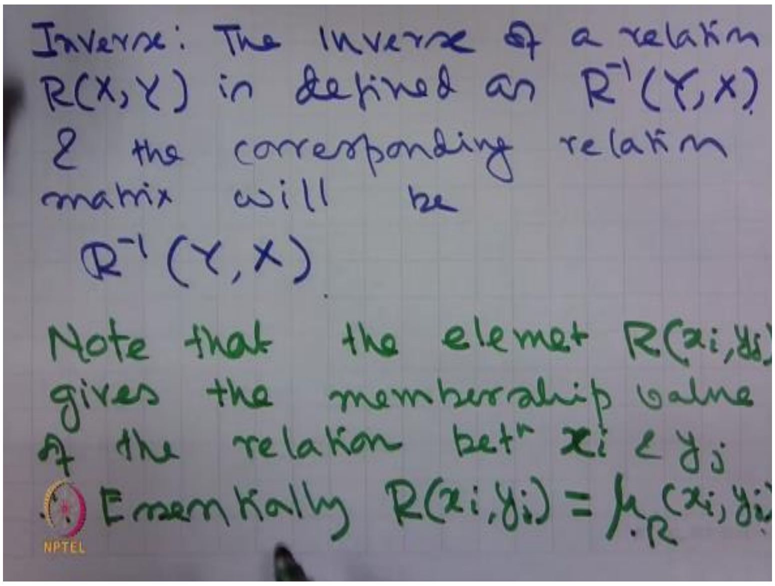


Given a relationship we can talk about the inverse.

The inverse of a relation $R(X, Y)$ is defined as $R^{-1}(Y, X)$ and the corresponding relation matrix will be $R^{-1}(Y, X)$

Note that the elements $R(x_i, y_j)$ gives the membership value of the relation between $x_i$ and $y_j$. Therefore, essentially $R\left(x_i, y_j\right) = \mu_R\left(x_i, y_j\right)$

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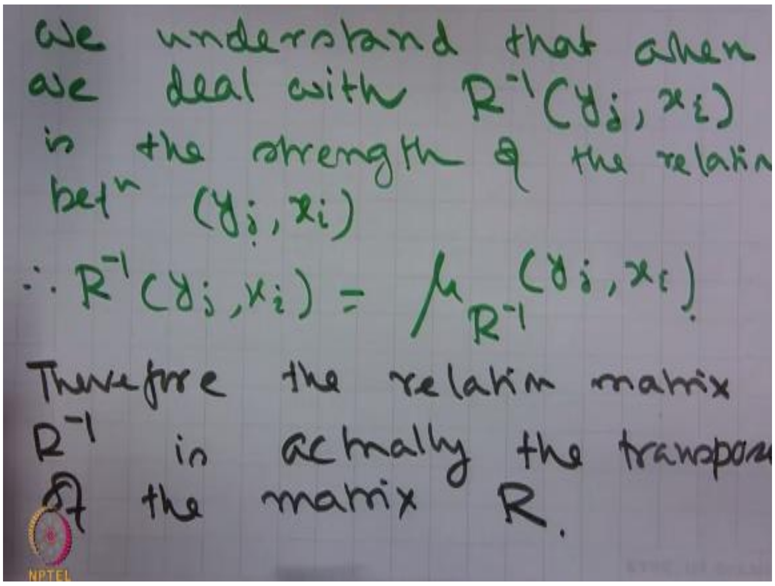


If we remember this, then we understand that when we talk about $R^{-1}(y_j, x_i)$ is the strength of the relation between $y_j$ and $x_i$.

Therefore, $R^{-1}( y_j, x_i) = \mu_{R^{-1}}(y_j, x_i)$

And therefore the relation matrix $R^{-1}$ is actually the transpose of the matrix $R$.

$$R^{-1} = R^t$$

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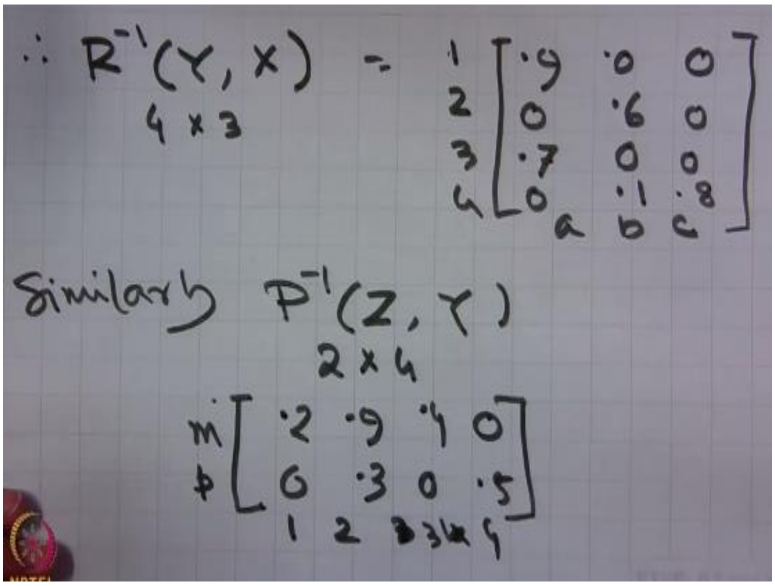


Therefore, with respect to our example

$R(X,Y) =$

$$\begin{array}{cc} \begin{array}{c}a\\b\\c\end{array} & \begin{bmatrix}0.9&0&0.7&0\\0&0.6&0&0.1\\0&0&0&0.8\end{bmatrix}\\ & \begin{array}{cccc}1&2&3&4\end{array}\end{array}$$

And $R^{-1}(Y,X) =$

$$\begin{array}{cc} \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix}0.9&0&0\\0&0.6&0\\0.7&0&0\\0&0.1&0.8\end{bmatrix}\\ & \begin{array}{ccc}a&b&c\end{array}\end{array}$$

Similarly, $P(Y,Z) =$

$$\begin{array}{cc} \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix}0.2&0\\0.9&0.3\\0.4&0\\0&0.5\end{bmatrix}\\ & \begin{array}{cc}m&p\end{array}\end{array}$$

And $P^{-1}(Z,Y) =$

$$\begin{array}{cc} \begin{array}{c}m\\p\end{array} & \begin{bmatrix}0.2&0.9&0.4&0\\0&0.3&0&0.5\end{bmatrix}\end{array}$$

1 2 3 4

Thus corresponding to both the relations $R$ and $P$ we get corresponding inverse relationship.

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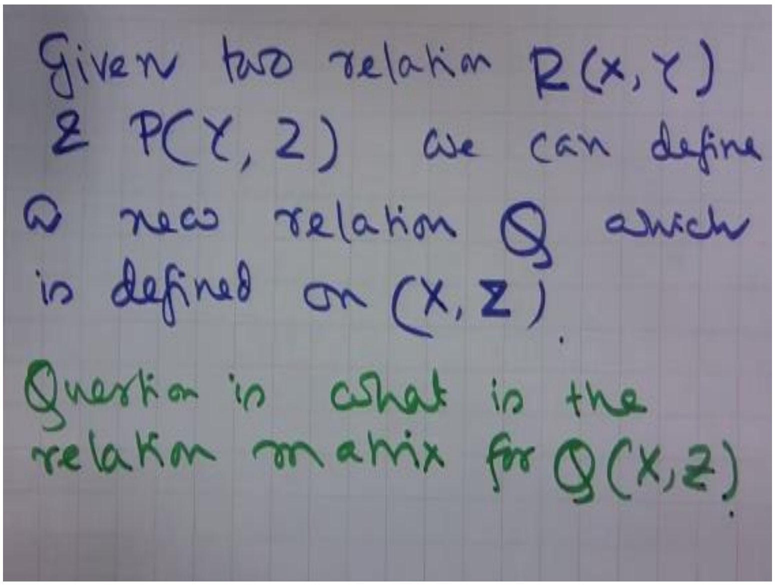


Given two relations $R(X,Y)$ and $P(Y,Z)$ we can define a new relation $Q$ which is defined on $(X,Z)$.

Question is:

What is the relation matrix for $Q(X,Z)$?

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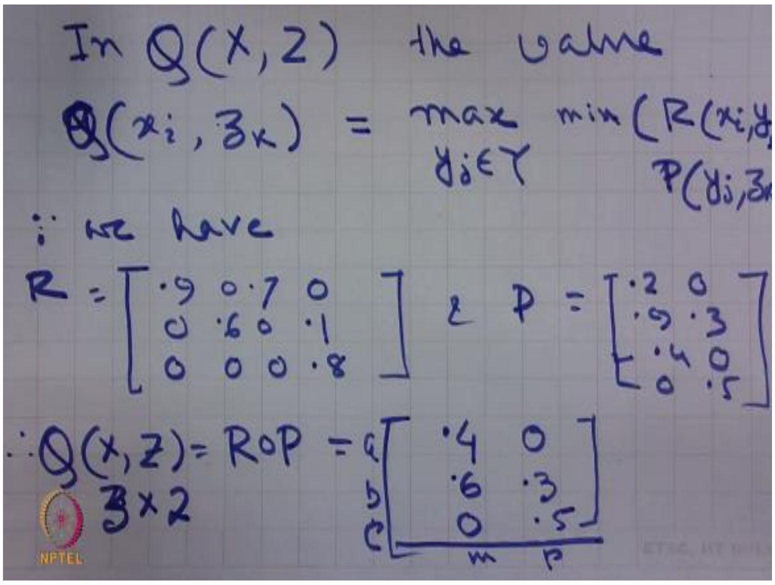


In $Q(X,Z)$ the value $Q(x_i, z_k)$ that is the strength of relation between the $i^{th}$ element of set $X$ and the $k^{th}$ element of set $Z$ is defined as

$$Q(x_i, z_k) = \max_{y_j \in Y}\{\min(R(x_i, y_j), P(y_j, z_k))\}$$

$$\because R = \begin{bmatrix} 0.9 & 0 & 0.7 & 0 \\ 0 & 0.6 & 0 & 0.1 \\ 0 & 0 & 0 & 0.8 \end{bmatrix} \text{ and } P = \begin{bmatrix} 0.2 & 0 \\ 0.9 & 0.3 \\ 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$\therefore Q(X, Z)_{3\times 2} = R \circ P =$

$$\begin{array}{cc} a & \\ b & \begin{bmatrix} 0.4 & 0 \\ 0.6 & 0.3 \\ 0 & 0.5 \end{bmatrix} \\ c & \\ & \begin{array}{cc} m & p \end{array} \end{array}$$

The operation is very similar to matrix multiplication, but there we take the product and add it. Here instead of multiplication we are looking at the minimum and instead of final addition we are looking at the maximum.

Thus, even if the elements of two sets $X, Z$ that are not directly related but, there is a set $Y$ such that the set $X$ is related to $Y$ and set $Y$ is related to $Z$. Then, we can get the relationship between $X$ and $Z$.

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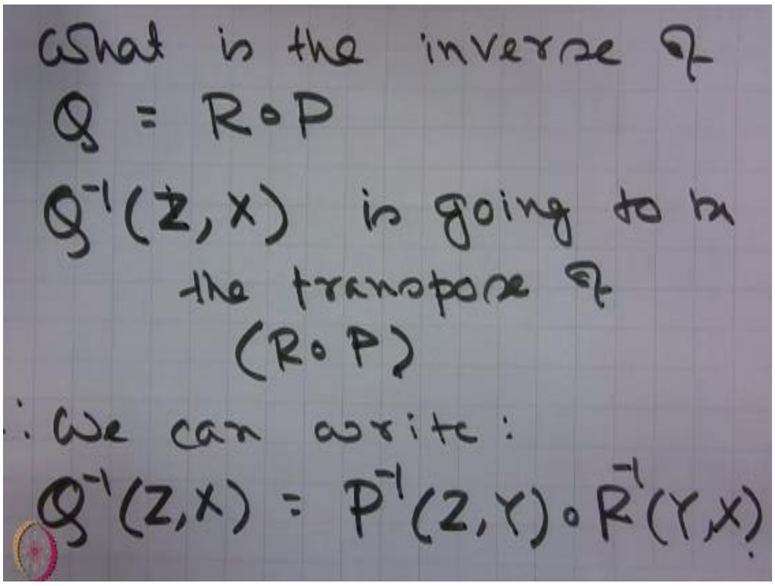


What is the inverse of $Q = R \circ P$?

$Q^{-1}(Z, X)$ is going to be the transpose of $R \circ P$ .

Therefore, we can write $Q^{-1}(Z, X) = P^{-1}(Z, Y) \circ R^{-1}(Y, X)$

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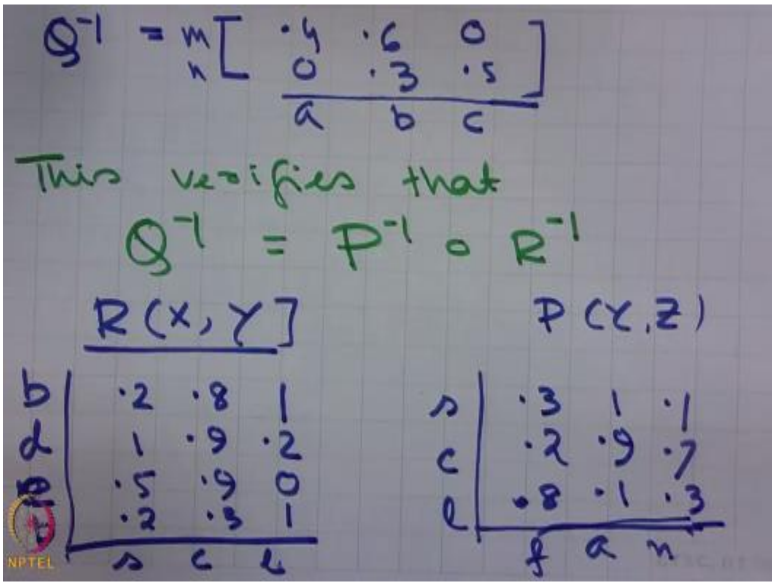


$$\because Q = \begin{bmatrix} 0.4 & 0 \\ 0.6 & 0.3 \\ 0 & 0.5 \end{bmatrix}$$

So we from here we can write $Q^{-1} = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0 & 0.3 & 0.5 \end{bmatrix}$

We already had $R^{-1}$ and $P^{-1}$

So let us first compute $P^{-1} \circ R^{-1}$

$$= \begin{bmatrix} 0.2 & 0.9 & 0.4 & 0 \\ 0 & 0.3 & 0 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0.7 & 0 & 0 \\ 0 & 0.1 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} \max\{\min(0.2,0.9), \min(0.9,0), \min(0.4,0.7), \min(0,0)\} & \max\{\min(0.2,0), \min(0.9,0.6), \min(0.4,0), \min(0,0.1)\} & \max\{\min(0.2,0), \min(0.9,0), \min(0.4,0), \min(0,0.8)\} \\ \max\{\min(0,0.9), \min(0.3,0), \min(0,0.7), \min(0.5,0)\} & \max\{\min(0,0), \min(0.3,0.6), \min(0,0), \min(0.5,0.1)\} & \max\{\min(0,0), \min(0.3,0), \min(0,0), \min(0.5,0.8)\} \end{bmatrix}$$

$$= \begin{bmatrix} \max\{0.2,0,0.4,0\} & \max\{0,0.6,0,0\} & \max\{0,0,0,0\} \\ \max\{0,0,0,0\} & \max\{0,0.3,0,0.1\} & \max\{0,0,0,0.5\} \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0 & 0.3 & 0.5 \end{bmatrix}$$

So this verifies that $Q^{-1} = P^{-1} \circ R^{-1}$

In the last class I have given you two matrix $R(X,Y)$ and $P(Y,Z)$

$$R(X,Y)$$

| | | | |
|---|---|---|---|
| $B$ | 0.2 | 0.8 | 1 |
| $D$ | 1 | 0.9 | 0.2 |
| $E$ | 0.5 | 0.9 | 0 |
| $P$ | 0.2 | 0.3 | 1 |
| | $S$ | $C$ | $L$ |

$$P(Y, Z)$$

$$\begin{array}{c|ccc} S & 0.3 & 1 & 0.1 \\ C & 0.2 & 0.9 & 0.7 \\ L & 0.8 & 0.1 & 0.3 \\ \hline & F & A & M \end{array}$$

I suggest that with respect to these two matrices you verify that

$$Q^{-1} = P^{-1} \circ R^{-1}$$

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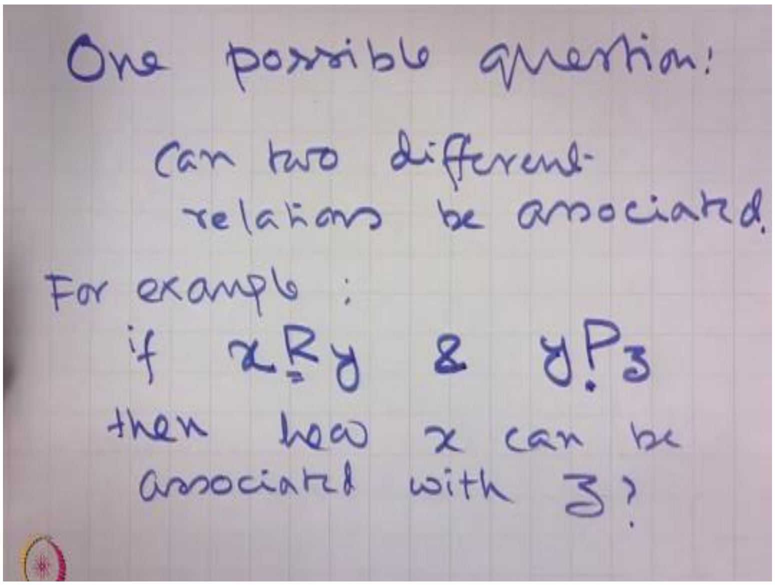


One possible question:

Can two different relations be associated?

For example, if $xRy$ and $yPz$ then how $x$ can be associated with $z$.

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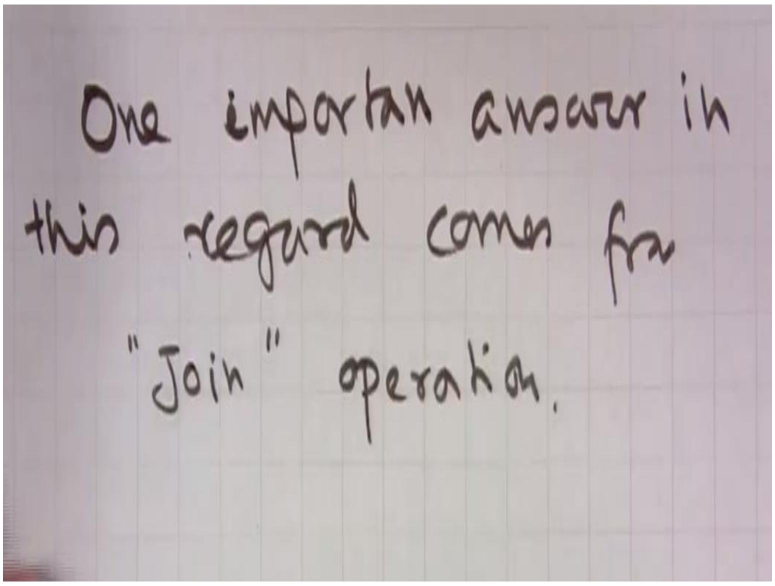


One important answer in this regard comes from '$join$' operation.

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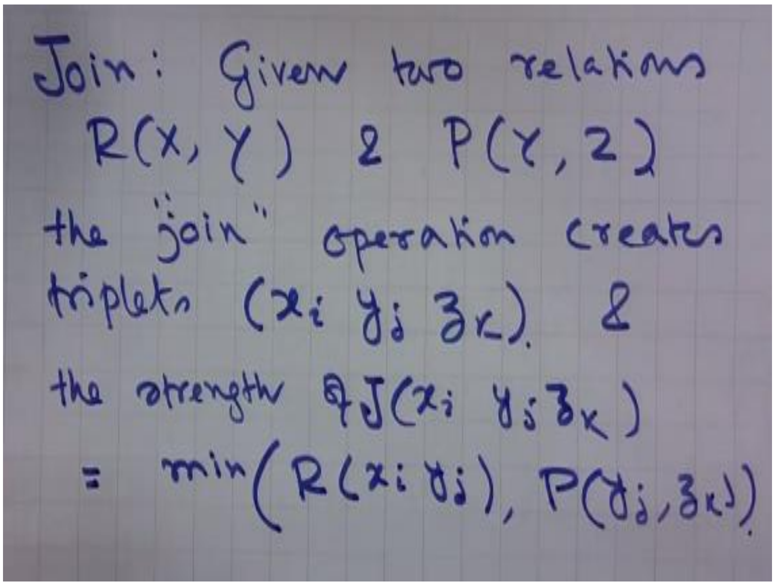
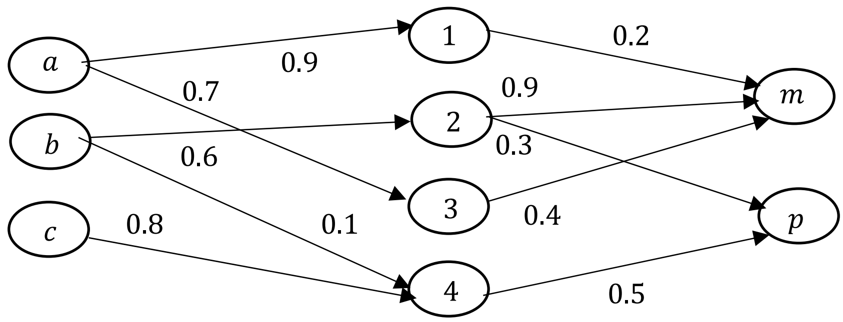



Given two relations $R(X, Y)$ and $P(Y, Z)$ the $join$ operation creates triplets $\left(x_i, y_j, z_k\right)$ and the strength of $J\left(x_i y_j, z_k\right) = \min\left(R\left(x_i, y_j\right), P\left(y_j, z_k\right)\right)$

In order to demonstrate that let us go back to the *Sagittal Diagram* that I have shown some time back. From here we can easily get the join of $R(X, Y)$ and $P(Y, Z)$.




The advantage of Sagittal Diagram is that we do not draw the arrows with zero strength. Therefore, it is very clear which triplets are related

For example:

$join$ for $(a, 1, m)$ is going to be minimum of 0.9 and 0.2 that is 0.2.

Similarly, another possible relation is $(a, 3, m)$ and its value is going to be minimum of 0.7 and 0.4 which is going to be 0.4.

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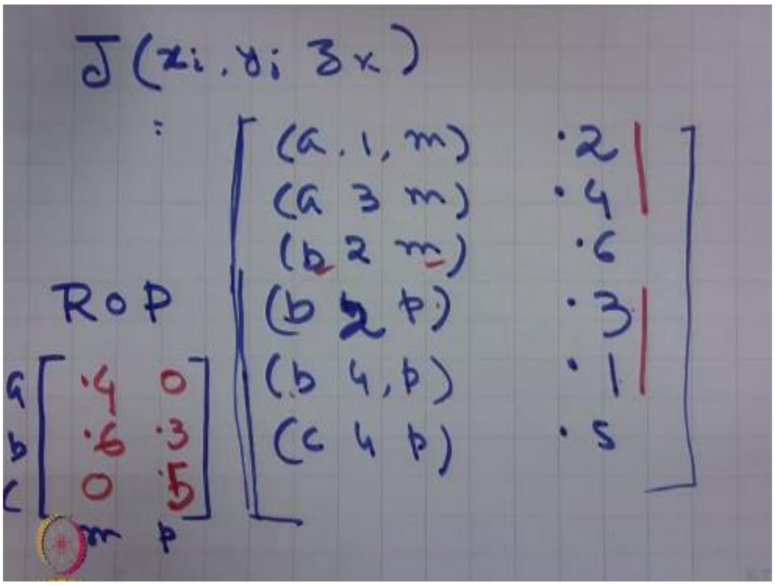


$$\therefore J(x_i, y_j, z_k) = \begin{bmatrix} (a, 1, m) & 0.2 \\ (a, 3, m) & 0.4 \\ (b, 2, m) & 0.6 \\ (b, 2, p) & 0.3 \\ (b, 4, p) & 0.1 \\ (c, 4, p) & 0.5 \end{bmatrix}$$

So this gives us the $join$ operation between $X, Y$ and $Z$ corresponding to each triplet we get the corresponding strength.

The question is:

Are composition and $join$ related in some way?

Because in composition we are looking at only the $X$ and Z but in join we are looking at all the three $X, Y$ and $Z$. So, to get the composition we can use the join matrix.

$$R \circ P(a, m) = \max\big(J(a, 1, m), J(a, 3, m)\big) = 0.4$$

$$R \circ P(b, m) = J(b, 2, m) = 0.6$$

$$R \circ P(b, p) = \max\big(J(b, 2, p), J(b, 4, p)\big) = 0.3$$

$$R \circ P(c, p) = J(c, 4, p) = 0.5$$

$$R \circ P = \begin{bmatrix} 0.4 & 0 \\ 0.6 & 0.3 \\ 0 & 0.5 \end{bmatrix}$$

So $join$ and composition are not completely independent we can easily get the join operation from the composition as well.

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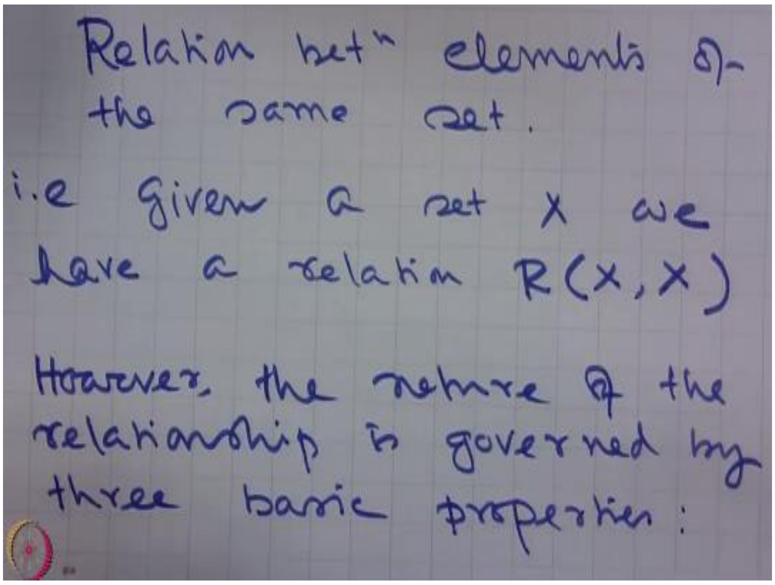


With that background let us look at a special type of relations.

Relation between elements of the same set that is, given a set $X$ we have a relation $R(X, X)$.

We have already seen such an example in our last class when I was talking about the distance relation between cities of India and then we have along the columns the set of cities along the rows the same set of cities and the $i, j^{th}$ element is looking at a measure of the distant property between the cities $x_i$ and $y_j$.

Like that many different relations can be defined between the elements of the same set. However, the nature of the relationship is governed by 3 basic properties.

What are these?

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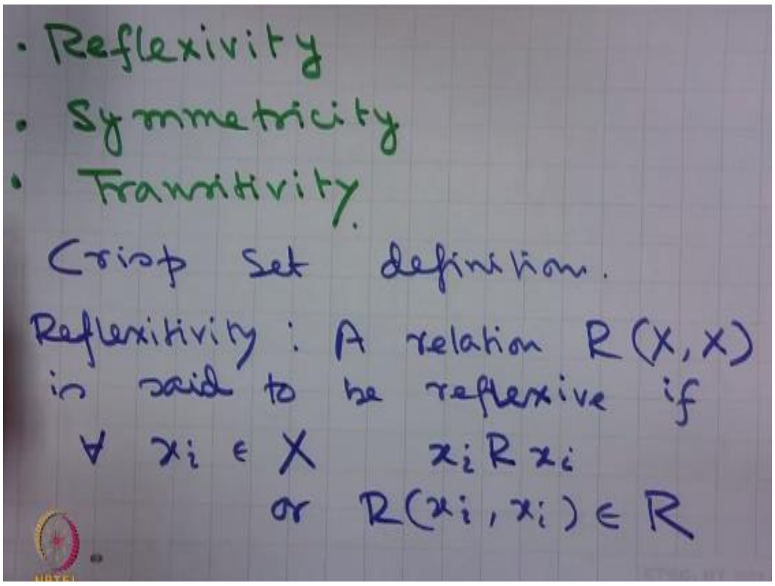


- Reflexivity
- Symmetricity
- Transitivity

Let me first define this with respect to crisp sets.

- Reflexivity: A relation $R(X,X)$ is said to be reflexive if for all $x_i \in X$, $x_iRx_i$ or $R(x_i,x_i) \in R$

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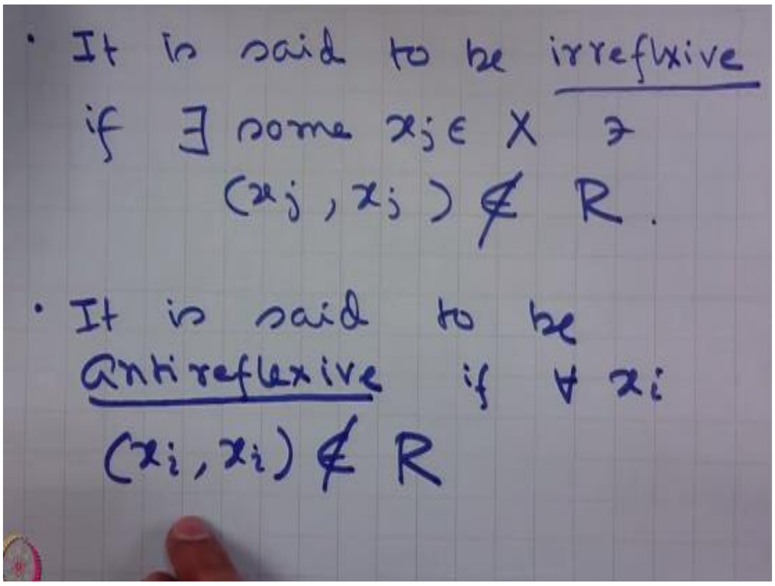

- Irreflexive: A relation $R(X,X)$ is said to be irreflexive if there exists some $x_j \in X$ such that $(x_j, x_j) \notin R$

That means if there exists at least one element such that it is not related to itself then we call the relationship irreflexive.

- Anti-reflexive: A relation $R(X,X)$ is said to be anti-reflexive if for all $x_i$, $(x_i, x_i) \notin R$

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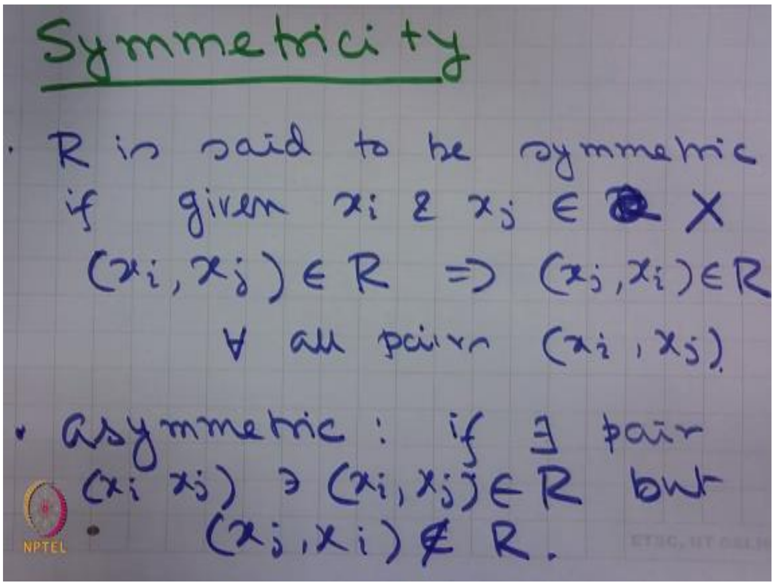


Now let us look at symmetricity

- Symmetric: A relation $R(X,X)$ is said to be symmetric if given $x_i, x_j \in X$, $(x_i, x_j) \in R \Rightarrow (x_j, x_i) \in R \quad \forall$ pairs $(x_i, x_j)$.

Quite naturally if a relation is symmetric then corresponding relation matrix is going to be a symmetric matrix.

- Asymmetric: A relation $R(X,X)$ is called asymmetric if there exists $(x_i, x_j)$ such that $(x_i, x_j) \in R$, but $(x_j, x_i) \notin R$.

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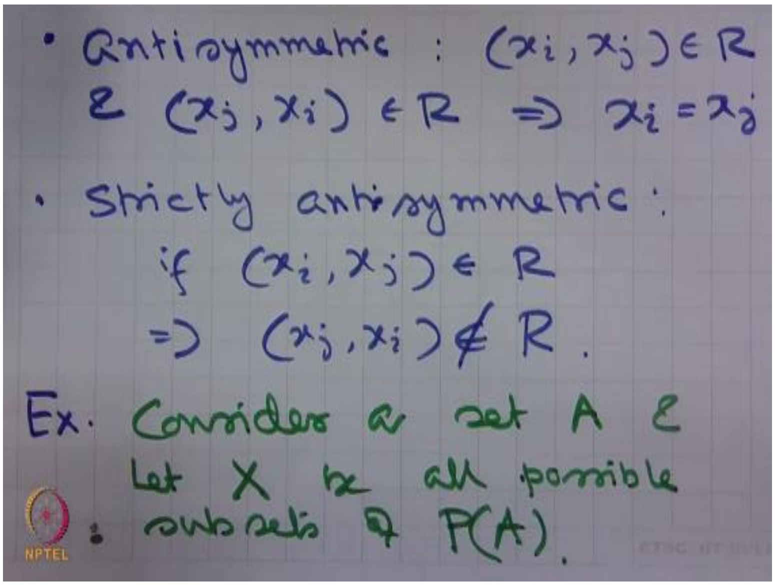


- Anti-symmetric: A relation $R(X, X)$ is said to be antisymmetric if $(x_i, x_j) \in R$ and $(x_j, x_i) \in R \Rightarrow x_i = x_j$
- Strictly Anti-symmetric: A relation $R(X, X)$ is called strictly anti-symmetric if $(x_i, x_j) \in R \Rightarrow (x_j, x_i) \notin R$

Let us give an example:

Consider a set $A$ and let $X$ be all possible subsets of $P(A)$

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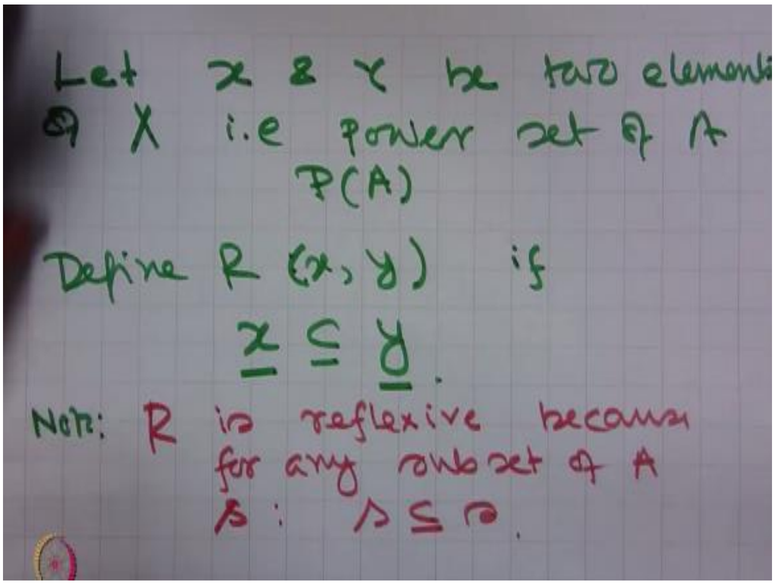


Let $x$ and $y$ be two elements of $X$ that is power set of $A$ which we denote as $P(A)$.

Define $R(x, y)$, if $x \subseteq y$

That means $x \subseteq A$, $y \subseteq A$ and we are saying if $x \subseteq y$ then $x$ is related to $y$.

Note: $R$ is reflexive because for any subset($\mathscr{s}$) of $A$,

$$\mathscr{s} \subseteq \mathscr{s}$$

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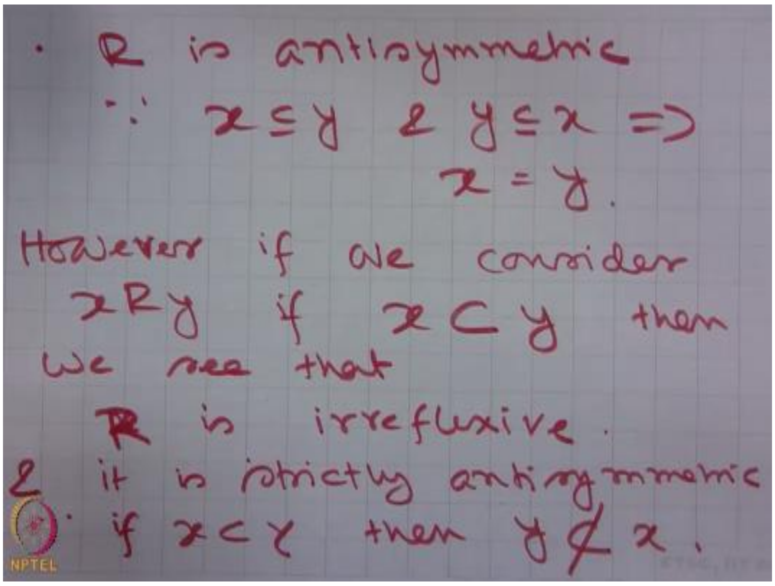


$R$ is anti-symmetric.

$\because x \subseteq y$ and $y \subseteq x \Rightarrow x = y$

However, if we consider $xRy$ if $x \subset y$ then we see that

- $R$ is irreflexive, because no set can be strictly contained in itself
- $R$ is strictly anti-symmetric $\because$ if $x \subset y$ then $y \not\subset x$

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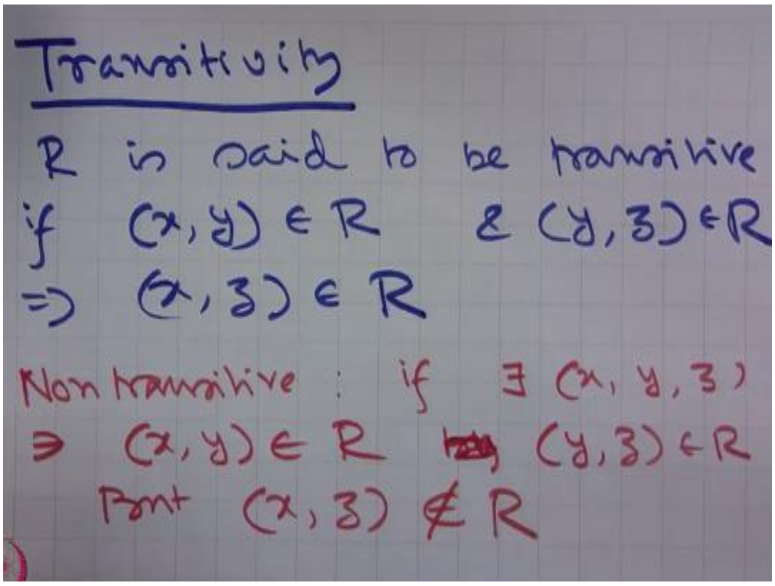


The third property is called transitivity.

- Transitive: A relation $R(X, X)$ is said to be transitive if $(x, y) \in R$ and $(y, z) \in R$

$$\Rightarrow (x,z) \in R$$

- Non-transitive: A relation $R(X,X)$ is called non-transitive if there exists $x, y, z \in X$ such that $(x,y) \in R$ and $(y,z) \in R$, but $(x,z) \notin R$.

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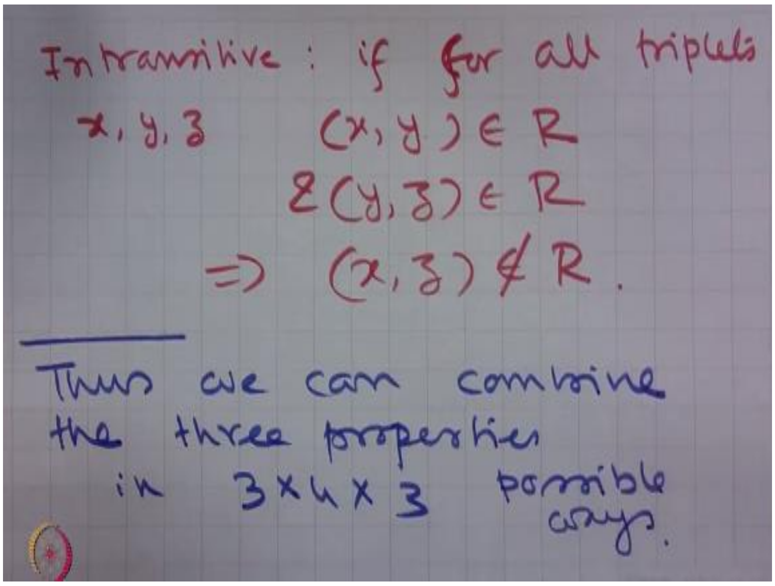


- Non-transitive: A relation $R(X,X)$ is called intransitive. If for all triplets $x, y, z$; $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \notin R$

So we have got reflexivity there are 3 possibilities there, we have seen symmetricity there are 4 possibilities and we have transitivity there also we have 3 possibilities.

Thus, we can combine the three properties in $3 \times 4 \times 3$ possible ways depending upon which particular combination we choose we can define a different type of relation on the same set $X$.

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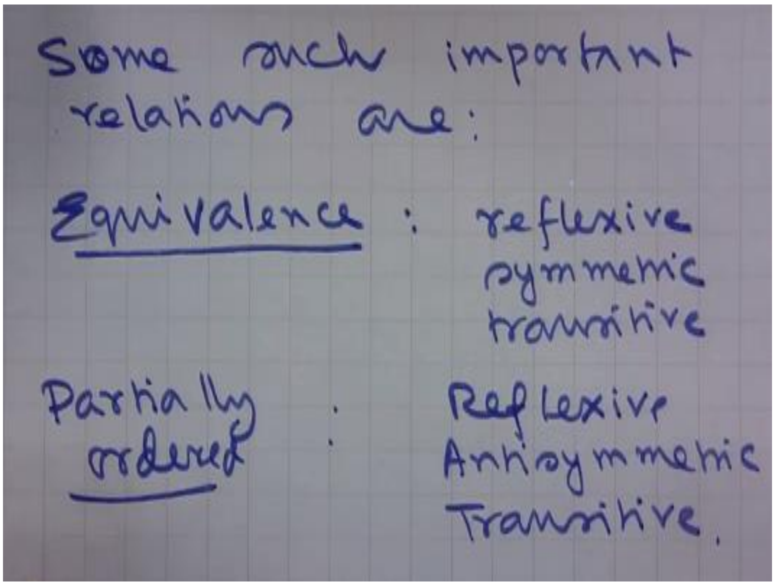

Some such important relations are equivalence relation which is reflexive, symmetric and transitive.

Another one is partially ordered which is reflexive, anti-symmetric and transitive.

Okay students I stopped here today. In the next class I shall start with equivalence relation and then also we will show how the crisp relationship can be extended to fuzzy sets by slightly changing the definitions of reflexivity, symmetricity and transitivity. Thank you so much.