Introduction to Fuzzy Sets Arithmetic and Logic
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Lecture - 18
Fuzzy Sets Arithmetic and Logic

Welcome students to the MOOCs course on introduction to Fuzzy Sets Arithmetic and Logic. This is lecture number 18 and as I said towards the end of the last class that in this lecture we shall start studying fuzzy relations.

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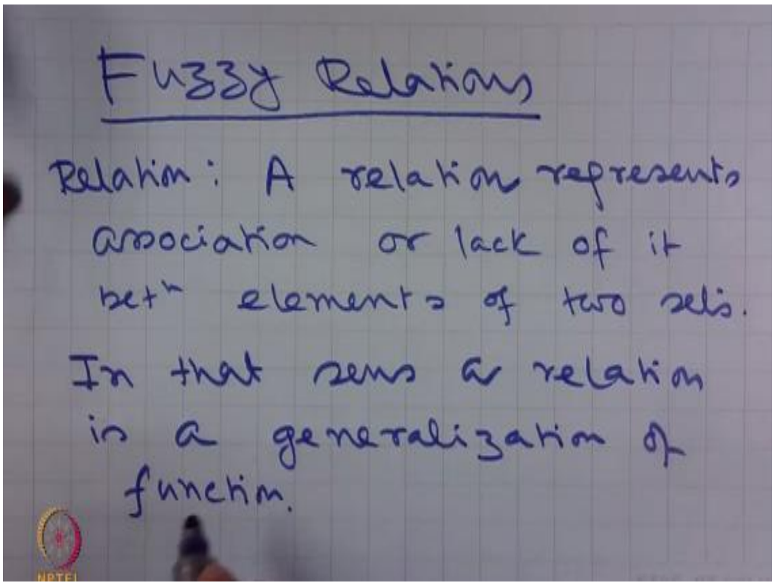

What is the relation?

A relation represents association or lack of it between elements of two sets.

In that sense a relation is a generalization of function.

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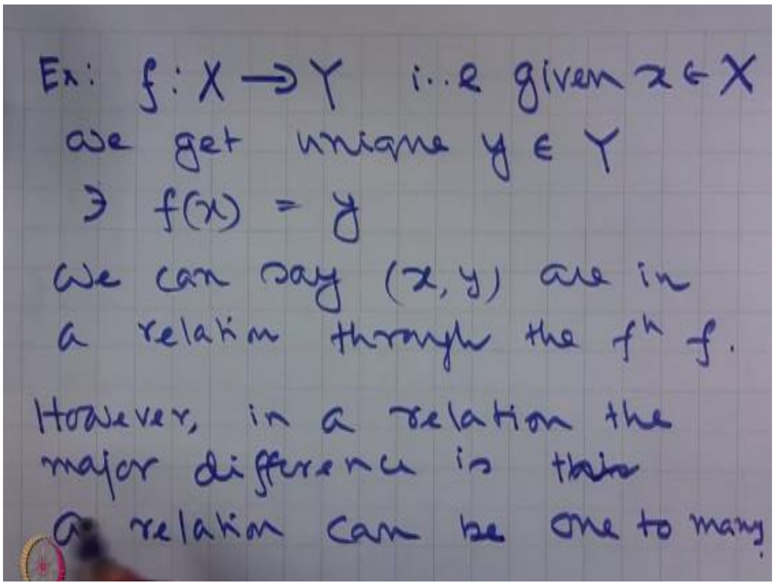

So let me give you an example:

If we define a function $f: X \to Y$ that is, given $x \in X$ we get unique $y \in Y$ such that $f(x) = y$

So we can say (x, y) are related or are in a relation through the function f.

However, in a relation the major difference is that, a relation can be one-to-many.

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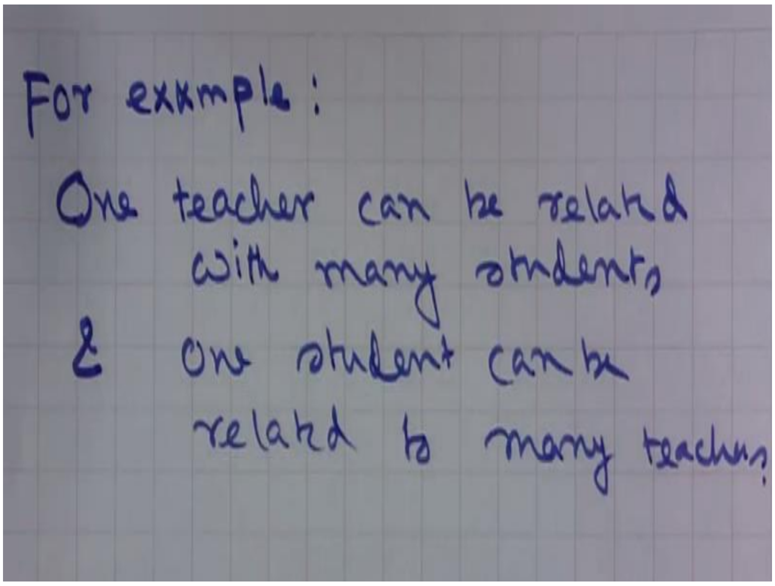

So for example:

One teacher can be related with many students, but, and one student can be related to many teachers. Thus, in both directions we can have one-to-many relationship.

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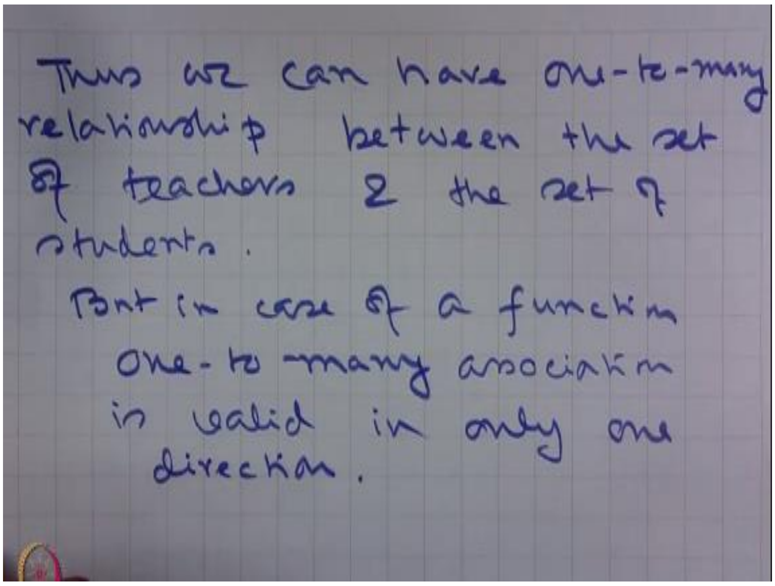

Thus, we can have one-to-many relationship between the set of teachers and the set of students. But, in case of a function one-to-many association is valid in only one direction.

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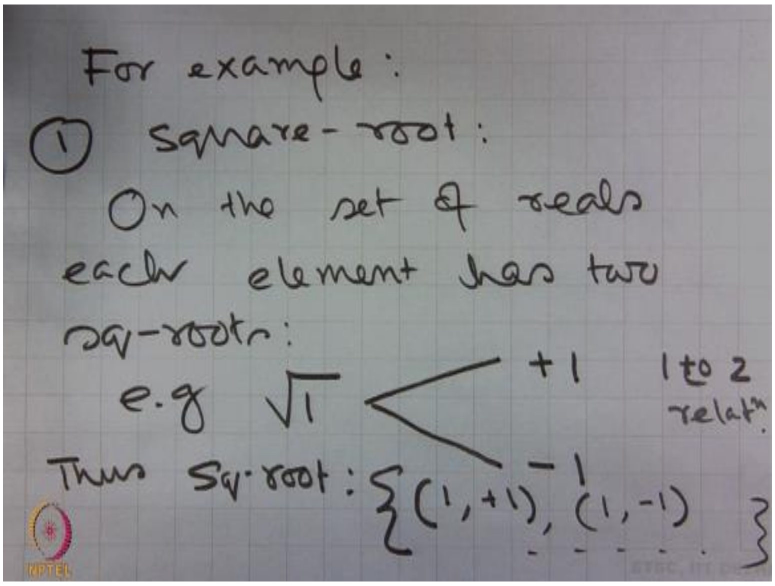

For example:

1. Consider square root.

On the set of reals each element has two square roots.

For example:

$$\sqrt{1} = \begin{cases} +1 \\ -1 \end{cases}$$

Thus if square root is the relation then its members are $\{(1,+1),(1,-1)\ldots\ldots\}$ for all other reals that we are considering.

But for each of $+1$ and -1 the relation in the reverse direction is only with 1.

Thus, it is a one-to-two relation which is a part of one-to-many relation.

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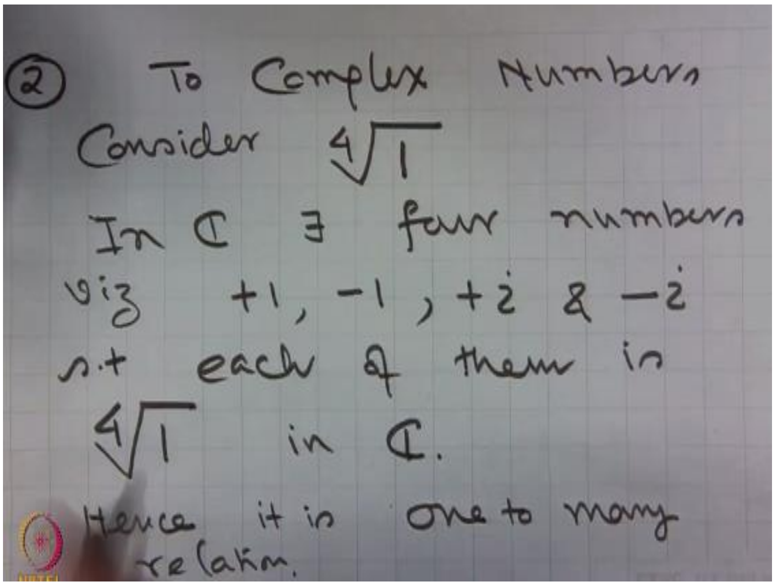

So let me give you an example,

2. Let us extend it to complex numbers.

Consider $\sqrt[4]{1}$

We know that in $\mathbb{C}$ there exists four numbers namely

$$+1, -1, +i, -i$$

such that each of them is $\sqrt[4]{1}$ in $\mathbb{C}$.

Again as before for each one of them if we raise them to the power 4 we get only 1. Hence it is one-to-many relations.

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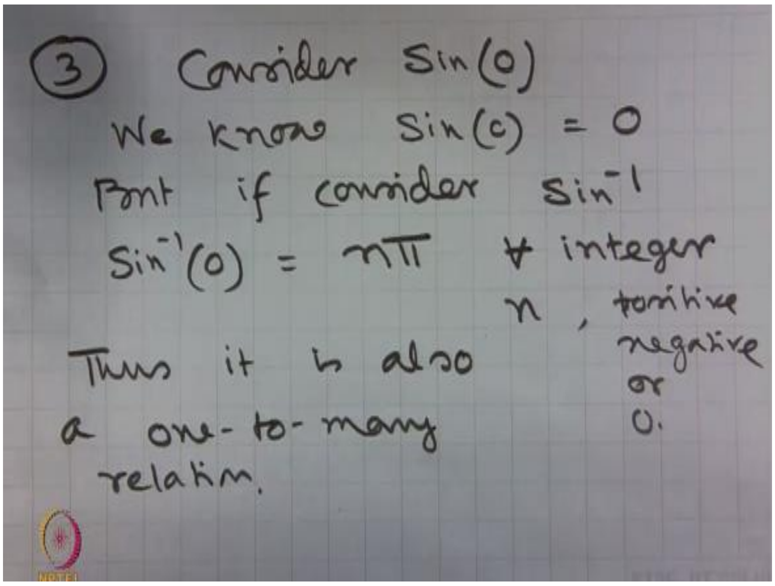

3. In a similar way, consider $\sin(0)$

We know $\sin(0) = 0$

But if we consider $\sin^{-1}(0) = n\pi$ for all integer $n \in \mathbb{Z}$

Thus, it is also a one-to-many relation.

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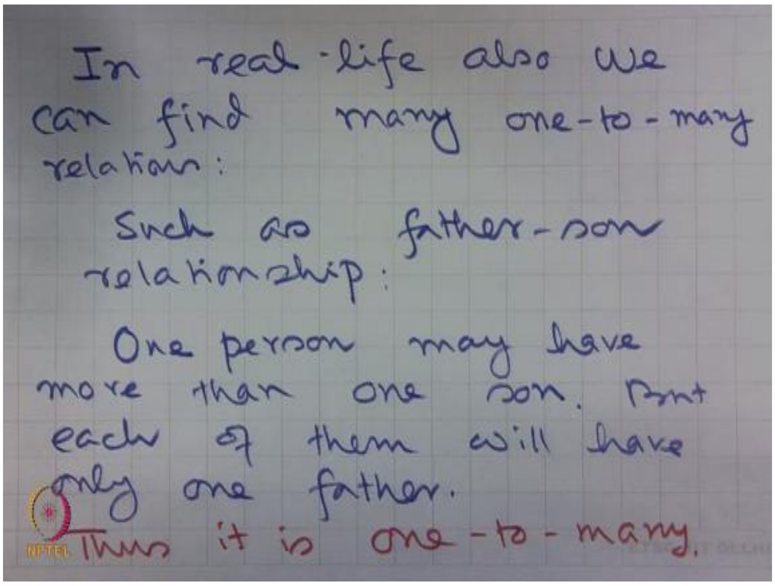

In real life also we can find many one-to-many relations such as father-son relationship.

One person may have more than one son, but each of them will have only one father. Thus, it is one-to-many.

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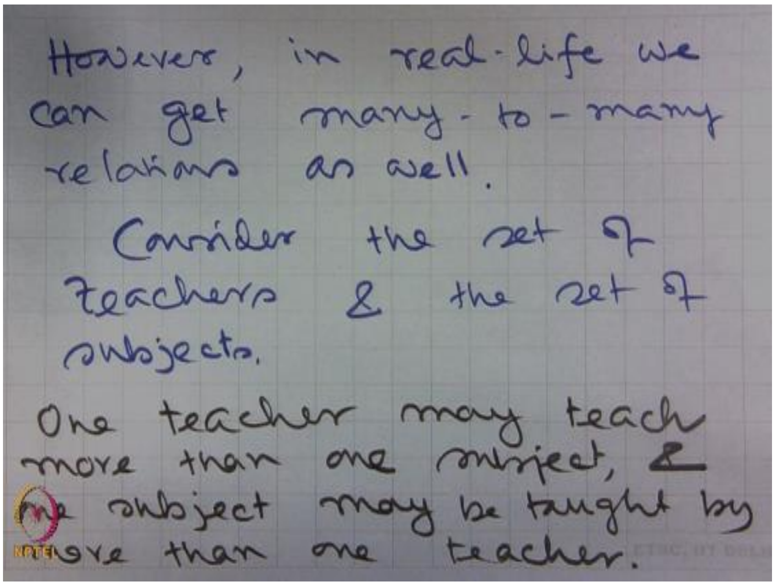

However, in real life we can get many-to-many relations as well.

Just similar to what I did earlier consider the set of teachers and the set of subjects.

One teacher may teach more than one subject and one subject may be taught by more than one teacher.

Thus, it is a many-to-many relation.

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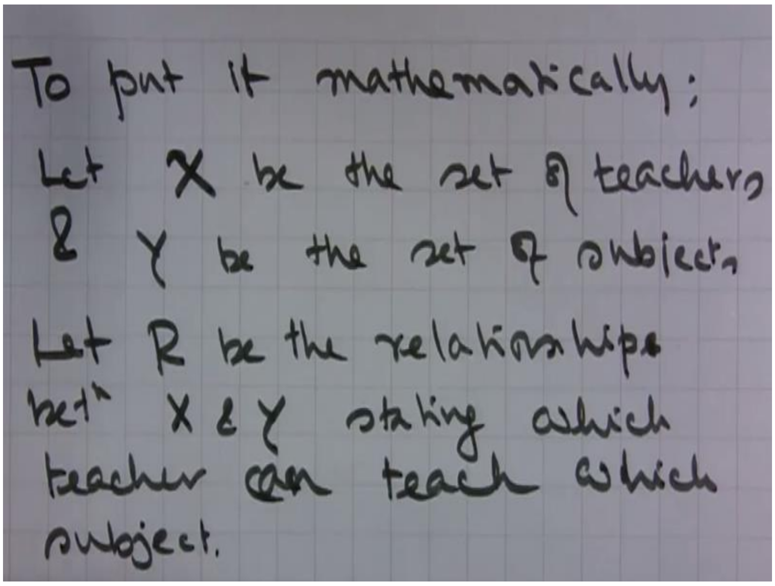

To put it mathematically:

Let X be the set of teachers and Y be the set of subjects.

Let R be the relationship between X and Y stating which teacher can teach which subject.

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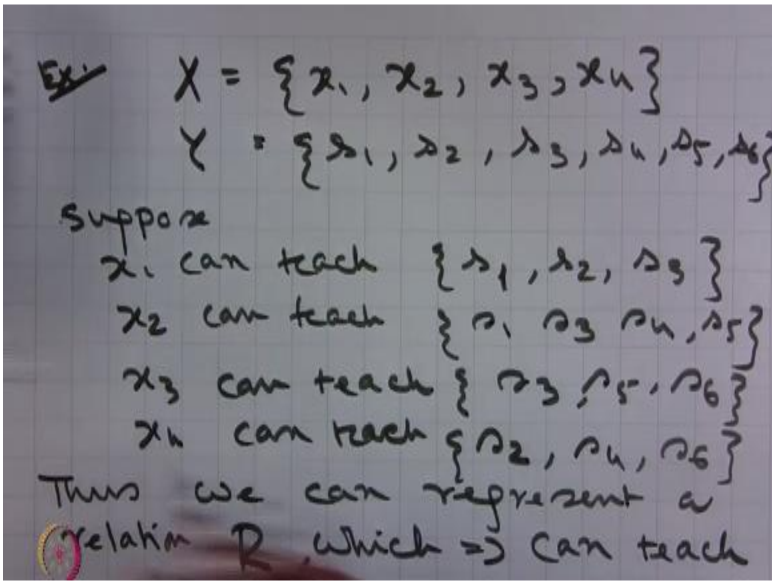

So let me give you an example:

Let X be the teachers $\{x_1, x_2, \ldots, x_4\}$ and Y be the subjects $\{s_1, s_2 \ldots, s_6\}$

Suppose:

x_1 can teach $\{s_1, s_2, s_3\}$

x_2 can teach $\{s_1, s_3, s_4, s_5\}$

x_3 can teach $\{s_3, s_5, s_6\}$

x_4 can teach $\{s_2, s_4, s_6\}$

Thus, we can represent a relation say R which may mean, '$can\ teach$' with the help of pairs.

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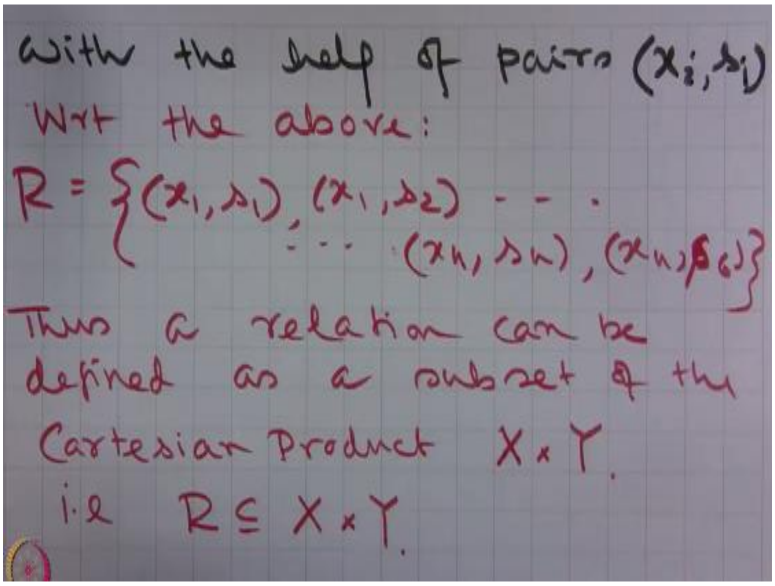

The pairs are of the form (x_i, s_j).

So with respect to the above R can be the pairs

$R = \{(x_1, s_1), (x_1, s_2), \dots (x_4, s_5), (x_4, s_6)\}$.

Thus, a relation can be defined as a subset of the Cartesian Product $X \times Y$ that is

$$R \subseteq X \times Y$$

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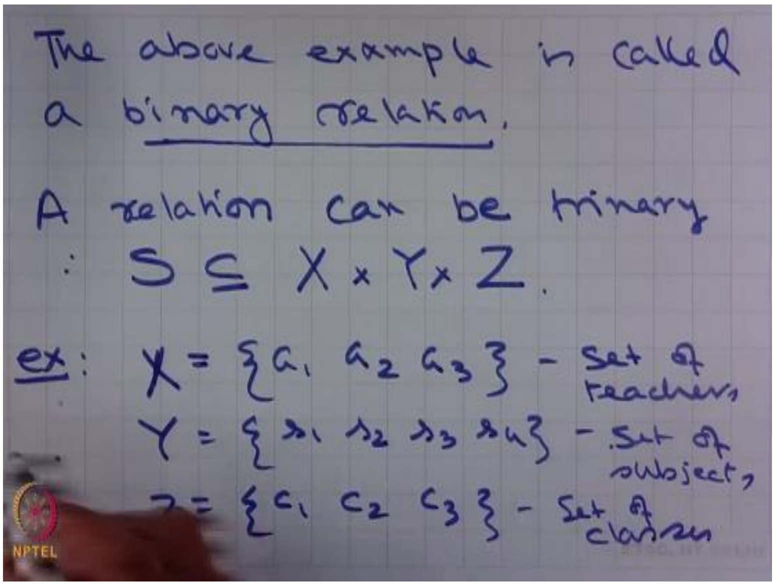

The above example is called a binary relation.

Why it is called a binary?

Because you are associating two sets in general, in a relation we may associate any number of sets.

So a relation can be trinary, that means the relation if we call it S, it may be contained in the Cartesian product of three sets.

$$S \subseteq X \times Y \times Z$$

For example, let X be set of teachers, Y be set of subjects and Z be set of classes such that

$$X = \{a_1, a_2, a_3\}$$

$$Y = \{s_1, s_2, s_3, s_4\}$$

$$Z = \{c_1, c_2, c_3\}$$

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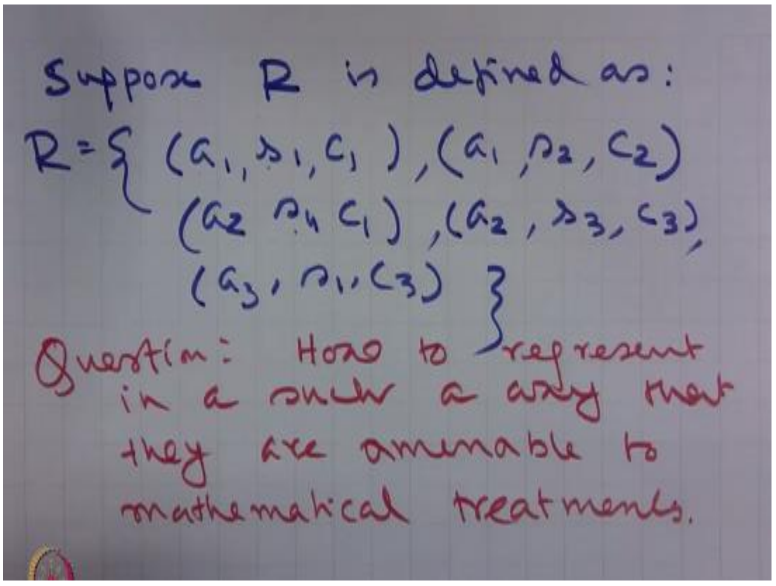

And suppose we define a relation R is the set of three tuples

$$R = \{(a_1, s_1, c_1), (a_1, s_2, c_2), (a_2, s_4, c_1), (a_2, s_3, c_3), (a_3, s_1, c_3)\}$$

Here, each (a_i, s_j, c_k) means that teacher a_i can teach subject s_j for the class c_k

Question is how to represent in such a way that they are amenable to mathematical treatments?

So there are many ways of representing a relation.

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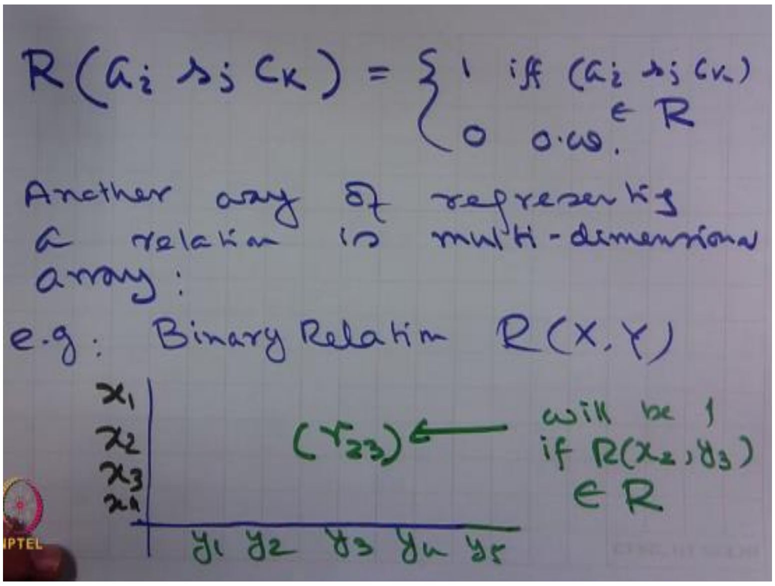

So with respect to this theory relation can be represented as:

$$R(a_i, s_j, c_k) = \begin{cases} 1 & (a_i, s_j, c_k) \in R \\ 0 & \text{otherwise} \end{cases}$$

Another way of representing a relation is multi-dimensional array.

For example, Consider the binary relation $R(X, Y)$

So if we have

x_1	$r_{1,1}$	$r_{1,2}$	$r_{1,3}$	$r_{1,4}$	$r_{1,5}$
x_2	$r_{2,1}$	$r_{2,2}$	$r_{2,3}$	$r_{2,4}$	$r_{2,5}$
x_3	$r_{3,1}$	$r_{3,2}$	$r_{3,3}$	$r_{3,4}$	$r_{3,5}$
x_4	$r_{4,1}$	$r_{4,2}$	$r_{4,3}$	$r_{4,4}$	$r_{4,5}$
	y_1	y_2	y_3	y_4	y_5

Here, each $r_{i,j}$ will be 1 if $R(x_i, y_j) \in R$

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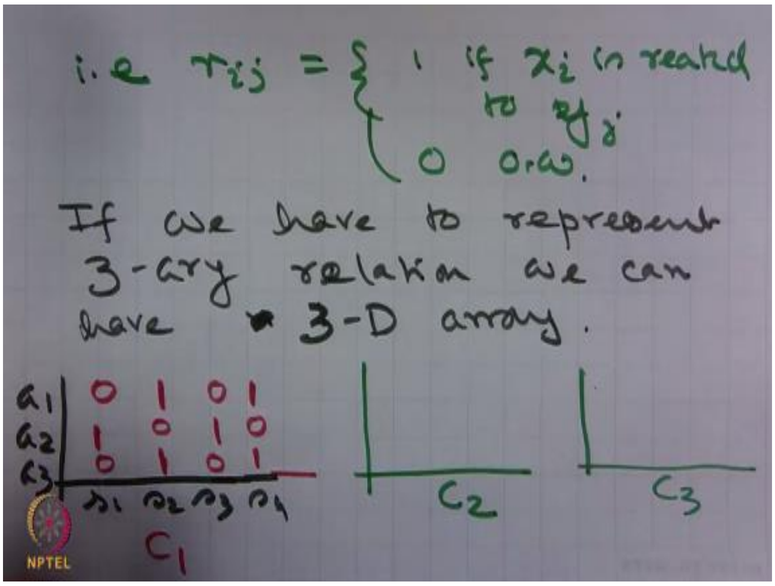


Or in other words,

$$r_{i,j} = \begin{cases} 1 & \text{if } x_i \text{ is related to } y_j \\ 0 & \text{otherwise} \end{cases}$$

So for a binary relation we can get a matrix.

If we have to represent 3-ary relations we can have 3-D array

So for example:

a_1	0	1	0	1
a_2	1	0	1	0
a_3	0	1	0	1
	s_1	s_2	s_3	s_4

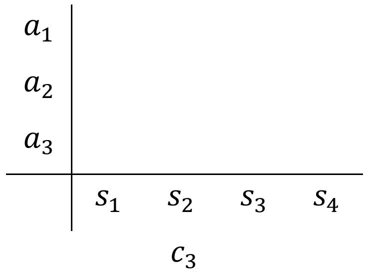
c_1

In a similar way we can get another 2-D matrix for c_2

a_1				
a_2				
a_3				
	s_1	s_2	s_3	s_4

c_2

And another 2-D matrix for c_3.



Now we can represent this information in a 3 dimensional array.

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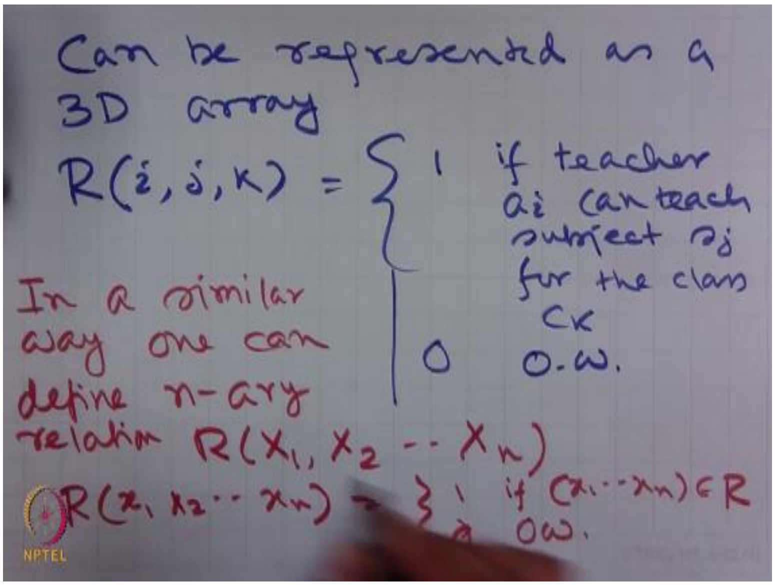

So, it can be represented as a 3-D array and

$$R(i,j,k) = \begin{cases} 1 & \text{if teacher } a_i \text{ can teach subject } s_j \text{ for the class } c_k \\ 0 & \text{otherwise} \end{cases}$$

In a similar way, one can define n-ary relation $R(X_1, X_2, \dots, X_n)$

$$R(x_1, x_2, \dots x_n) = \begin{cases} 1 & (x_1, \dots\ x_n) \in R \\ 0 & \text{otherwise} \end{cases}$$

And each $x_i \in X_i$

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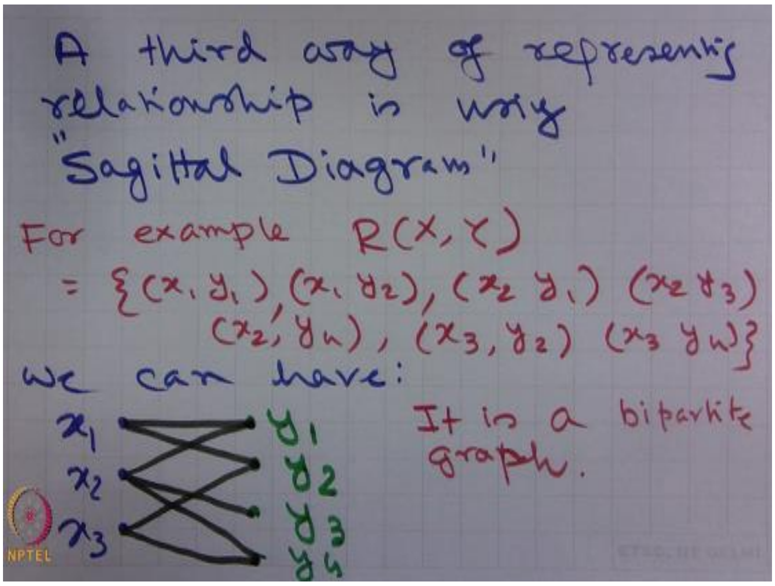
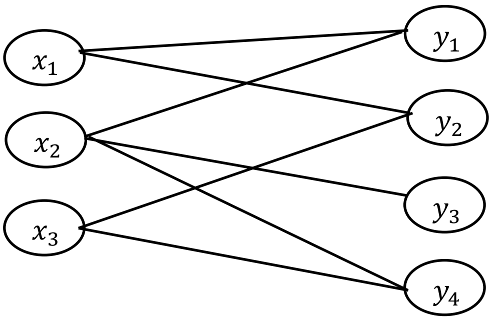


A third way of representing relationship is using a $Sagittal\ Diagram$

For example:

$$R(X, Y) = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_3), (x_2, y_4), (x_3, y_2), (x_3, y_4)\}$$

So it is a bipartite graph,



But it gives us a complete picture of which x_i is related with which y_j.

So in my discussions depending upon the necessity I shall change the representation to solve certain problems.

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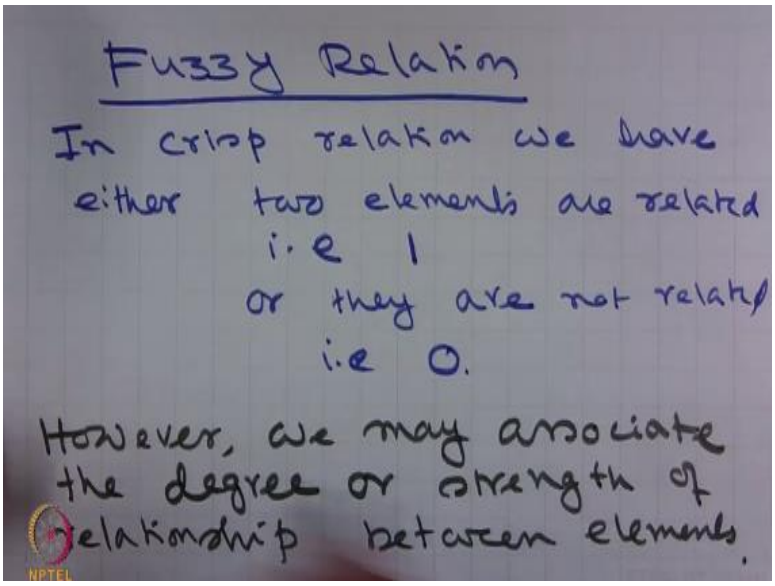

Now what is a Fuzzy Relation?

In crisp relation, we have either two elements are related that is 1 or they are not related that is 0.

However, we may associate the degree or strength of relationship between elements and that gives the concept of fuzzy relation.

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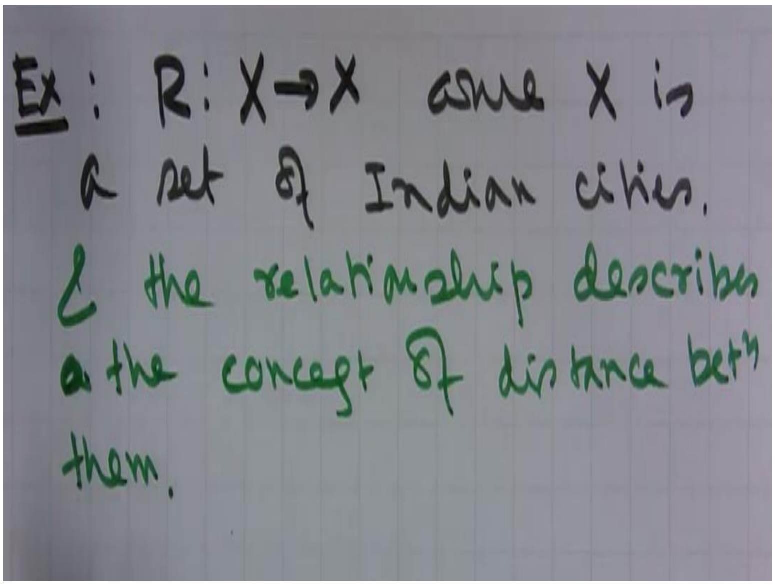

So let me give you an example.

Suppose I am looking at a relation $R: X \rightarrow X$ where X is a set of Indian cities and the relationship describes the concept of distance between them.

So if two cities are considered to be very distant or very far from each other then they will have a stronger relationship. However, if they are close that degree of their relationship will diminish.

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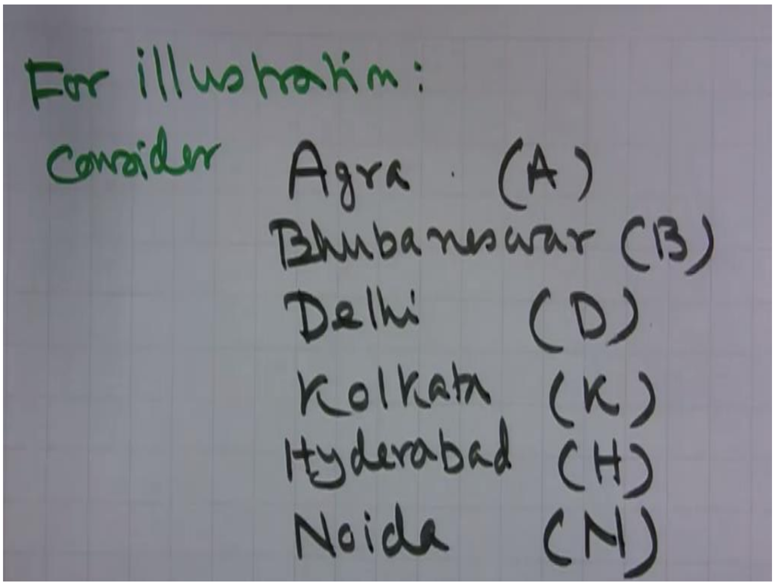

So for illustration: Consider

- Agra (A)
- Bhubaneswar (B)
- Delhi (D)
- Kolkata (K)
- Hyderabad (H)
- Noida (N)

Suppose we want to represent our intuitive idea about the distance between them by using fuzzy relation.

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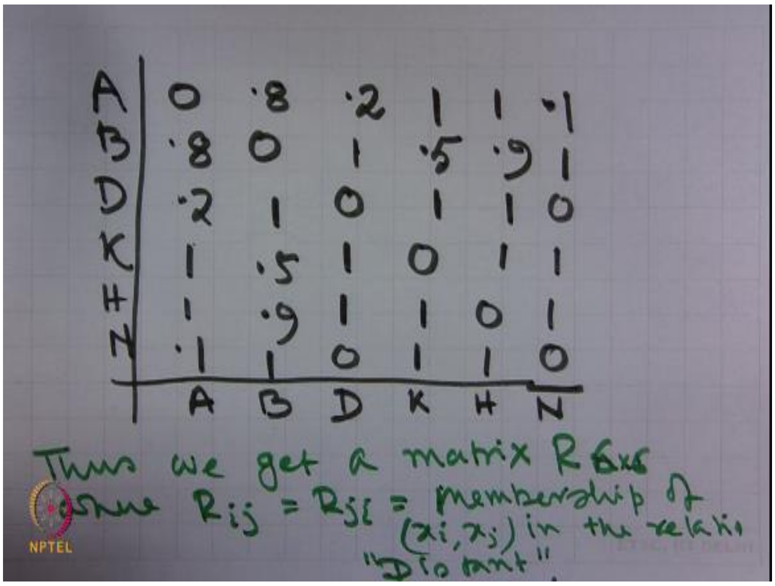

So we can have a matrix like the following

A	0	0.8	0.2	1	1	0.1
B	0.8	0	1	0.5	0.9	1
D	0.2	1	0	1	1	0
K	1	0.5	1	0	1	1
H	1	0.9	1	1	0	1
N	0.1	1	0	1	1	0
	A	B	D	K	H	N

Thus we get a symmetric matrix $R_{6\times6}$ where

$R_{i,j} = R_{j,i} =$ membership of the cities x_i and x_j in the relation say 'distant'.

Okay so that gives us the basic concept of fuzzy relation where in with respect to crisp relations we have used only 0 and 1 denoting the absence or presence of the relationship.

Here we are generalizing it and we are associating with each pair (x_i, x_j) membership value between 0 to 1 which gives the degree of association or the strength of the association between the elements x_i and y_j.

So with that background now I will study different types of relationships that one can handle or one faces while solving practical problems.

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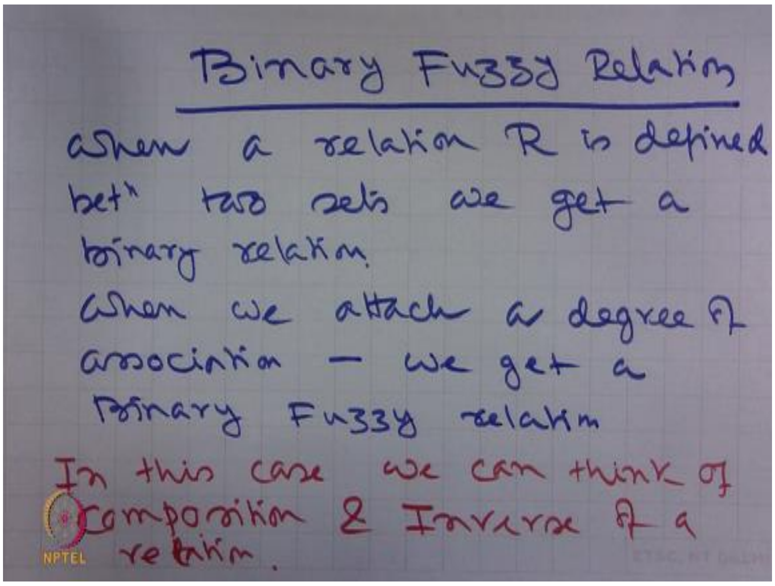

The simplest of them is binary fuzzy relation.

So as you know when a relation R is defined between two sets we get a binary relation.

When we attach a degree of association we get a Binary Fuzzy Relation.

And in this case we can think of composition and inverse of a relation.

As I said a relation is a more generalization of a function therefore we can think of extending the concept of inverse and the concept of composition from the domain of functions to the domain of relations.

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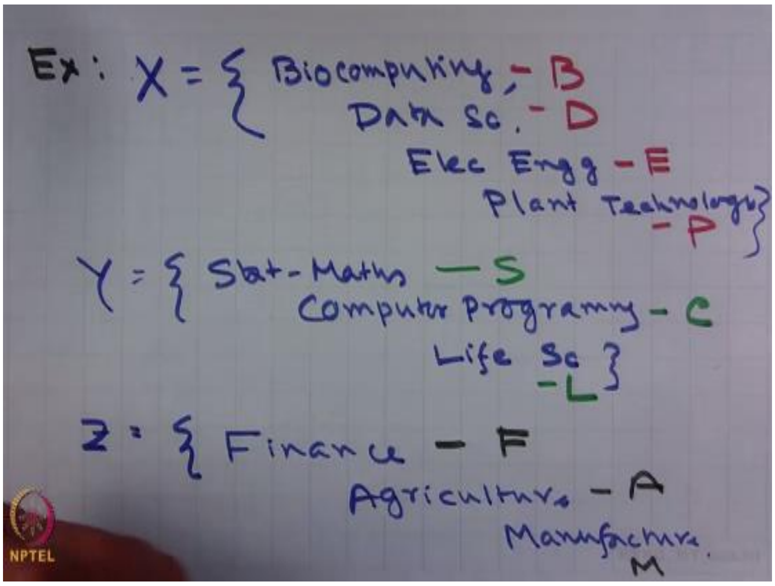

So for example:

Suppose we have courses

$$X = \{\text{Biocomputing } (B), \text{Data Science } (D), \text{Electrical Engineering } (E),$$
$$\text{Plant Technology } (P)\}$$

Let Y denote subjects

$$Y = \{\text{ Statistics and Mathematics } (S), \text{Computer Programming } (C), \text{Life Science } (L)\}$$

And Z be job profiles

$$Z = \{\text{Finance } (F), \text{Agriculture } (A), \text{Manufacture } (M)\}$$

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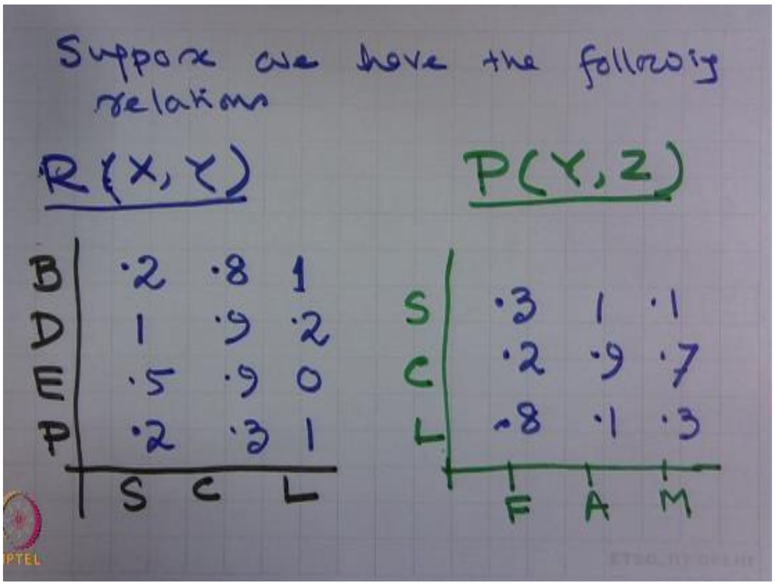

And suppose we have the following relations.

First relation is between X, Y, that is academic program and subject. Suppose the relation R denotes the importance of the subject for this program.

$R(X,Y)$

B	0.2	0.8	1
D	1	0.9	0.2
E	0.5	0.9	0
P	0.2	0.3	1
	S	C	L

So this gives a degree or a measure of our intuitive idea of the importance of a subject in a program.

In a similar way we have another relationship which is P which is between Y and Z that means the relationship between subjects and different job sectors. How important is the subject for a particular job sector.

$P(Y,Z)$

S	0.3	1	0.1
C	0.2	0.9	0.7
L	0.8	0.1	0.3
	F	A	M

Thus we have two different binary relationships.

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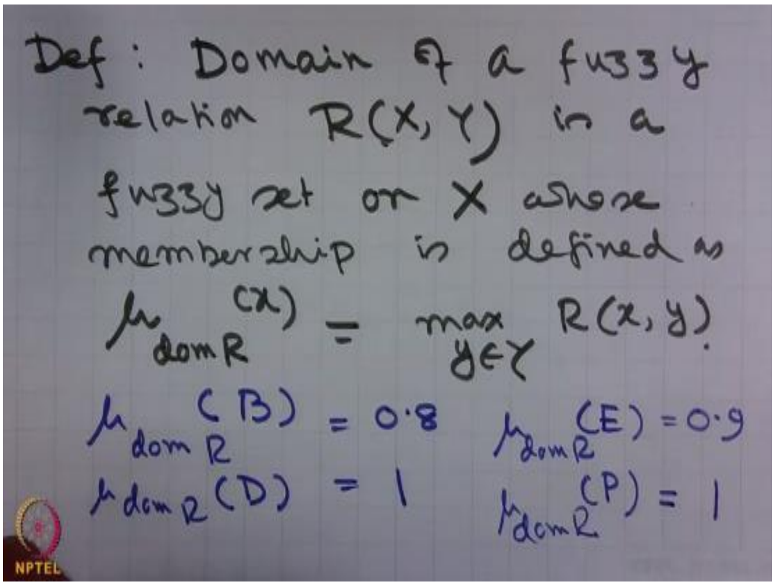

So let us go for some definition:

Domain of a fuzzy relation $R(X, Y)$ is a fuzzy set on X whose membership is defined as

$$\mu_{domR}(x) = \max_{y \in Y} R(x, y)$$

[Note that we are working on binary fuzzy relation so we have just two sets]

So illustration:

$R(X, Y)$

	S	C	L
B	0.2	0.8	1
D	1	0.9	0.2
E	0.5	0.9	0
P	0.2	0.3	1

Therefore, we can write that:

$$\mu_{domR}(B) = 0.8$$

$$\mu_{domR}(D) = 1$$

$$\mu_{domR}(E) = 0.9$$

$$\mu_{domR}(P) = 1$$

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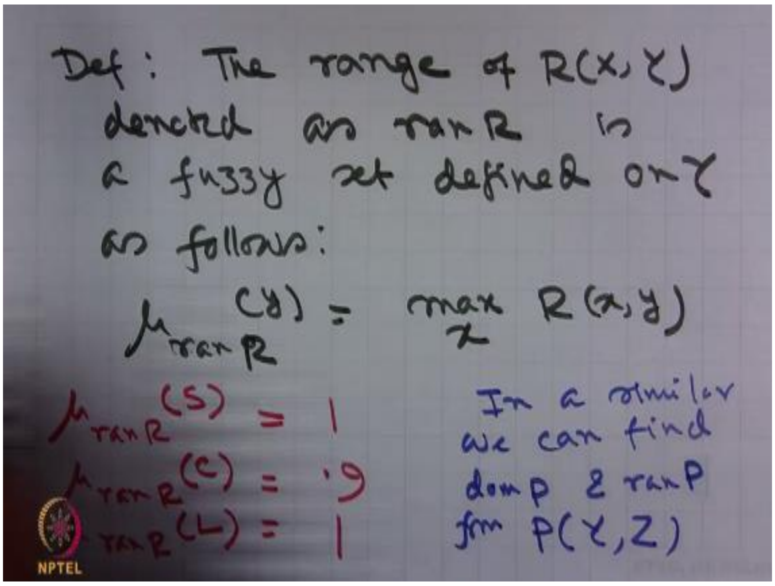

Another important definition

The Range of $R(X, Y)$ are denoted as $ranR$ is a fuzzy set defined on Y as follows:

$$\mu_{ranR}(y) = \max_{x} R(x, y)$$

That means for each y we will look at the strength of its relationship with different values of x and we shall choose the maximum of them.

So illustration:

$R(X, Y)$

	S	C	L
B	0.2	0.8	1
D	1	0.9	0.2
E	0.5	0.9	0
P	0.2	0.3	1

Therefore, we can write that:

$$\mu_{ranR}(S) = 1$$

$$\mu_{ranR}(C) = 0.9$$

$$\mu_{domR}(L) = 1$$

In a similar way we can find $domP$ and $ranP$ from $P(Y, Z)$

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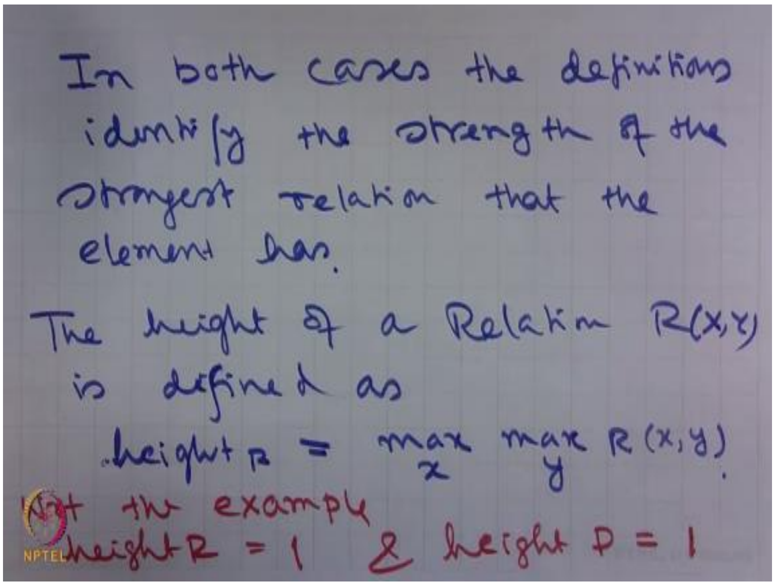

So, in both cases the definitions identify the strength of the strongest relation that the element has.

The height of a relation $R(X, Y)$ is defined as

$$height(R) = \max_{x} \max_{y} R(x, y)$$

Therefore, with respect to our examples:

$$height(R) = 1$$

$$height(P) = 1$$

Okay students I stopped here today.

So in this class I have given you the concept of relations and its extension in the form of Fuzzy Relations and we have illustrated with examples different relations and also we have studied certain basic definitions.

In the next class I shall continue with Binary Fuzzy Relation and also I shall take care of some other types of relations that we can investigate while studying Fuzzy Relations. Okay students. Thank you.