Introduction to Fuzzy Sets Arithmetic & logic Prof. Niladri Chatterjee Department of Mathematics Indian Institute of Technology – Delhi

Lecture – 17 Fuzzy Sets Arithmetic and Logic

Welcome students to the 17th lecture of the MOOCs course on Introduction to Fuzzy Sets, Arithmetic and Logic.

In today's class we shall be doing solving of fuzzy equations as you can well understand that solution of equations is very important particularly when we are designing system that needs us to handle fuzzy mathematics to a significant extent.

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Solving Fuzzy Equations i) Linear Eq: A+X = B ii) Multiplicative A·X = ashere A 2 B are frozy numbers, and x is also expected to be a fuggy number

In particular we shall look at two different types of equations

- 1. Linear Equation A + X = B
- 2. Multiplicative $A \cdot X = B$

where A and B are fuzzy numbers and X is also expected to be a fuzzy number

Why I am saying expected?

Because sometimes there may be a trivial solution where X is only one real number.

We shall see that later.

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O A + X = BIf A 2 B are reals we get straightforward sol" X = B-A. Does it work for fussy norm

So let us first start with

1) A + X = B

Now if *A* and *B* are reals we get straightforward solution X = B - A

Does it work for fuzzy numbers?

The answer is NO.

Why?

The reason is very simple.

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Let up illustrate uping interval arithmetric. Suppose Support of A = [a, a] Support & B = [b, b2] Then $B-A = [b_1 - a_2, b_2 - a_1]$: $A + (B - A) = [b_1 - a_2 + a_1, b_2 - a_1 + a_2]$

Let us illustrate using interval arithmetic.

Suppose support of $A = [a_1, a_2]$ and support of $B = [b_1, b_2]$

Because you know, we have already done these things that it will be most negative the value will be $b_1 - a_2$ and most positive value that it can have is $b_2 - a_1$

Then $B - A = [b_1 - a_2, b_2 - a_1]$

Therefore, $A + (B - A) = [b_1 - a_2 + a_1, b_2 - a_1 + a_2] \neq [b_1, b_2]$

So this shows that we cannot get a solution by simply subtracting A from B.

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we can easily see that one possible solt is [b,-a, , b2-a2] $[a_1, a_2] + [b_1 - a_1, b_2 - a_2]$ = [b_1, b_2] There is a catch! [b_1 - a_1, b_2 - a_2] need the Onst be a balid interval.

However, we can easily see that one possible solution is $[b_1 - a_1, b_2 - a_2]$ It is obvious that $[a_1, a_2] + [b_1 - a_1, b_2 - a_2] = [b_1, b_2]$ But it is not so simple.

but it is not so simp

There is a catch!

 $[b_1 - a_1, b_2 - a_2]$ need not be a valid interval.

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EX: Consider A = [-3 -: Support of A = [-3, 9] Support of B = [-1, 3] ·· [b,-a, b2-a2] [-1-(-3), 3-9] = [2,-6] we nee that NOT a

Example: Consider $A = \begin{bmatrix} -3 & 1 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix}$ Therefore, support of $A = \begin{bmatrix} -3 & 9 \end{bmatrix}$, support of $B = \begin{bmatrix} -1 & 3 \end{bmatrix}$ Therefore, $\begin{bmatrix} b_1 - a_1, b_2 - a_2 \end{bmatrix} = \begin{bmatrix} -1 - (-3), 3 - 9 \end{bmatrix} = \begin{bmatrix} 2, -6 \end{bmatrix}$ And we see that this is not a valid interval.

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However, if the given equation in [-3 1 2] + X = [-1 6 8] them [b,-a, , b2-a2] = [-1+3, 8-2] = [2,6] which is a valid interval

However, if the given equation is

$[-3 \ 1 \ 2] + X = [-1 \ 6 \ 8]$

then $[b_1 - a_1, b_2 - a_2] = [-1 + 3, 8 - 2] = [2, 6]$ which is a valid interval. (**Refer Slide Time: 10:15**)

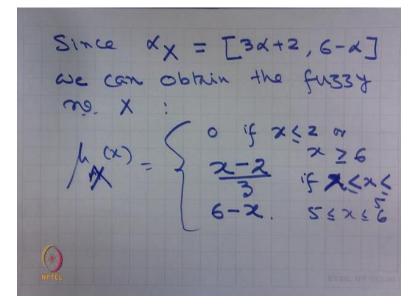
This validity should hold for all d-cuts , Because are known $d_A + d_X = d_B$ Consider again A=[-3 12] B=[-1 68] ×A = [-3+4×, 2-×] AB = E - 1 + 7A, 8 - 2A]AX = E3A + 2, 6 - A]

But the catch is that this validity should hold for all α -cuts.

Because we know ${}^{\alpha}A + {}^{\alpha}X = {}^{\alpha}B$ So consider again $A = [-3 \ 1 \ 2]$ and $B = [-1 \ 6 \ 8]$ Therefore, ${}^{\alpha}A = [-3 + 4\alpha, 2 - \alpha]$ and ${}^{\alpha}B = [3\alpha + 2, 6 - \alpha]$ Therefore, we get by subtracting ${}^{\alpha}A$ from ${}^{\alpha}B$

$$^{\alpha}X = [3\alpha + 2, 6 - \alpha]$$

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Since ${}^{\alpha}X = [3\alpha + 2, 6 - \alpha]$ we can obtain the fuzzy number *X* and

$$\mu_X(x) = \begin{cases} 0 & x \le 2 \text{ or } x \ge 6\\ \frac{x-2}{3} & 2 \le x \le 5\\ 6-x & 5 \le x \le 6 \end{cases}$$

Therefore, in this case we get a solution.

Thus the conclusion is that all the time you may not get a solution for A + X = B when A and B are fuzzy numbers, we have to check the validity and then only we can be sure that the fuzzy equation can be solved.

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(2) Solution for A·X = B ashere A & B are fuzzy numbers. As before are go by internal arithmetic Let $A = [a_1, a_2]$ B = [b,, b2] iere are 3 possibilities for A & B

Let us now look at solution for

2) $A \cdot X = B$ where A and B are fuzzy numbers.

As before we go by interval arithmetic.

Let the support of $A = [a_1, a_2]$ and support of $B = [b_1, b_2]$

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Illy 1) 0 ≤ 0, ≤ 02 DO SOL ii) $G_1 \leq 0 \leq G_2$ Q1 5 Q2 50 (iii) The equation A.X = . may have 9 different possibilities wr

Now there are three possibilities for A and B

i.	$0 \le a_1 \le a_2$	i.	$0 \le b_1 \le b_2$
ii.	$a_1 \leq 0 \leq a_2$	ii.	$b_1 \leq 0 \leq b_2$
iii.	$a_1 \le a_2 \le 0$	iii.	$b_1 \leq b_2 \leq 0$

Now, for A, any of the 3 situations can be a possibility. Similarly, for B, any of the 3 situations may be a possibility.

Therefore, the equation $A \cdot X = B$ may have 9 different possibilities with respect to the support of *A* and *B*.

Therefore, whether a solution actually exists or not, in order to find that we have to look at the particular situation of the given problem and then we have to see whether a solution is possible or not because we shall see that in all of the 9 cases solutions may not be possible.

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Casel: O < a, < az O < b, < bz We observe that $\begin{bmatrix} \alpha_1, \alpha_2 \end{bmatrix} * \begin{bmatrix} b_1 \\ \alpha_1 \end{bmatrix} * \begin{bmatrix} b_2 \\ \alpha_1 \end{bmatrix}$ will give the interval [b, bz One may think that [b] may be the support of

So let us look at Case 1

When $0 \le a_1 \le a_2$ and $0 \le b_1 \le b_2$

We can observe that

 $[a_1, a_2] * \left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]$ will give the interval $[b_1, b_2]$

Therefore, one may think that $\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]$ may be the support for the unknown quantity *X*. (Refer Slide Time: 20:58)

Ex1
$$[2,3] * [x_1, x_2] = [5,9]$$

 \therefore By the above observation
 $X = [\frac{5}{2}, \frac{9}{3}] = [\frac{2\cdot 5}{3}, \frac{3}{3}]$
And this is accepted.
Ex2 $[2,3] * [x_1, x_2] = [4,5]$
 $\therefore \frac{9}{61} = 2 & \frac{92}{62} = \frac{5}{3}$
 $\therefore \frac{91}{61} = 2 & \frac{92}{62} = \frac{5}{3}$

So let us consider examples, and we are in the interval arithmetic mode and the equation to be solved suppose is

$$[2,3] * [x_1, x_2] = [5,9]$$

Therefore, by the above observation $X = \left[\frac{5}{2}, \frac{9}{3}\right] = [2.5, 3]$

And this is accepted because this is a valid interval. Example 2:

$$[2,3] * [x_1, x_2] = [4,5]$$

Therefore, $\frac{b_1}{a_1} = 2$ and $\frac{b_2}{a_2} = \frac{5}{3}$

Thus, *X* can be $\left[2, \frac{5}{3}\right]$ which is not a valid interval.

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Thus in the 2nd case we don't have a solution. Therefore, the condition that ashen both A2BERt them soll exists if $\frac{b_1}{c_1} \leq \frac{b_2}{c_2}$ As before, this should hold offer all x-cuta

Thus, in the second case we do not have a solution.

Therefore, the condition that when both $A, B \in \mathbb{R}^+$ then solution exists if

$$\frac{b_1}{a_1} \le \frac{b_2}{a_2}$$

As before this should hold for all α -cuts.

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Hence the conditions for existence of solution for A+X , BSRt , ashen he. A & E (0, 1] to ensure walid

Hence the conditions for existence of solution for $A \cdot X = B$ when A, B belongs to \mathbb{R}^+ are:

(a)
$$\frac{\alpha_{b_1}}{\alpha_{a_1}} \le \frac{\alpha_{b_2}}{\alpha_{a_2}}$$
 for all $\alpha \in (0, 1]$

This is needed so that validity of the solution is ensured

(b) If $\alpha < \beta$

$$\frac{{}^{\alpha}b_1}{{}^{\alpha}a_1} \le \frac{{}^{\beta}b_1}{{}^{\beta}a_1} \le \frac{{}^{\beta}b_2}{{}^{\beta}a_2} \le \frac{{}^{\alpha}b_2}{{}^{\alpha}a_2}$$

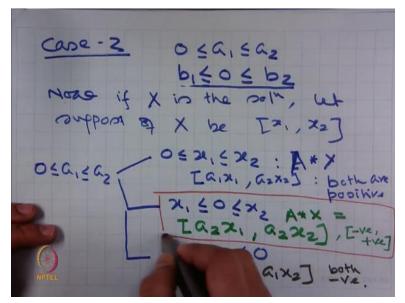
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i.e if $dA = L^{\alpha}a_1 / \alpha a_2$ $KB = L^{\alpha}b_1 / \alpha b_2$ the above inequalities a hold 4 K < R is will evenre that BX SXX

That is if ${}^{\alpha}A = [{}^{\alpha}a_1, {}^{\alpha}a_2]$ and ${}^{\alpha}B = [{}^{\alpha}b_1, {}^{\alpha}b_2]$

Then, the above inequalities should hold for all $\alpha < \beta$ because this will ensure that $\beta X \subseteq \alpha X$

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Now let us look at Case 2

 $0 \le a_1 \le a_2$ and $b_1 \le 0 \le b_2$

Now if X is the solution, let support of X be $[x_1, x_2]$

Therefore, we can have three situations:

1. $0 \le x_1 \le x_2$

Support of $A \cdot X = [a_1x_1, a_2x_2]$. Both are positive.

2. $x_1 \le 0 \le x_2$

Support of $A \cdot X = [a_2x_1, a_2x_2] = [-ve, +ve]$

3. $x_1 \le x_2 \le 0$

Support of $A \cdot X = [a_2x_1, a_ax_2]$. Both are negative.

As $b_1 \le 0 \le b_2$, this simple observation suggests that if there is a solution then $x_1 \le 0 \le x_2$ is the case.

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.: If sol exists then $[\alpha_2 x_1, \alpha_2 x_2] = [b_1, b_2]$ $\begin{bmatrix} x_1, x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 62 \end{bmatrix}, \begin{bmatrix} b_2 \\ 62 \end{bmatrix}$ Therefore in this case we whall always get a sol PTE

Therefore, if a solution exists then $[a_2x_1, a_2x_2] = [b_1, b_2]$ Therefore, $[x_1, x_2] = \left[\frac{b_1}{a_2}, \frac{b_2}{a_2}\right]$

Therefore, in this case we shall always get a solution.

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2 b1 5 b2 50 . To obtain a solu we much 2, 5 22 50 : [a,, a2] * [x, ; x2] = [b, , b2] => [a2x1, a1x2] = [b1, b2] Cr X2 = .02

Let us now consider Case 3,

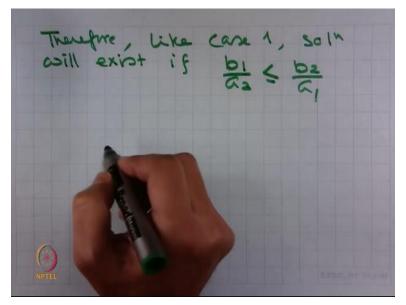
 $0 \le a_1 \le a_2$ and $b_1 \le b_2 \le 0$

Therefore, to obtain a solution, we must have $x_1 \le x_2 \le 0$

Because A is completely positive B is the completely negative therefore, unless X is all negative we cannot get a solution in this case.

Therefore, $[a_1, a_2] * [x_1, x_2] = [b_1, b_2]$ $\Rightarrow [a_2x_1, a_1x_2] = [b_1, b_2]$ Or $x_1 = \frac{b_1}{a_2}$ and $x_2 = \frac{b_2}{a_1}$

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Therefore, like Case 1 solution will exist if

$$\frac{b_1}{a_2} \le \frac{b_2}{a_1}$$

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Let up take examples:
a)
$$[4,9] * [71,72] = [-5,-3]$$

 $\therefore b_1 = -5 & b_2 = -3$
 $=-55 & -75$
 $\therefore This is Not a said
internal.
 $\therefore There is no poly.$
b) $[-2,6] * [X_1, X_2] = [-4,-1]$
 $\therefore b_1 = -\frac{1}{6} & b_2 = -\frac{1}{2}$$

So let us take examples,

(a)
$$[4,9] * [x_1, x_2] = [-5, -3]$$

$$\therefore \frac{b_1}{a_2} = -\frac{5}{9} = -0.55 \text{ and } \frac{b_2}{a_1} = -\frac{3}{4} = -0.75$$

Therefore, this is not a valid interval. Therefore, no solution exists

(b) $[2, 6] * [x_1, x_2] = [-4, -1]$

$$\therefore \frac{b_1}{a_2} = -\frac{4}{6} = \text{ and } \frac{b_2}{a_1} = -\frac{1}{2}$$

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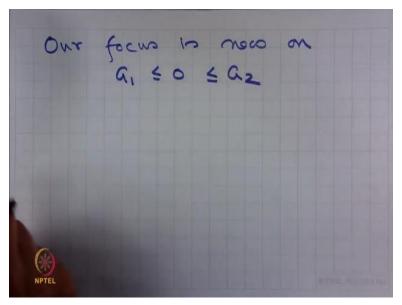
= [一号,一支] ashich is a leased interval ... Solt avill exist 0

Therefore, the interval is $\left[-\frac{2}{3}, -\frac{1}{2}\right]$ which is a valid interval. Therefore, solution will exist. (**Refer Slide Time: 38:27**)

In a symmetric asy one can deal with the three situations pertaining to 6, 502 50

In a symmetric way one can deal with the three situations pertaining to $a_1 \le a_2 \le 0$ So I leave it as an exercise to see when solution will exist and when it will not.

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So our focus is now on $a_1 \le 0 \le a_2$

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So Case 1:

 $0 \leq b_1 \leq b_2$

As before we have three cases:

1. $0 \le x_1 \le x_2$

Lower bound $a_1 x_1 \leq 0$. \therefore Rejected

2. $x_1 \le 0 \le x_2$

 $a_1 x_2$ will be $< 0 \therefore$ Rejected

3. $x_1 \le x_2 \le 0$

The upper bound = $a_2 x_2 < 0$.

Therefore, if $a_1 \le 0 \le a_2$ but $0 \le b_1 \le b_2$ there does not exist any solution.

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In a similar any are con show that a. 50 E G2 & b. 5 b2 50 port have any solk.

In a similar way we can show that $a_1 \le 0 \le a_2$ and $b_1 \le b_2 \le 0$ do not have any solution.

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The most complicated situation in: Q1505Q2 bi EO E bz. We have look at the following quantitien : b_1 : pomite b_2 : pomitive a_2 : pomitive b_2 : pomitive b_2 : negative b_2 : negative a_1

Therefore, the most complicated situation is when $a_1 \le 0 \le a_2$ and $b_1 \le 0 \le b_2$

So we have to look at the following quantities:

$$\frac{b_1}{a_2}$$
: Positive $\frac{b_2}{a_2}$: Positive $\frac{b_1}{a_2}$: Negative $\frac{b_2}{a_1}$: Negative

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We shall have to look at ahuch
alternative can give us the
conport.

$$[a_1, a_2] = [-3, 2]$$

 $[b_1, b_2] = [-4, 9]$
 $\therefore b_1 = 4$
 $b_1 = -3$
 $b_2 = -3$
 $b_3 = -3$
 $b_4 = -2$
 $b_5 = -2$
 $b_6 = -2$
 $b_6 = -2$

Therefore, we shall have to look at which alternative can give us the support. Let us give an example

$$[a_1, a_2] = [-3, 2]$$

 $[b_1, b_2] = [-4, 9]$

Therefore,

$$\frac{b_1}{a_2} = \frac{4}{3} \qquad \qquad \frac{b_2}{a_2} = \frac{9}{2}$$
$$\frac{b_1}{a_2} = \frac{4}{-2} = -2 \qquad \qquad \frac{b_2}{a_1} = -\frac{9}{3} = -3$$

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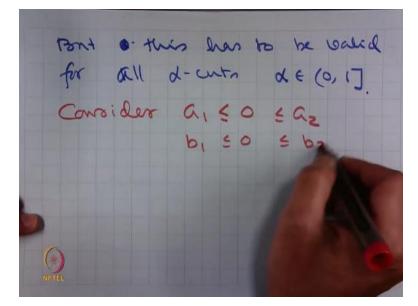
internals for por following is of fre poroibil [-2] [x, x2" う E-2, 43] ジ E-2, 43] [-3] · ~([-2, -3] (vii) Abo (iii) (Viii) (In) iv) 「生」 X)

Therefore, we check the following intervals for possibility of $[x_1, x_2]$:

i.	$\left[-2,\frac{4}{3}\right]$	vi.	{-3}
	$\left[-2,\frac{9}{2}\right]$	vii.	[-2,-3]
		viii.	$\left[\frac{4}{3}\right]$
iii.	$\left[-3,\frac{4}{3}\right]$	ix.	
iv.	$\left[-3,\frac{9}{2}\right]$		
v.	{-2}	х.	$\left[\frac{4}{3}, \frac{9}{2}\right]$

So if the solution exists any one of them should give us the answer.

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As before here we are looking at only the interval arithmetic but these has to be valid for all α -cuts

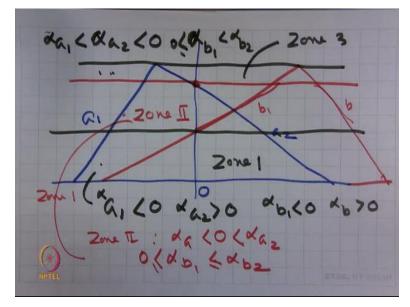
 $\alpha \in (0,1]$

So how to find it for all α -cuts?

I give you a trick.

Consider the situation when $a_1 \le 0 \le a_2$ and $b_1 \le 0 \le b_2$

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We draw the following diagram.

So like that wherever the lines are crossing the y –axis we have to split it into different zones with respect to each zone we get the conditions and the way we have solved earlier we have to see if

for this zone solution exists or not like that after analyzing all the cases we can ascertain if we can get a solution for the given multiplicative fuzzy equation.

Okay students I stop here today, if you want I can upload a small document where it will analyze all cases and will give you clues to in which situation one can find a solution.

Okay thank you so much. With this I complete my lectures on fuzzy arithmetic in the next class I shall start with fuzzy relations. Thank You.