Introduction to Fuzzy Sets Arithmetic & logic
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Lecture – 17
Fuzzy Sets Arithmetic and Logic

Welcome students to the 17th lecture of the MOOCs course on Introduction to Fuzzy Sets, Arithmetic and Logic.

In today's class we shall be doing solving of fuzzy equations as you can well understand that solution of equations is very important particularly when we are designing system that needs us to handle fuzzy mathematics to a significant extent.

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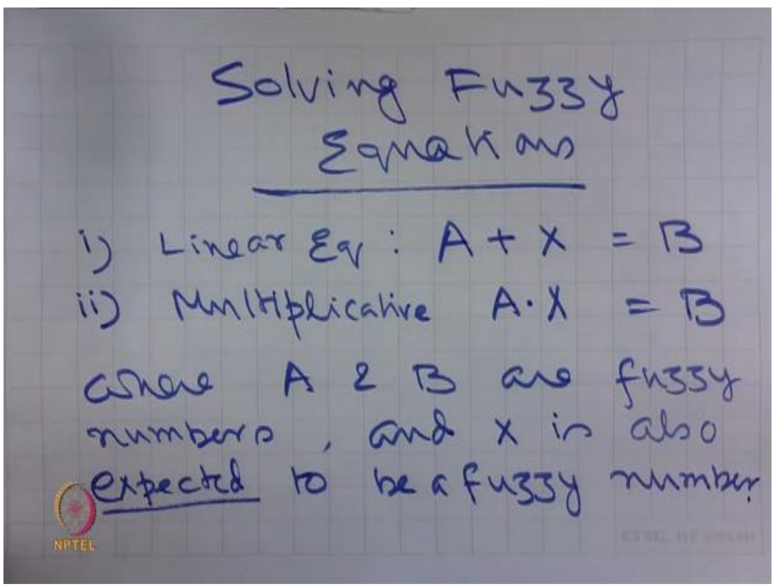

In particular we shall look at two different types of equations

1. Linear Equation $A + X = B$
2. Multiplicative $A \cdot X = B$

where A and B are fuzzy numbers and X is also expected to be a fuzzy number

Why I am saying expected?

Because sometimes there may be a trivial solution where X is only one real number.

We shall see that later.

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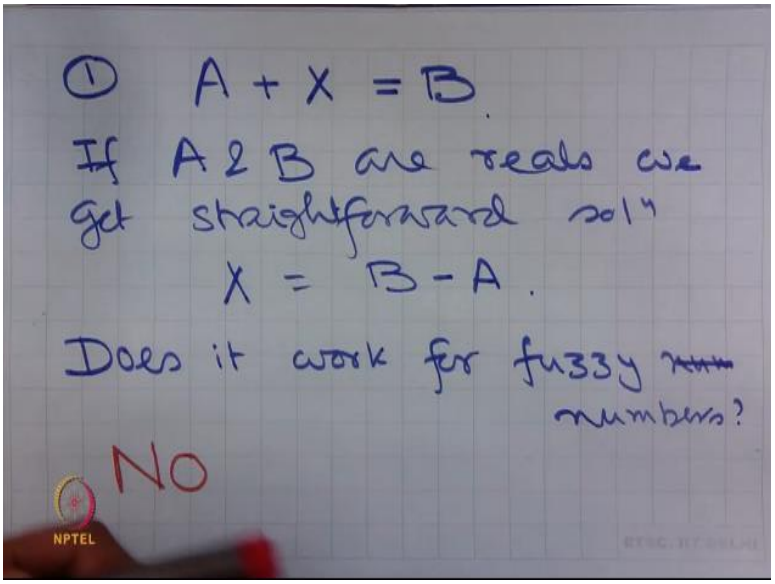

So let us first start with

1) $A + X = B$

Now if A and B are reals we get straightforward solution $X = B - A$

Does it work for fuzzy numbers?

The answer is NO.

Why?

The reason is very simple.

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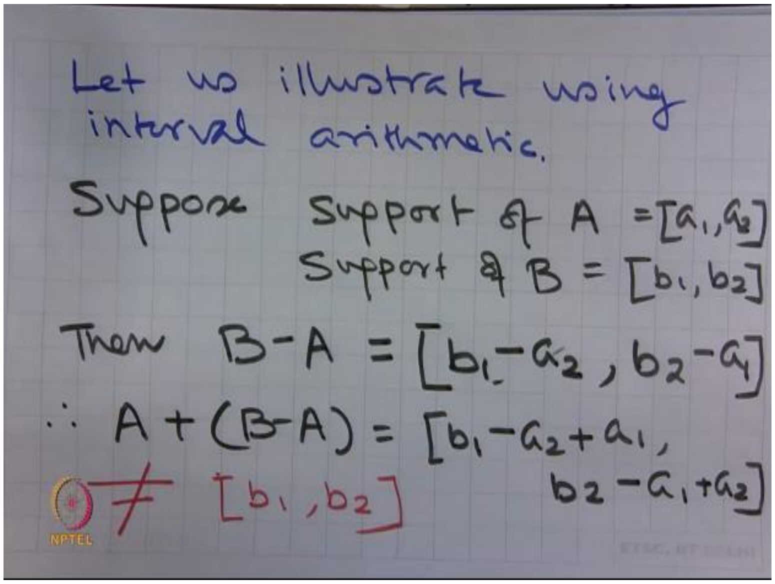

Let us illustrate using interval arithmetic.

Suppose support of $A = [a_1, a_2]$ and support of $B = [b_1, b_2]$

Because you know, we have already done these things that it will be most negative the value will be $b_1 - a_2$ and most positive value that it can have is $b_2 - a_1$

Then $B - A = [b_1 - a_2, b_2 - a_1]$

Therefore, $A + (B - A) = [b_1 - a_2 + a_1, b_2 - a_1 + a_2] \neq [b_1, b_2]$

So this shows that we cannot get a solution by simply subtracting A from B.

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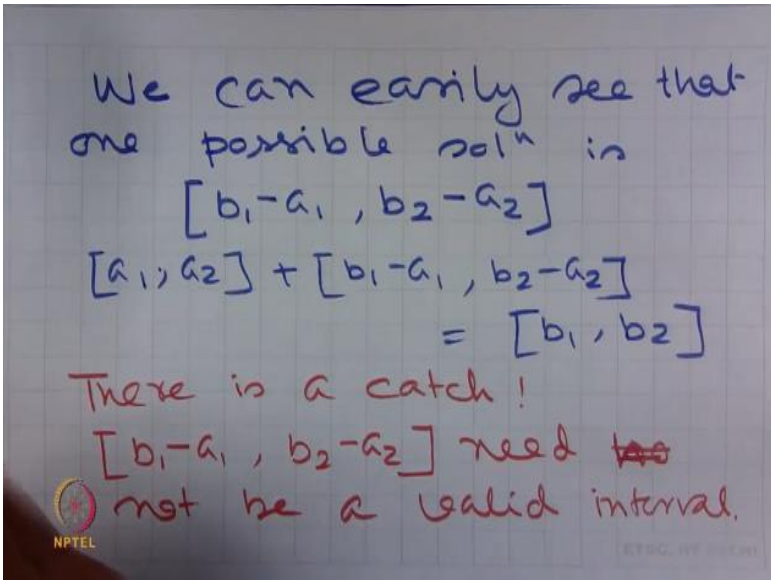

However, we can easily see that one possible solution is $[b_1 - a_1, b_2 - a_2]$

It is obvious that $[a_1, a_2]$+$[b_1 - a_1, b_2 - a_2] = [b_1, b_2]$

But it is not so simple.

There is a catch!

$[b_1 - a_1, b_2 - a_2]$ need not be a valid interval.

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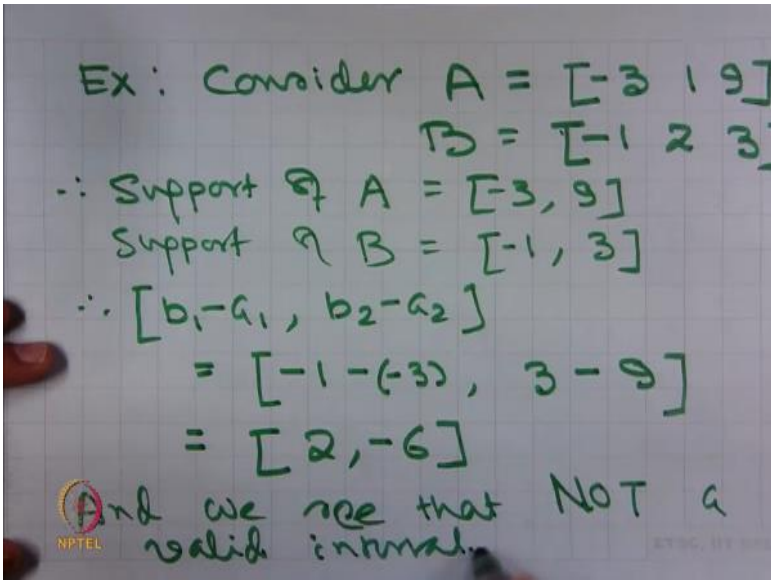

Example: Consider $A = [-3\ \ 1\ \ 9]$ and $B = [-1\ \ \ 2\ \ 3]$

Therefore, support of $A = [-3,\ 9]$, support of $B = [-1,\ \ 3]$

Therefore, $[b_1 - a_1, b_2 - a_2] = [\,-1 - (-3)\ \ ,3 - 9] = [2, -6]$

And we see that this is not a valid interval.

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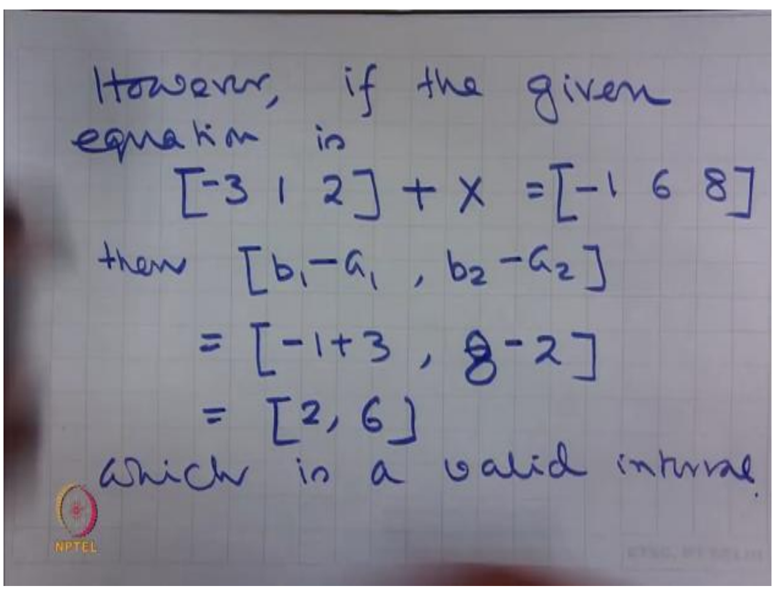

However, if the given equation is

$$[-3\ \ 1\ \ 2] + X = [-1\ \ 6\ \ 8]$$

then $[b_1 - a_1, b_2 - a_2] = [-1 + 3\ ,8 - 2] = [2, 6]$ which is a valid interval.

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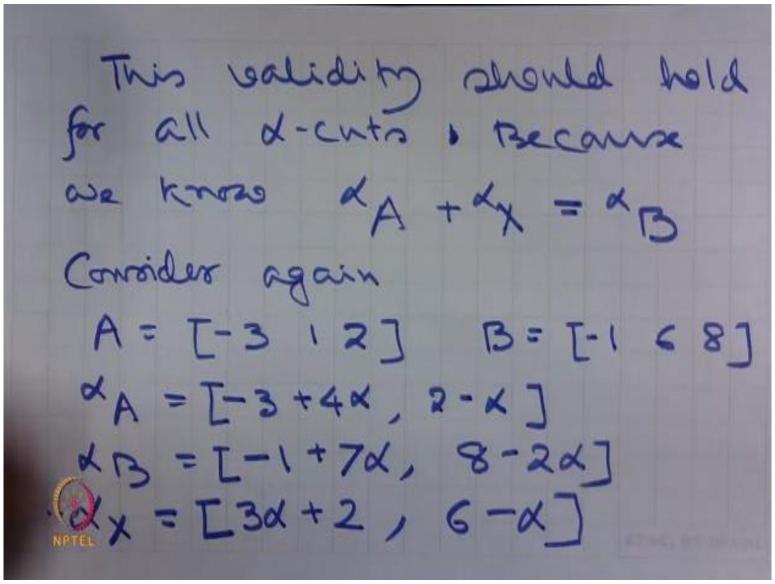

But the catch is that this validity should hold for all α-cuts.

Because we know ${}^{\alpha}A + {}^{\alpha}X = {}^{\alpha}B$

So consider again $A = [-3\ \ 1\ \ 2]$ and $B = [-1\ \ 6\ \ 8]$

Therefore, ${}^{\alpha}A = [-3 + 4\alpha, 2 - \alpha]$ and ${}^{\alpha}B = [3\alpha + 2, 6 - \alpha]$

Therefore, we get by subtracting ${}^{\alpha}A$ from ${}^{\alpha}B$

$${}^{\alpha}X = [3\alpha + 2, 6 - \alpha]$$

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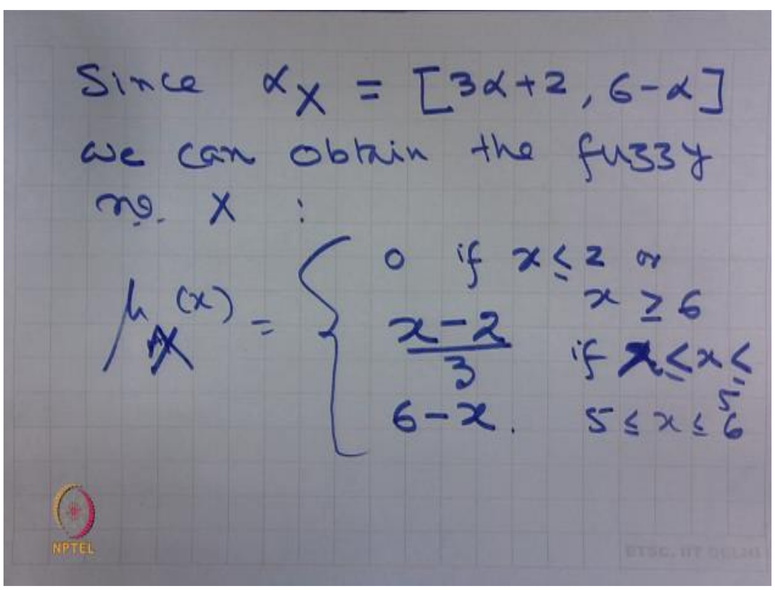

Since ${}^{\alpha}X = [3\alpha + 2, 6 - \ \alpha]$ we can obtain the fuzzy number X and

$$\mu_X(x) = \begin{cases} 0 & x \le 2 \text{ or } x \ge 6 \\ \frac{x-2}{3} & 2 \le x \le 5 \\ 6 - x & 5 \le x \le 6 \end{cases}$$

Therefore, in this case we get a solution.

Thus the conclusion is that all the time you may not get a solution for $A + X = B$ when A and B are fuzzy numbers, we have to check the validity and then only we can be sure that the fuzzy equation can be solved.

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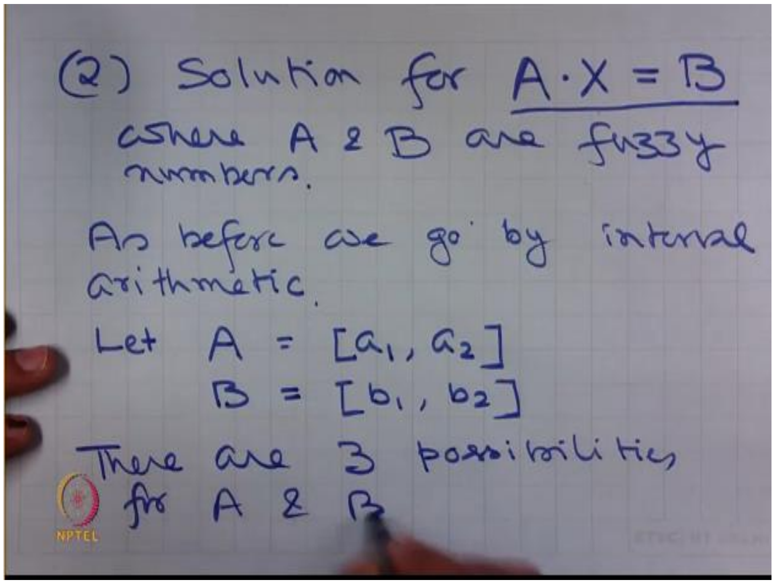

Let us now look at solution for

2) $A \cdot X = B$ where A and B are fuzzy numbers.

As before we go by interval arithmetic.

Let the support of $A = [a_1, a_2]$ and support of $B = [b_1, b_2]$

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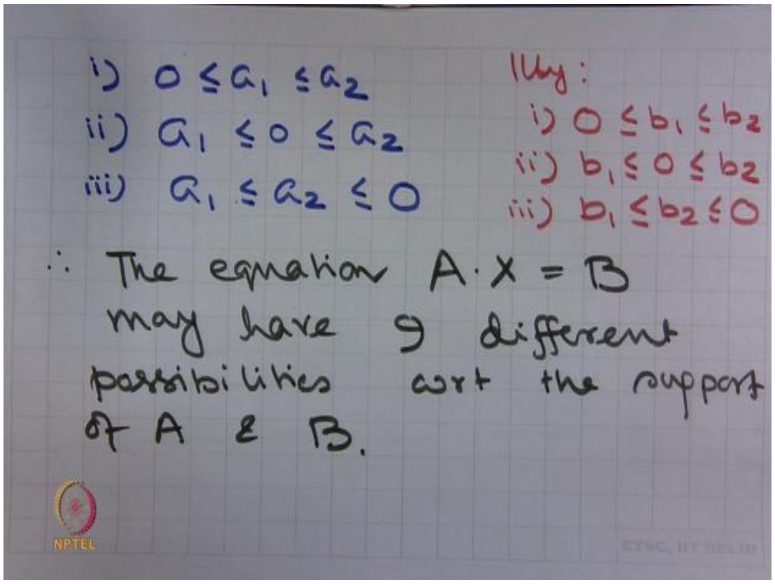

Now there are three possibilities for A and B

i. $0 \leq a_1 \leq a_2$

ii. $a_1 \leq 0 \leq a_2$

iii. $a_1 \leq a_2 \leq 0$

i. $0 \leq b_1 \leq b_2$

ii. $b_1 \leq 0 \leq b_2$

iii. $b_1 \leq b_2 \leq 0$

Now, for A, any of the 3 situations can be a possibility. Similarly, for B, any of the 3 situations may be a possibility.

Therefore, the equation $A \cdot X = B$ may have 9 different possibilities with respect to the support of A and B.

Therefore, whether a solution actually exists or not, in order to find that we have to look at the particular situation of the given problem and then we have to see whether a solution is possible or not because we shall see that in all of the 9 cases solutions may not be possible.

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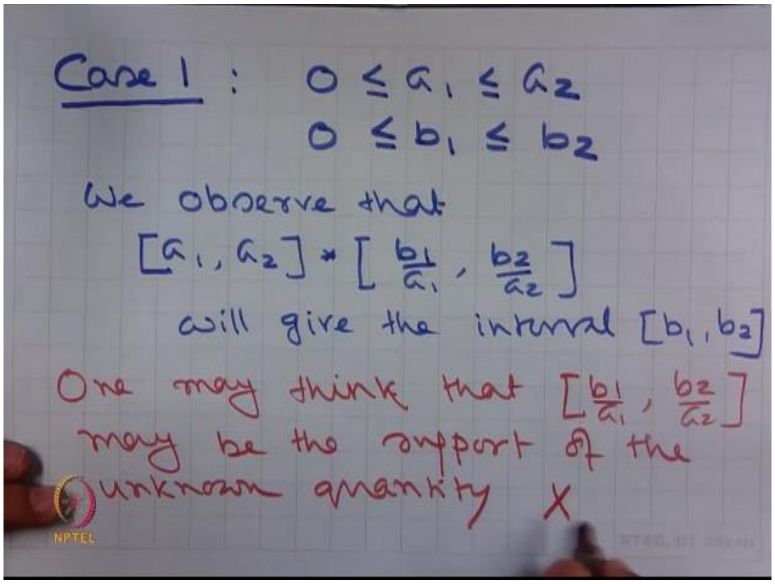

So let us look at Case 1

When $0 \le a_1 \le a_2$ and $0 \le b_1 \le b_2$

We can observe that

$[a_1, a_2] * \left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]$ will give the interval $[b_1,\ b_2]$

Therefore, one may think that $\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]$ may be the support for the unknown quantity X.

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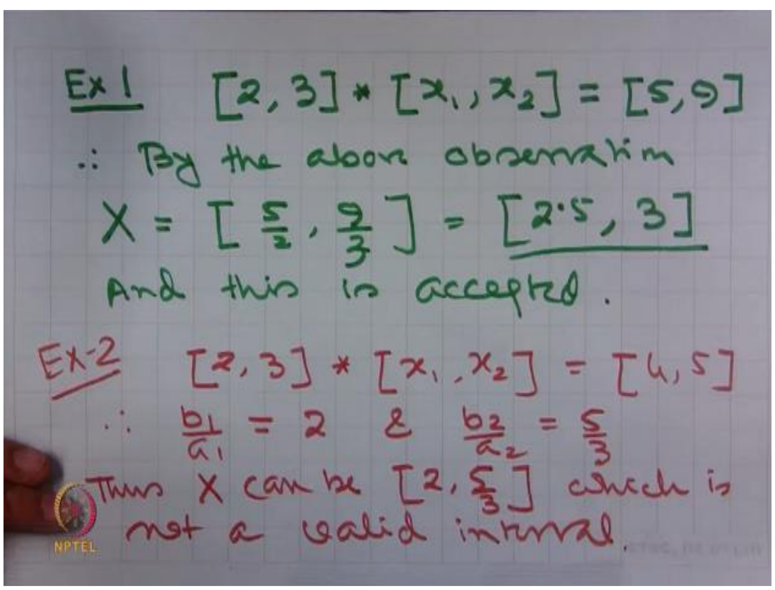

So let us consider examples, and we are in the interval arithmetic mode and the equation to be solved suppose is

$$[2, 3] * [x_1, x_2] = [5, 9]$$

Therefore, by the above observation $X = \left[\frac{5}{2}, \frac{9}{3}\right] = [2.5,\ 3]$

And this is accepted because this is a valid interval.

Example 2:

$$[2, 3] * [x_1, x_2] = [4, 5]$$

Therefore, $\frac{b_1}{a_1} = 2$ and $\frac{b_2}{a_2} = \frac{5}{3}$

Thus, X can be $\left[2, \frac{5}{3}\right]$ which is not a valid interval.

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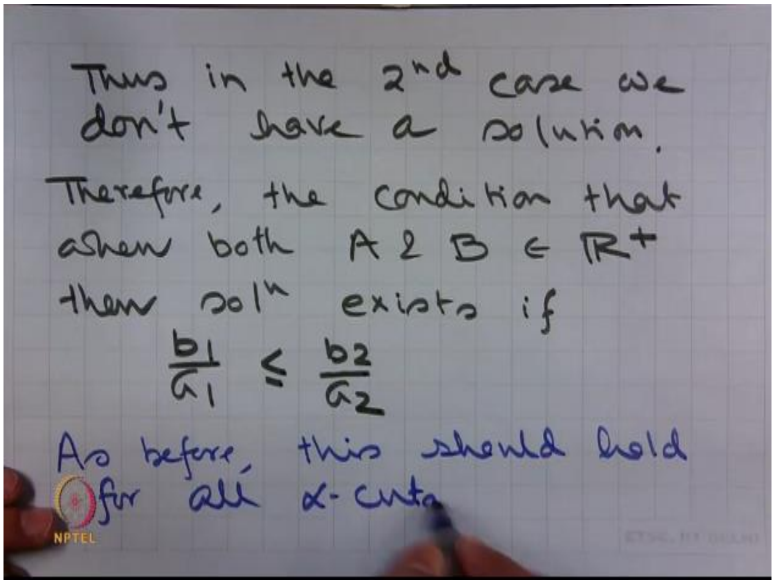

Thus, in the second case we do not have a solution.

Therefore, the condition that when both $A, B \in \mathbb{R}^+$ then solution exists if

$$\frac{b_1}{a_1} \le \frac{b_2}{a_2}$$

As before this should hold for all α-cuts.

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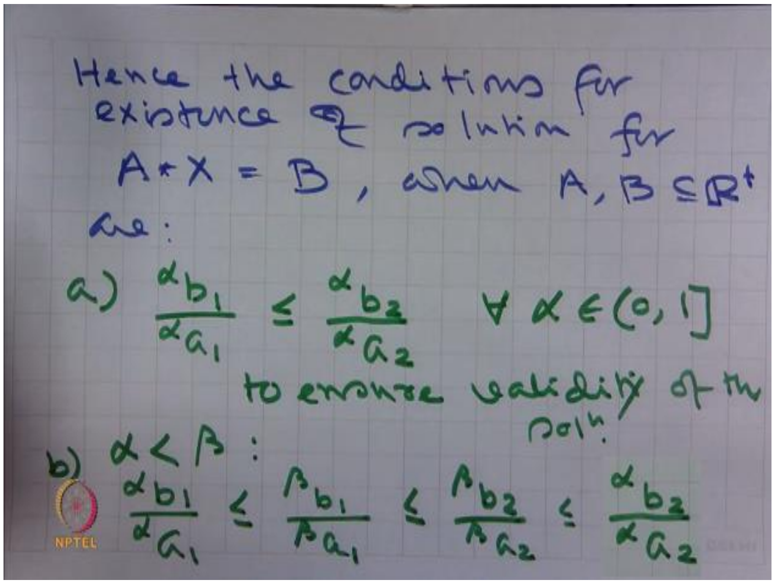

Hence the conditions for existence of solution for $A \cdot X = B$ when A, B belongs to $\mathbb{R}^+$ are:

(a) $\frac{^\alpha b_1}{^\alpha a_1} \le \frac{^\alpha b_2}{^\alpha a_2}$ for all $\alpha \in (0, 1]$

This is needed so that validity of the solution is ensured

(b) If $\alpha < \beta$

$$\frac{^\alpha b_1}{^\alpha a_1} \le \frac{^\beta b_1}{^\beta a_1} \le \frac{^\beta b_2}{^\beta a_2} \le \frac{^\alpha b_2}{^\alpha a_2}$$

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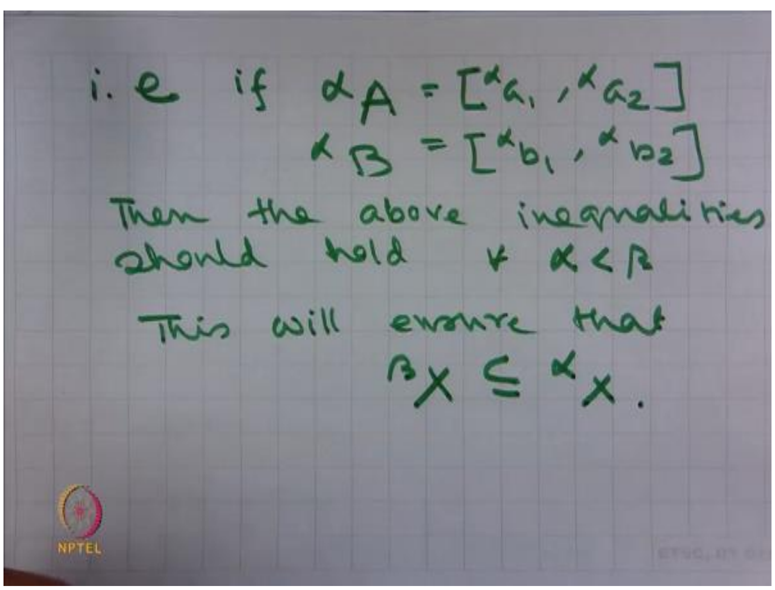

That is if $^\alpha A = [\,^\alpha a_1,\, ^\alpha a_2]$ and $^\alpha B = [\,^\alpha b_1,\, ^\alpha b_2]$

Then, the above inequalities should hold for all $\alpha < \beta$ because this will ensure that $^\beta X \subseteq {}^\alpha X$

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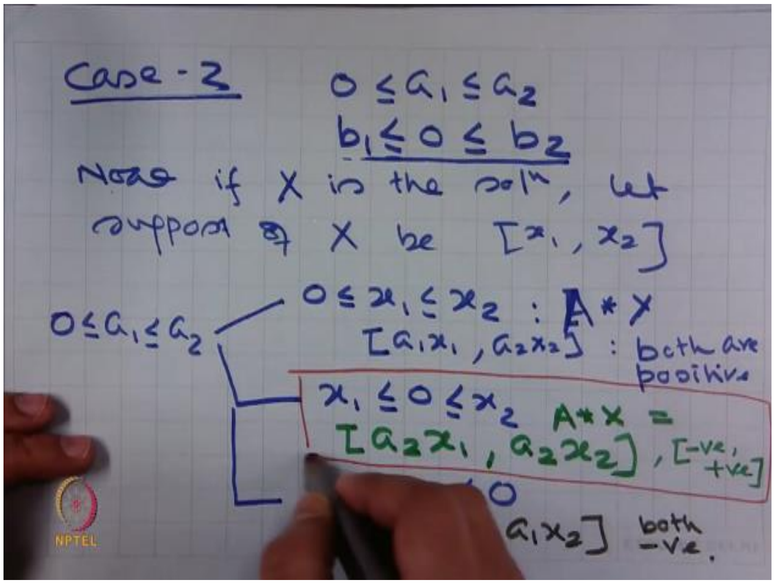

Now let us look at Case 2

$0 \leq a_1 \leq a_2$ and $b_1 \leq 0 \leq b_2$

Now if X is the solution, let support of X be $[x_1, x_2]$

Therefore, we can have three situations:

1. $0 \leq x_1 \leq x_2$

   Support of $A \cdot X = [a_1 x_1, a_2 x_2]$. Both are positive.

2. $x_1 \leq 0 \leq x_2$

   Support of $A \cdot X = [a_2 x_1, a_2 x_2] = [-ve, +ve]$

3. $x_1 \leq x_2 \leq 0$

   Support of $A \cdot X = [a_2 x_1, a_a x_2]$. Both are negative.

As $b_1 \leq 0 \leq b_2$, this simple observation suggests that if there is a solution then $x_1 \leq 0 \leq x_2$ is the case.

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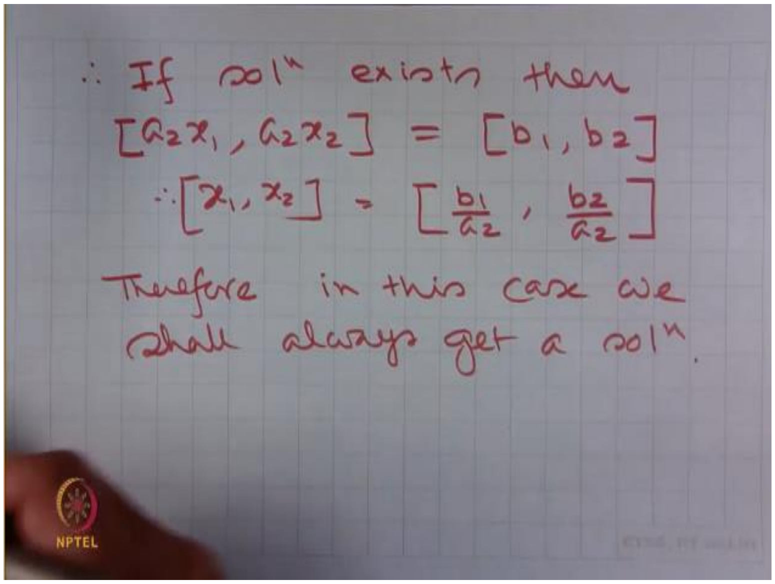

Therefore, if a solution exists then $[a_2x_1, a_2x_2] = [b_1, b_2]$

Therefore, $[x_1, x_2] = \left[\frac{b_1}{a_2}, \frac{b_2}{a_2}\right]$

Therefore, in this case we shall always get a solution.

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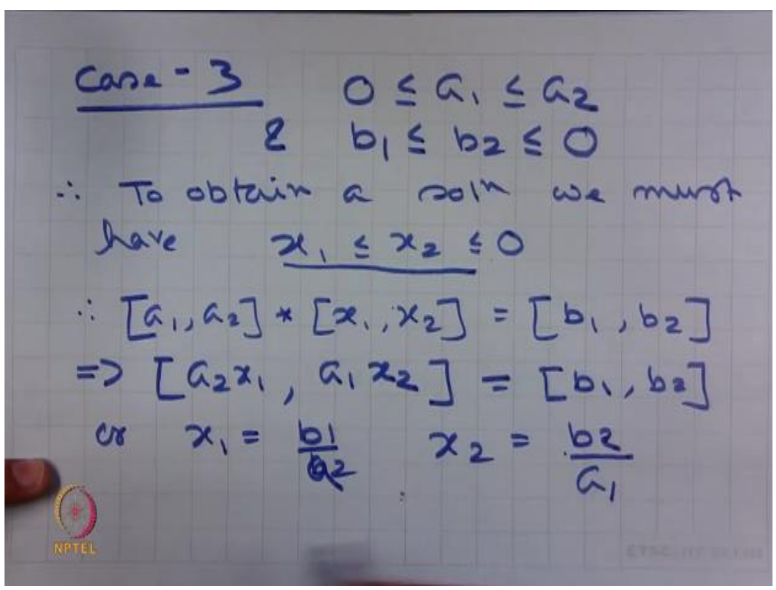

Let us now consider Case 3,

$0 \leq a_1 \leq a_2$ and $b_1 \leq b_2 \leq 0$

Therefore, to obtain a solution, we must have $x_1 \leq x_2 \leq 0$

Because A is completely positive B is the completely negative therefore, unless X is all negative we cannot get a solution in this case.

Therefore, $[a_1, a_2] * [x_1, x_2] = [b_1, b_2]$

$\Rightarrow [a_2 x_1, a_1 x_2] = [b_1, b_2]$

Or $x_1 = \frac{b_1}{a_2}$ and $x_2 = \frac{b_2}{a_1}$

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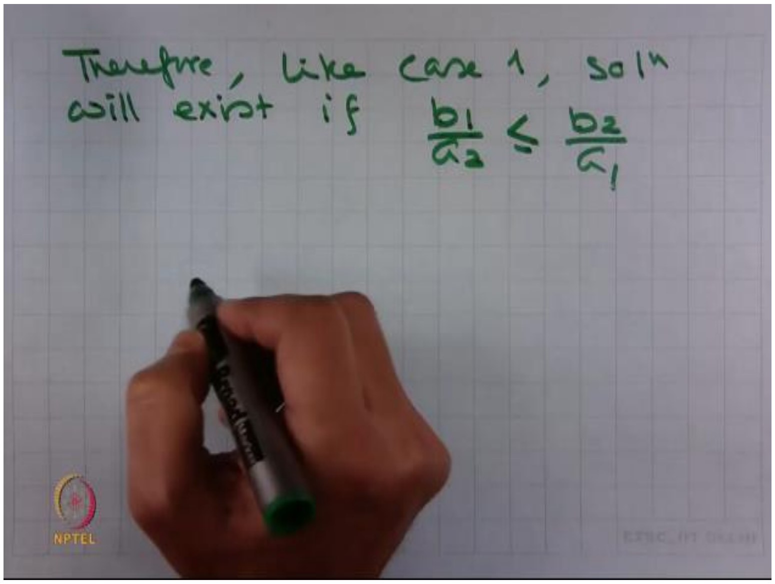

Therefore, like Case 1 solution will exist if

$$\frac{b_1}{a_2} \leq \frac{b_2}{a_1}$$

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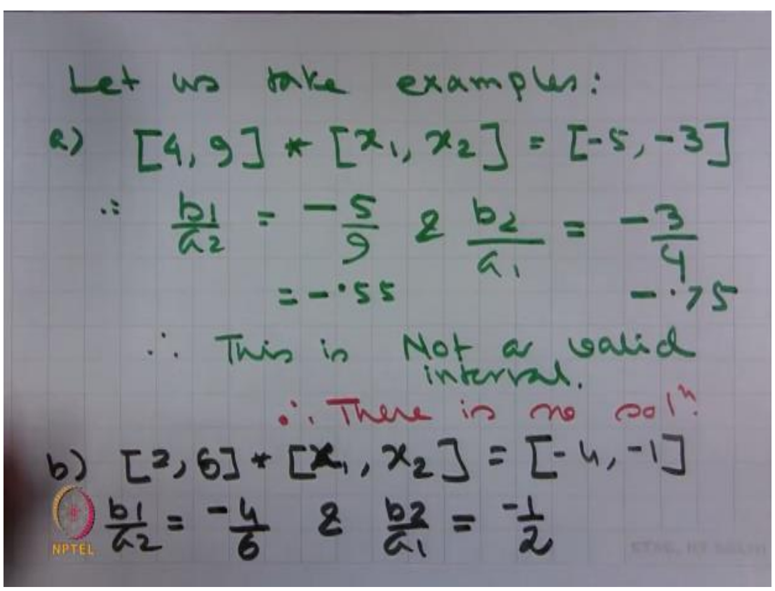

So let us take examples,

(a) $[4, 9] * [x_1, x_2] = [-5, -3]$

$$\therefore \frac{b_1}{a_2} = -\frac{5}{9} = -0.55 \text{ and } \frac{b_2}{a_1} = -\frac{3}{4} = -0.75$$

Therefore, this is not a valid interval. Therefore, no solution exists

(b) $[2, 6] * [x_1, x_2] = [-4, -1]$

$$\therefore \frac{b_1}{a_2} = -\frac{4}{6} = \text{ and } \frac{b_2}{a_1} = -\frac{1}{2}$$

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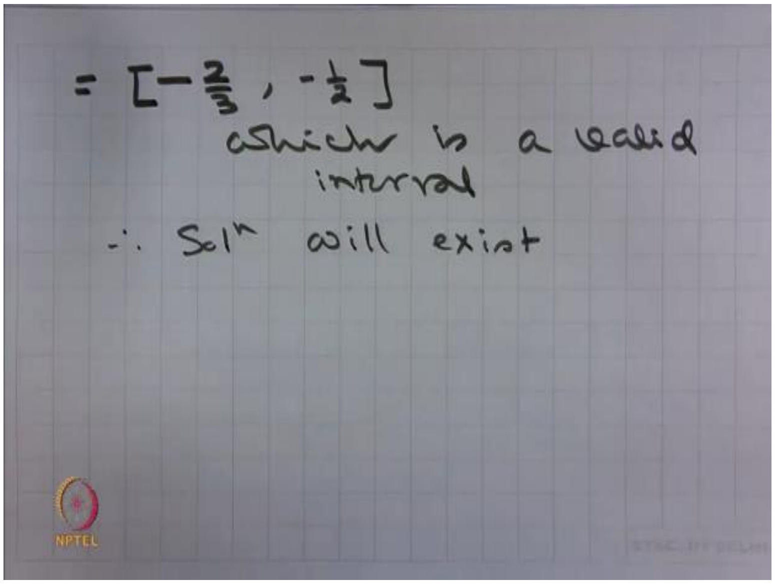

Therefore, the interval is $\left[-\frac{2}{3}, -\frac{1}{2}\right]$ which is a valid interval. Therefore, solution will exist.

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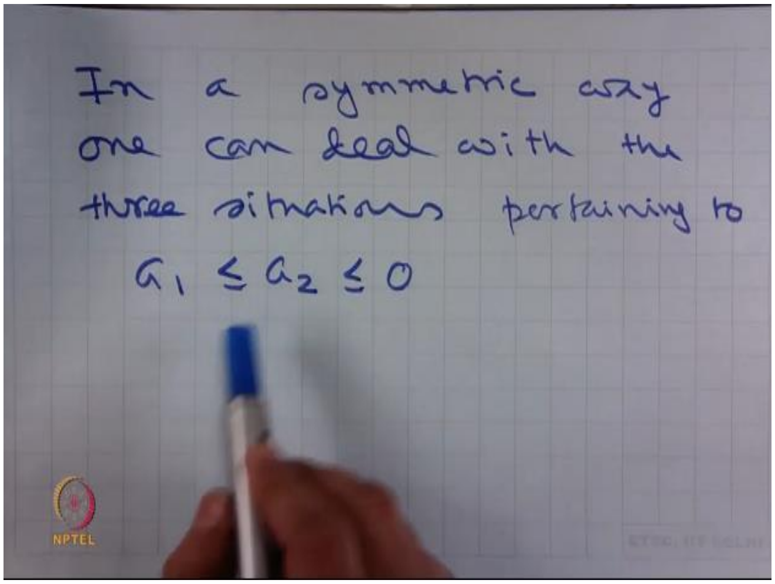

In a symmetric way one can deal with the three situations pertaining to $a_1 \leq a_2 \leq 0$

So I leave it as an exercise to see when solution will exist and when it will not.

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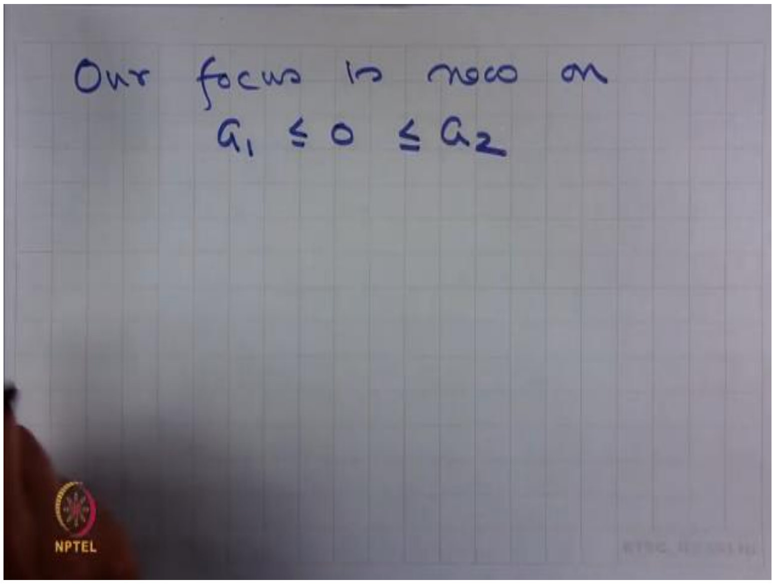

So our focus is now on $a_1 \leq 0 \leq a_2$

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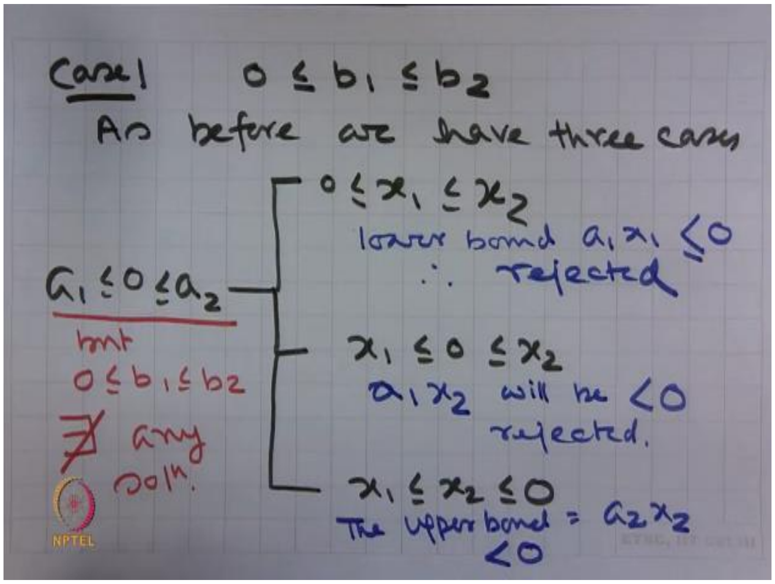

So Case 1:

$0 \leq b_1 \leq b_2$

As before we have three cases:

1. $0 \leq x_1 \leq x_2$

   Lower bound $a_1 x_1 \leq 0$. $\therefore$ Rejected

2. $x_1 \leq 0 \leq x_2$

   $a_1 x_2$ will be < 0 ∴ Rejected

3. $x_1 \leq x_2 \leq 0$

   The upper bound $= a_2 x_2 < 0$.

Therefore, if $a_1 \leq 0 \leq a_2$ but $0 \leq b_1 \leq b_2$ there does not exist any solution.

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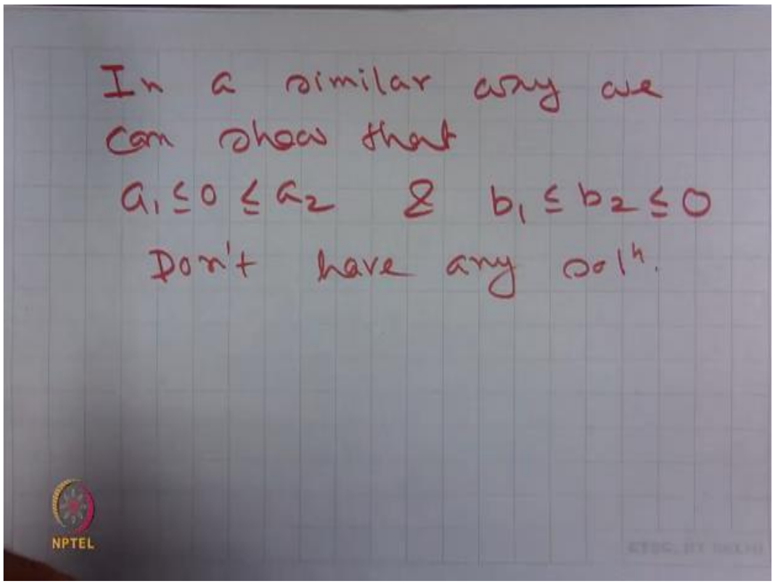

In a similar way we can show that $a_1 \leq 0 \leq a_2$ and $b_1 \leq b_2 \leq 0$ do not have any solution.

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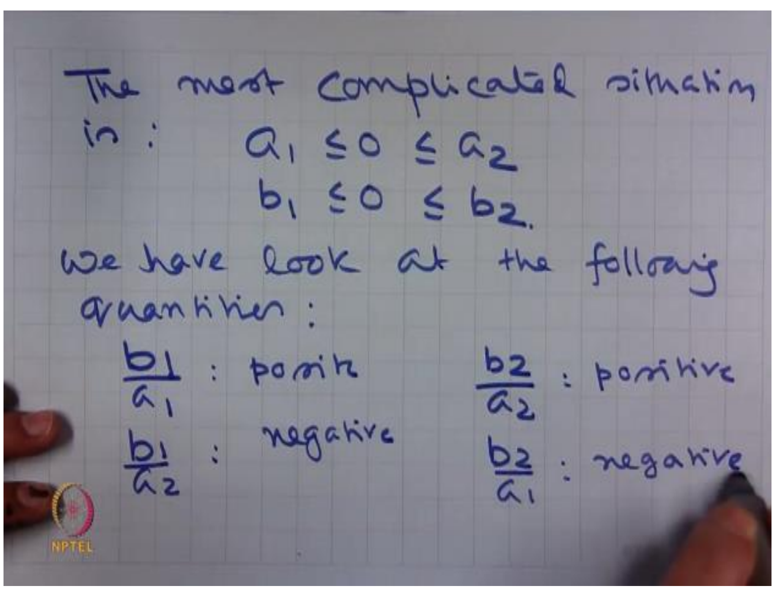

Therefore, the most complicated situation is when $a_1 \leq 0 \leq a_2$ and $b_1 \leq 0 \leq b_2$

So we have to look at the following quantities:

$\frac{b_1}{a_2}$: Positive $\frac{b_2}{a_2}$: Positive

$\frac{b_1}{a_2}$: Negative $\frac{b_2}{a_1}$: Negative

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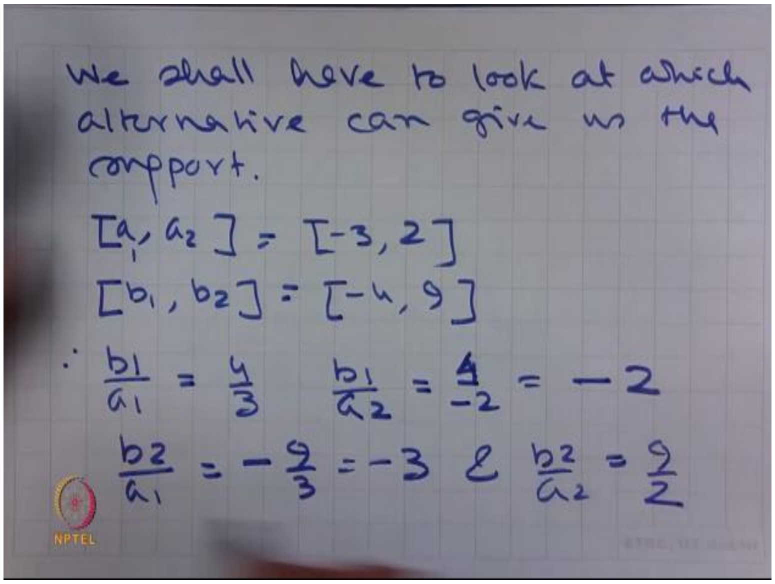

Therefore, we shall have to look at which alternative can give us the support.

Let us give an example

$$[a_1, a_2] = [-3, 2]$$

$$[b_1, b_2] = [-4, 9]$$

Therefore,

$$\frac{b_1}{a_2} = \frac{4}{3} \qquad \frac{b_2}{a_2} = \frac{9}{2}$$

$$\frac{b_1}{a_2} = \frac{4}{-2} = -2 \qquad \frac{b_2}{a_1} = -\frac{9}{3} = -3$$

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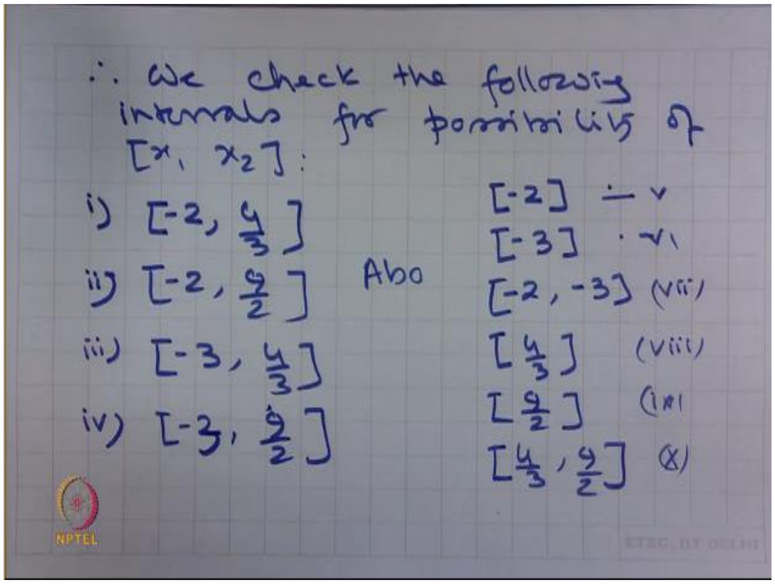

Therefore, we check the following intervals for possibility of $[x_1, x_2]$:

i. $\left[-2, \frac{4}{3}\right]$

ii. $\left[-2, \frac{9}{2}\right]$

iii. $\left[-3, \frac{4}{3}\right]$

iv. $\left[-3, \frac{9}{2}\right]$

v. $\{-2\}$

vi. $\{-3\}$

vii. $[-2, -3]$

viii. $\left[\frac{4}{3}\right]$

ix. $\left[\frac{9}{2}\right]$

x. $\left[\frac{4}{3}, \frac{9}{2}\right]$

So if the solution exists any one of them should give us the answer.

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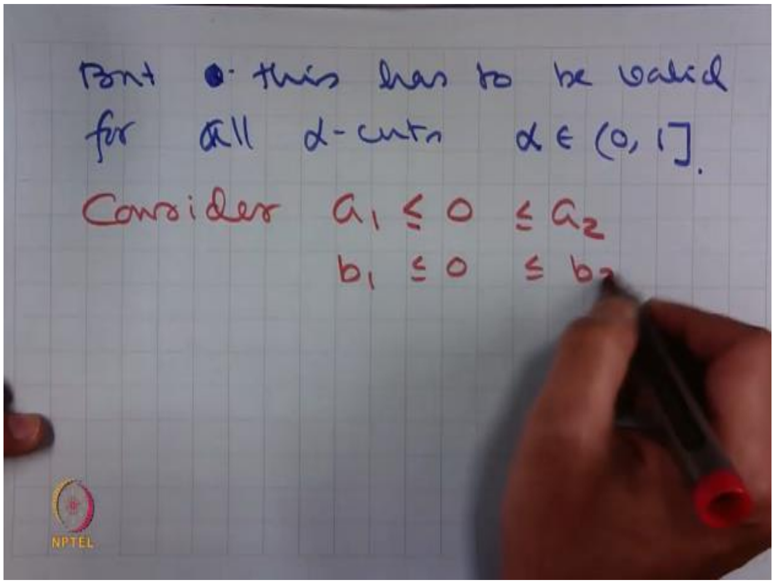

As before here we are looking at only the interval arithmetic but these has to be valid for all α-cuts $\alpha \in (0,1]$

So how to find it for all α-cuts?

I give you a trick.

Consider the situation when $a_1 \leq 0 \leq a_2$ and $b_1 \leq 0 \leq b_2$

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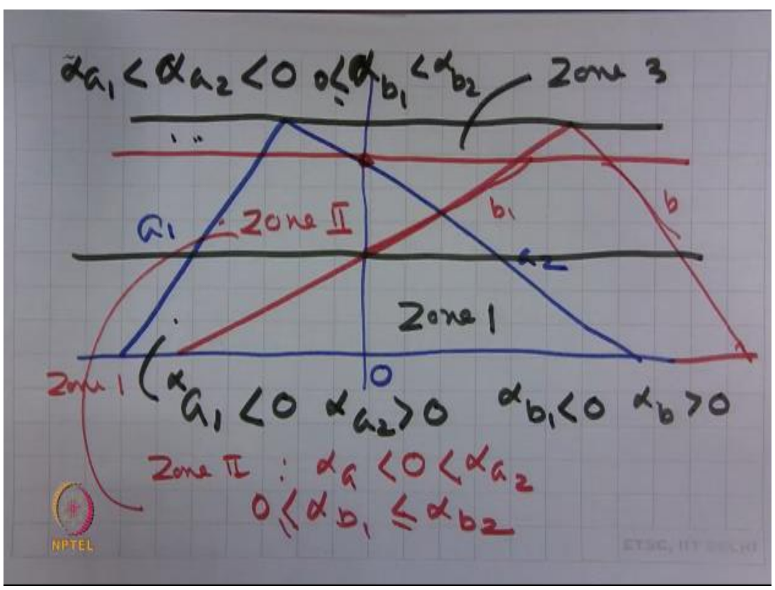

We draw the following diagram.

So like that wherever the lines are crossing the y −axis we have to split it into different zones with respect to each zone we get the conditions and the way we have solved earlier we have to see if for this zone solution exists or not like that after analyzing all the cases we can ascertain if we can get a solution for the given multiplicative fuzzy equation.

Okay students I stop here today, if you want I can upload a small document where it will analyze all cases and will give you clues to in which situation one can find a solution.
Okay thank you so much. With this I complete my lectures on fuzzy arithmetic in the next class I shall start with fuzzy relations. Thank You.