Introduction to Fuzzy Sets Arithmetic and Logic
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Lecture -16
Fuzzy Sets Arithmetic and Logic

Welcome students to the 16th lecture of the MOOCs course on Fuzzy Sets Arithmetic and Logic.

In the last class we have studied the extension principle which allows us to extend the notion of a function from crisp sets to the domain of fuzzy sets.

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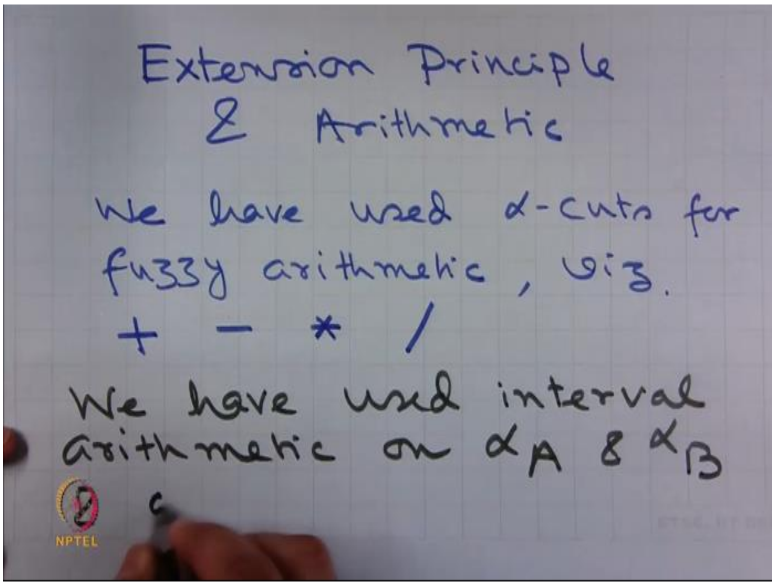

Today in this class we shall look at Extension principle and Arithmetic with respect to fuzzy sets.

Earlier we have done fuzzy arithmetic and if you remember we have used α-cuts sfor fuzzy arithmetic namely $+ \quad - \quad * \; /$

If you remember we have done them, we have used interval arithmetic on ${}^{\alpha}A$ and ${}^{\alpha}B$ and we have used that

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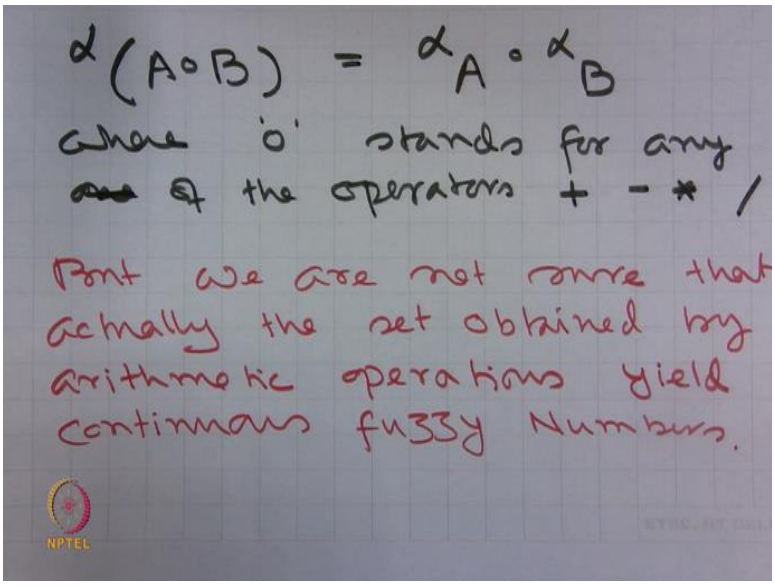

$^{\alpha}(A \circ B) = {}^{\alpha}A \circ {}^{\alpha}B$, where $\circ$ stands for any of the operators $+ \; - \; * \; /$

But, we are not sure that actually the set obtained by arithmetic operations yield continuous fuzzy numbers.

We have sort of assumed that result, when we are operating between two alpha cuts and that will indeed generate the α-cut of the final result.

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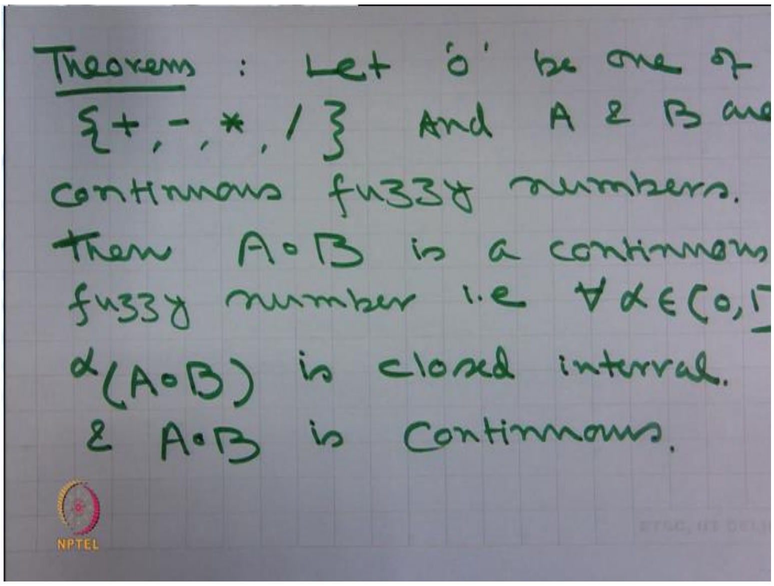

So the following theorem establishes that fact where we have used Extension Principle.

Theorem:

Let $\circ$ be one of $\{+, -, *, /\}$ and A and B are continuous fuzzy numbers.

Then $A \circ B$ is a continuous fuzzy number that is for all $\alpha \in (0, 1]$, $^{\alpha}(A \circ B)$ is a closed interval and $A \circ B$ is continuous.

If we show that, then we justify why you have used the interval operations on different α-cuts to get the final result.

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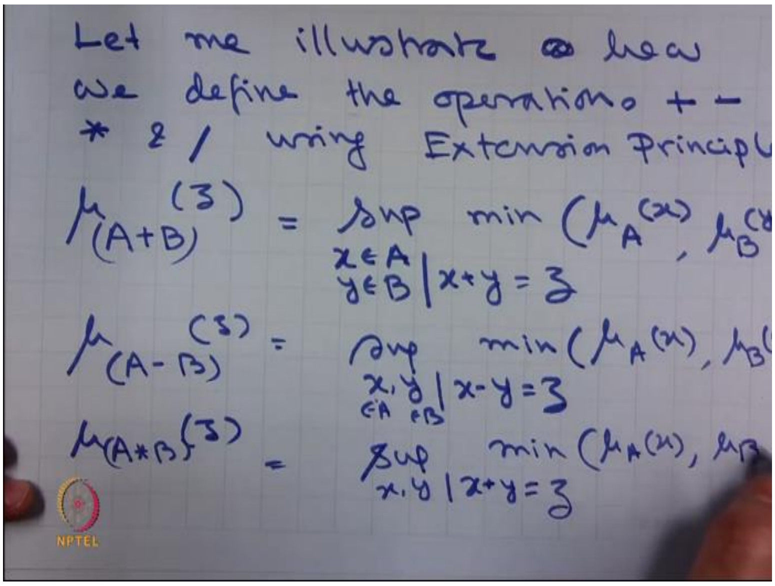

So let me illustrate how we define the operations $+, -, \ *, \ /$ using extension principle,

$$\mu_{A+B}(z) = \sup_{x \in A\ y \in B | x+y=z} \{\min(\mu_A(x), \mu_B(y))\}$$

$$\mu_{A-B}(z) = \sup_{x \in A\ y \in B | x-y=z} \{\min(\mu_A(x), \mu_B(y))\}$$

$$\mu_{A*B}(z) = \sup_{x \in A\ y \in B | x*y=z} \{\min(\mu_A(x), \mu_B(y))\}$$

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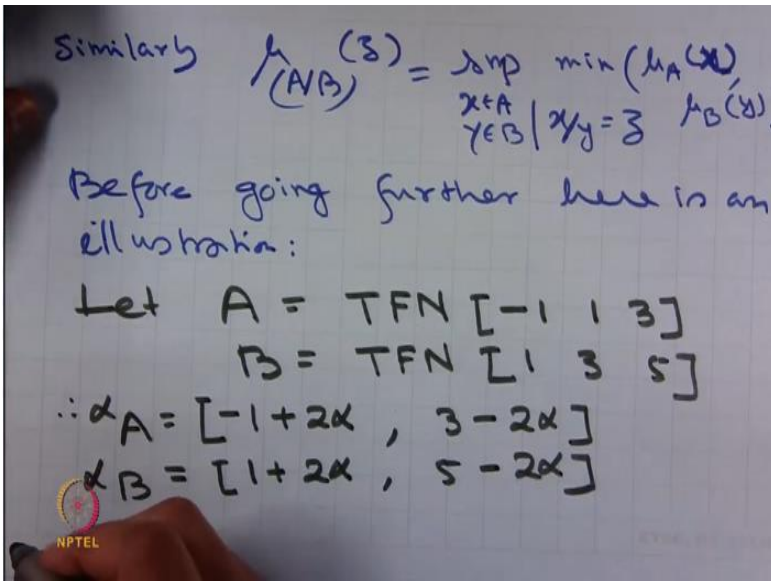

And similarly

$$\mu_{A/B}(z) = \sup_{x \in A\ y \in B | x/y=z} \{\min(\mu_A(x), \mu_B(y))\}$$

Before going further here is an illustration:

Let $A = [-1 \quad 1 \quad 3]$ and $B = [1 \quad 3 \quad 5]$

Therefore, ${}^{\alpha}A = [-1 + 2\alpha, 3 - 2\alpha]$ and ${}^{\alpha}B = [1 + 2\alpha, 5 - 2\alpha]$

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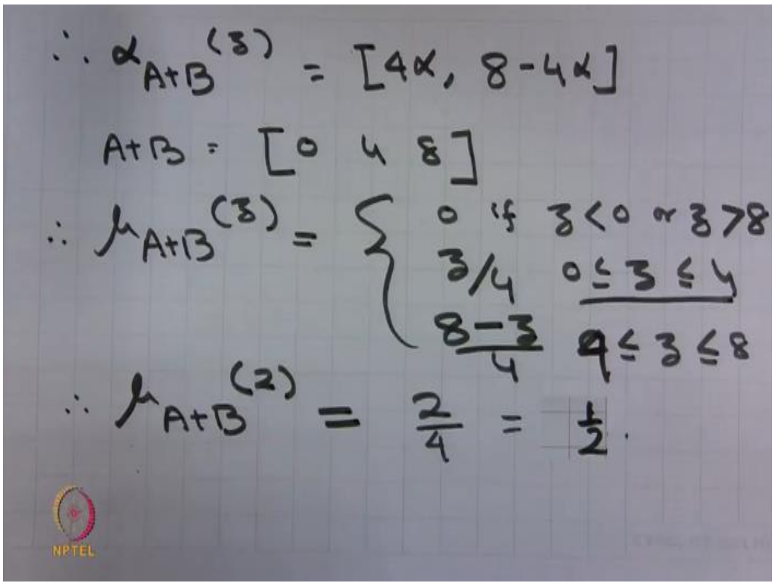

Therefore, by interval operation

$${}^{\alpha}(A + B) = [4\alpha, 8 - 4\alpha]$$

We have already seen that $A + B = [0 \quad 4 \quad 8]$

Therefore, $\mu_{A+B}(z) = \begin{cases} 0 & z < 0 \text{ or } z > 8 \\ \frac{z}{4} & 0 \le z \le 4 \\ \frac{8-z}{4} & 4 \le z \le 8 \end{cases}$

So , when $z = 8$ it is 0 when $z = 4$ it is 1.

Therefore, $\mu_{A+B}(2) = \frac{2}{4} = \frac{1}{2}$

Now we need to check by using the formula whether we can get a $\mu(2) = \frac{1}{2}$

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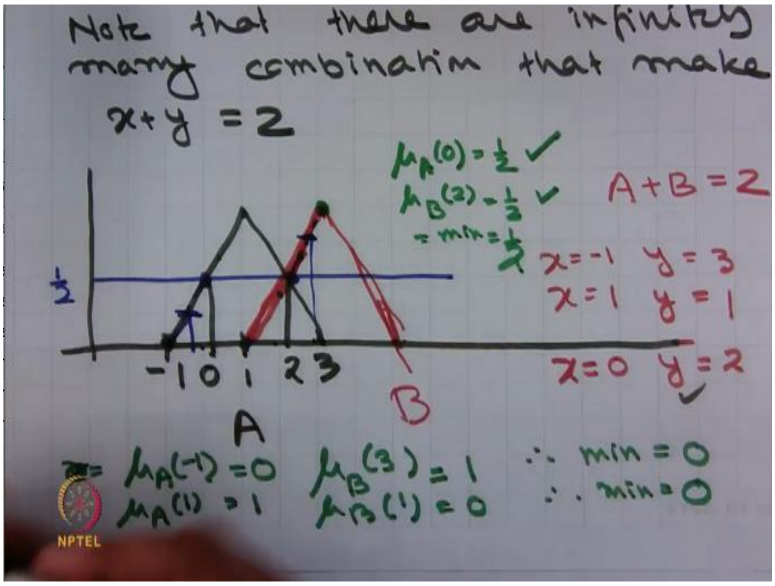

So let us do that.

Note that there are infinitely many combinations that make $x + y = 2$

Let us look at in how many ways we can get $A + B = 2$

When

$x = -1\ ; y = 3,$

$x = 1\ ; y = 1,$

$x = 0\ ; y = 2$

So let us see intuitively what is happening?

$\mu_A(-1) = 0, \quad \mu_B(3) = 1 \quad \therefore min = 0$

$\mu_A(1) = 1, \quad \mu_B(1) = 0 \quad \therefore min = 0$

Now let us move suppose I increase the value of x from -1

Therefore, the value of B has to be appropriately reduced from 3 and possibility that these two when add up will give me 2. Similarly, as we go up and as we go down with this line, We find that at this point both of them attain the value half.

Therefore, what is this point this is 0 and what is this point it is 2?

Therefore, $\mu_A(0) = \frac{1}{2}, \quad \mu_B(2) = \frac{1}{2}$

and if we go above this will fall below half and if we take a point above this then here the μ falls below half so in all the cases the μ half will be less than half but when $x = 0$ and $y = 2$, the minimum is half therefore we can see the maximum possible value.

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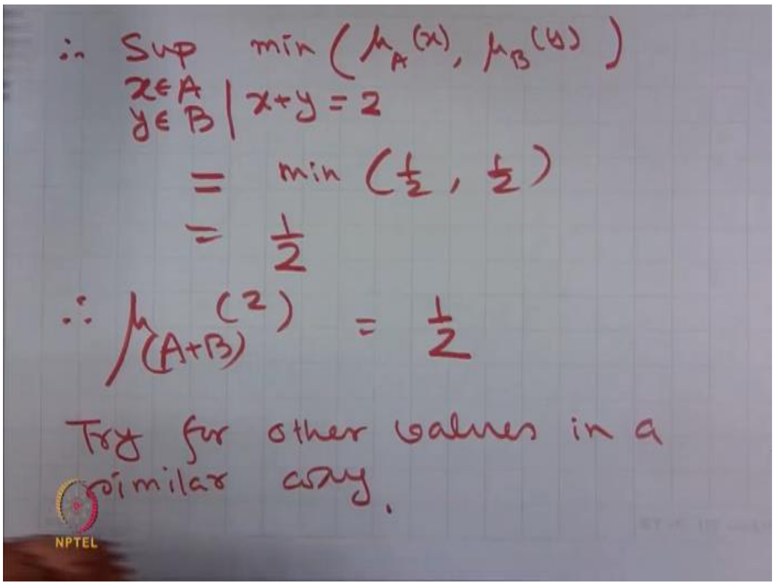

Or say therefore $\sup\limits_{x\in A, y\in B | x+y=2} \{\min(\mu_A(x), \mu_B(y))\} \;= \min\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}$

and in any case we have already seen this result sometime earlier but here we show that with the help of extension principle we obtain the same value.

I request you to try for other values in a similar way.

Okay, so with that background now, let me prove the theorem.

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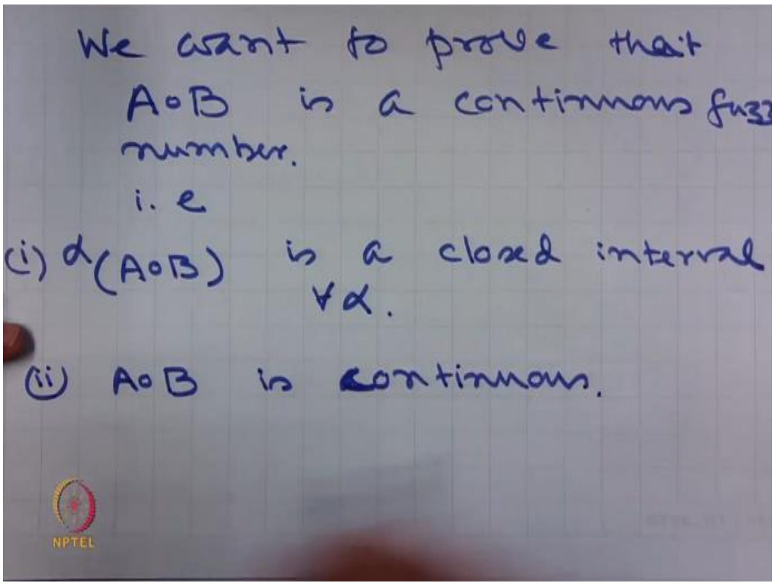

We want to prove that $A \circ B$ is a continuous fuzzy number that means

i. $^{\alpha}(A \circ B)$ is a closed interval $\forall\, \alpha$

ii. $A \circ B$ is continuous

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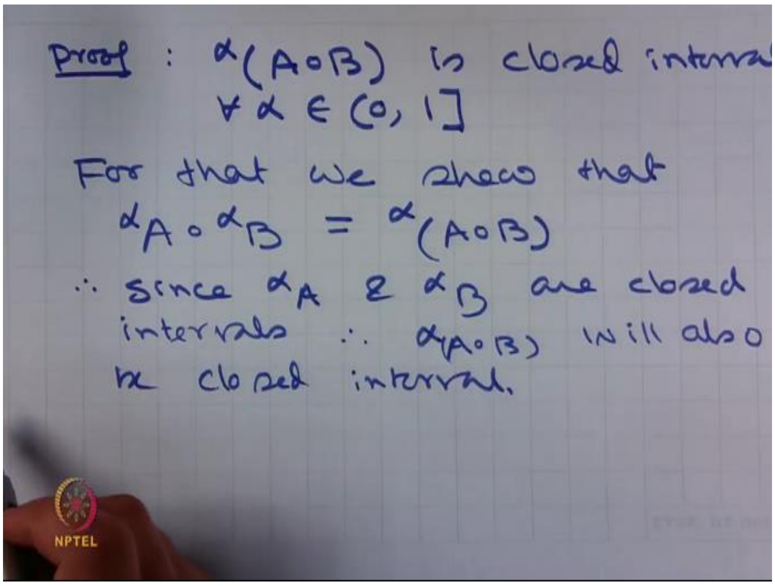

So let us first prove that ${}^{\alpha}(A \circ B)$ is closed interval $\forall \alpha \in (0, 1]$

For that we show that ${}^{\alpha}A \circ {}^{\alpha}B = {}^{\alpha}(A \circ B)$

Therefore, since ${}^{\alpha}A$ and ${}^{\alpha}B$ are closed intervals therefore, ${}^{\alpha}(A \circ B)$ will also be a closed interval and why do we say that these are closed interval?

Because it is already given that they are fuzzy numbers and we were doing interval arithmetic on that and we want to show that this is also going to be a closed interval.

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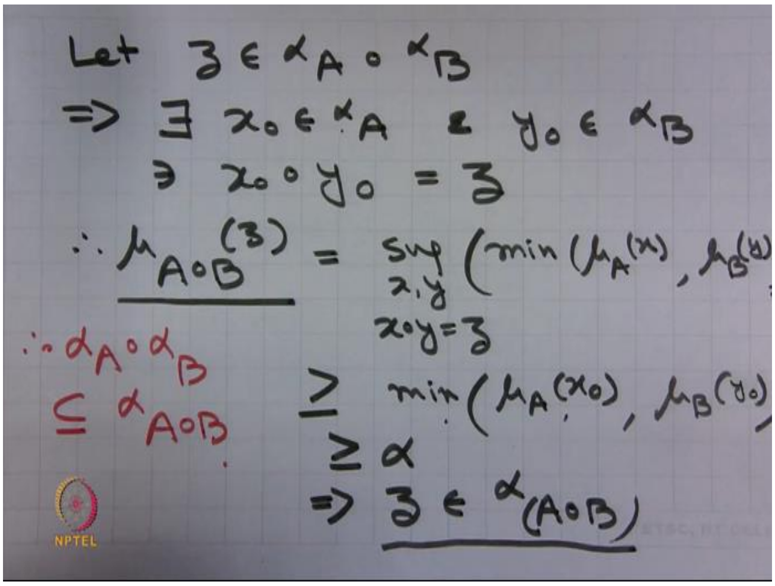

So let $z \in {}^{\alpha}A \circ {}^{\alpha}B$

$\Rightarrow \exists\, x_0 \in {}^{\alpha}A$ and $y_0 \in {}^{\alpha}B$ such that $x_0 \circ y_0 = z$

$$\therefore \mu_{A \circ B}(z) = \sup_{x \in A, y \in B | x \circ y = z} \{\min(\mu_A(x), \mu_B(y))\} \geq \min\{\mu_A(x_0), \mu_B(y_0)\}$$

And we know that both of them have membership value $\geq \alpha$ because they belong to their respective α-cuts therefore minimum has to be $\geq \alpha$

$$\therefore \mu_{A\circ B}(z) \geq \alpha \Rightarrow z \in {}^{\alpha}(A \circ B)$$

Therefore, we started with $z \in {}^{\alpha}A \circ {}^{\alpha}B$ and we find that $z \in {}^{\alpha}(A \circ B)$.

Therefore, we say that ${}^{\alpha}A \circ {}^{\alpha}B \subseteq {}^{\alpha}(A \circ B)$, so this is one way of the proof.

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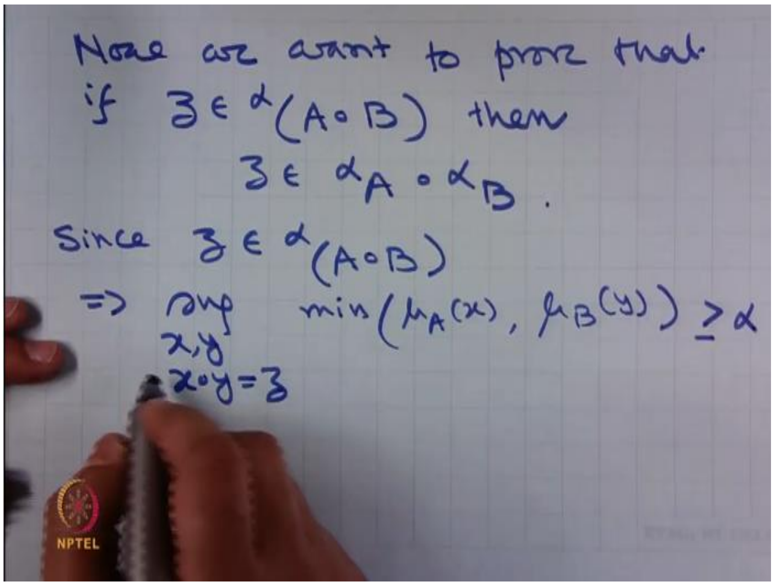

Now we want to prove that if $z \in {}^{\alpha}(A \circ B)$ then $z \in {}^{\alpha}A \circ {}^{\alpha}B$

This proof is slightly tricky; mathematically and it needs some knowledge of analysis, so try to understand it carefully.

Since $z \in {}^{\alpha}(A \circ B)$

$$\Rightarrow \sup_{x\in A, y\in B | x\circ y = z} \{\min(\mu_A(x), \mu_B(y))\} \geq \alpha$$

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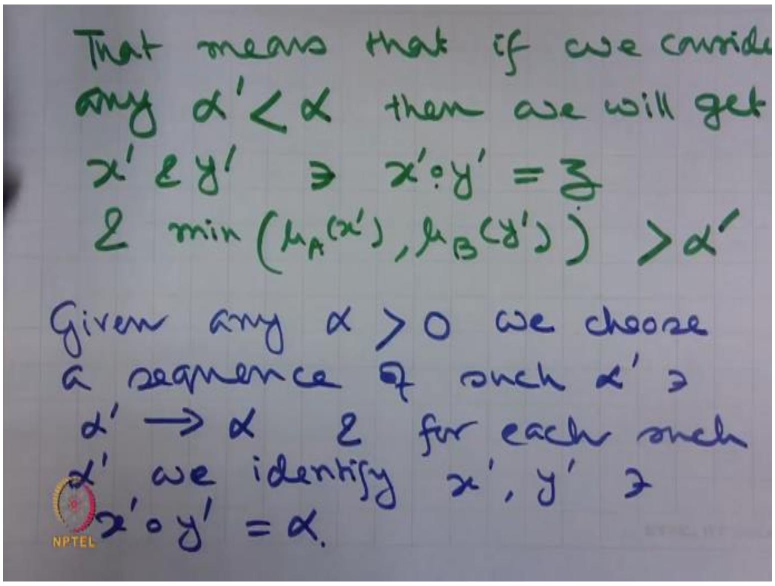

That means that if we consider any $\alpha' < \alpha$ then we will get x' and y' such that $x' \circ y' = z$ and $\min\{\mu_A(x'), \mu_B(y')\} > \alpha'$ because if we do not get such x' and y' then obviously the supremum of the minimum cannot be $\alpha > \alpha'$

Therefore the moment we choose any $\alpha' < \alpha$, we will get x', y'which will produce the same z under the operation but, the minimum of this has to be greater than α'.

Therefore, given any $\alpha > 0$, we choose a sequence of such α' such that α' converges to α and for each such α' we identify x', y' such that $x' \circ y' = \alpha$

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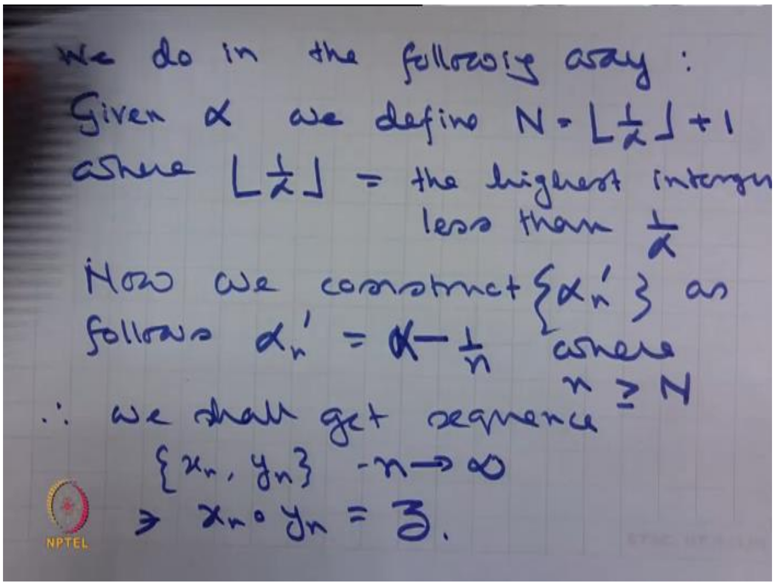

How do you do that?

We do in the following way:

Given α we define $N = \left\lfloor \frac{1}{\alpha} \right\rfloor + 1$ where, $\left\lfloor \frac{1}{\alpha} \right\rfloor$ = the highest integer that is less than $\frac{1}{\alpha}$

Now we construct $\{\alpha_n'\}$ as follows:

$\alpha_n' = \alpha - \frac{1}{n}$ where $n \geq N$

Therefore, as explained above we shall get a sequence $\{x_n, y_n\}$ $n \to \infty$ such that $x_n \circ y_n = z$

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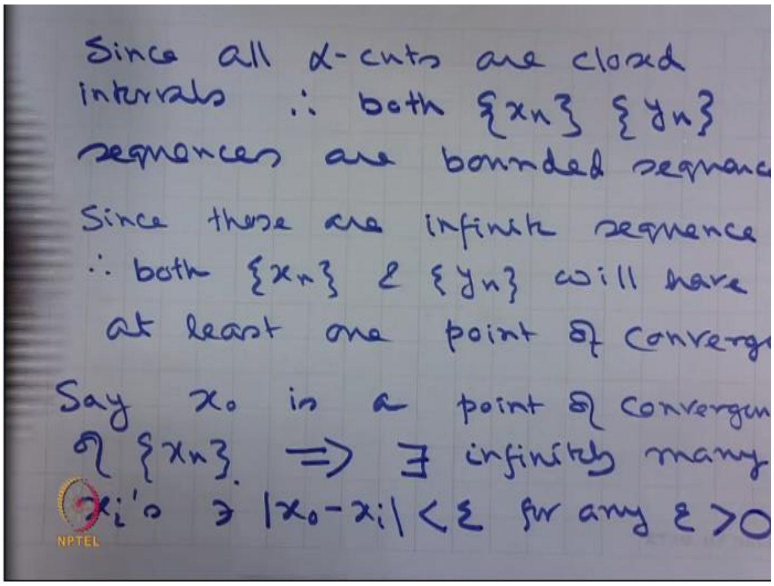

Since all α-cuts are closed intervals

Therefore, both $\{x_n\}$ and $\{y_n\}$ sequences are bounded sequence because we are taking numbers from within the α-cut and since these are infinite sequence therefore, both $\{x_n\}$ and $\{y_n\}$ will have at least one point of convergence.

Say x_0 is a point of convergence of $\{x_n\}$.

What does it mean?

It implies that there exists infinitely many $x_i's$ such that $|x_0 - x_i| < \epsilon$ for any $\epsilon > 0$.

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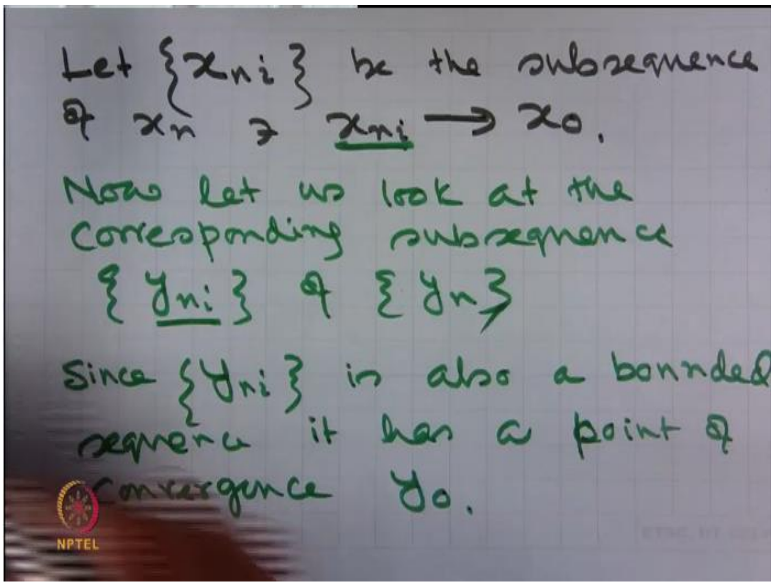

Let $\{x_{n_i}\}$ be the subsequence of x_n such that $x_{n_i} \to x_0$.

Now let us look at the corresponding subsequence namely $\{y_{n_i}\}$ of $\{y_n\}$.

So first we have chosen a subsequence which is converging to x_0 then we are only looking at those indices of that subsequence and we are looking at corresponding y values.

since $\{y_{n_i}\}$ is also a bounded sequence it has a point of convergence y_0.

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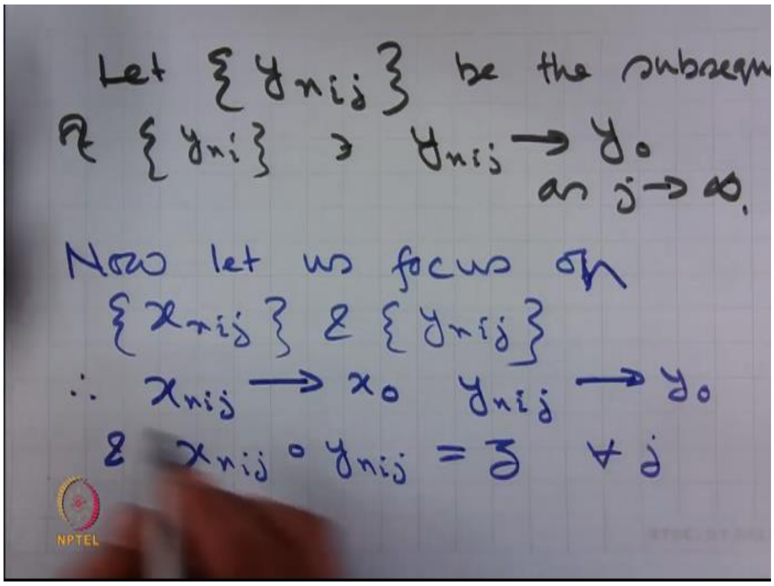

And let $\{y_{n_{i_j}}\}$ be the subsequence of $\{y_{n_i}\}$ such that $y_{n_{i_j}} \to y_0$ as $j \to \infty$.

Now let us focus on $\{x_{n_{i_j}}\}$ and $\{y_{n_{i_j}}\}$ and therefore, $x_{n_{i_j}} \to x_0$ and $y_{n_{i_j}} \to y_0$ and $x_{n_{i_j}} \circ y_{n_{i_j}} = z$ for all j.

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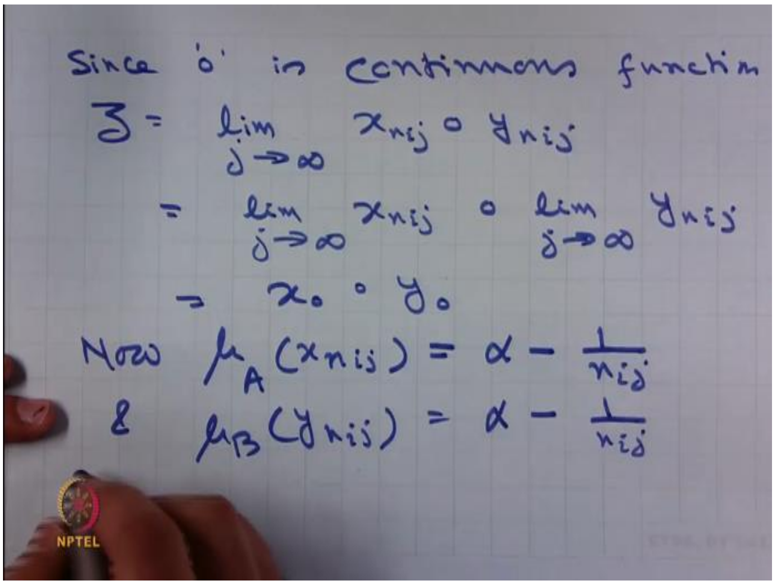

Now since $\circ$ is a continuous function.

$$z = \lim_{j\to\infty} x_{n_{i_j}} \circ\ y_{n_{i_j}}\ = \lim_{j\to\infty} x_{n_{i_j}} \circ\ \lim_{j\to\infty}\ y_{n_{i_j}}\ = x_0 \circ y_0$$

Now $\mu_A\left(x_{n_{i_j}}\right) = \alpha - \frac{1}{n_{i_j}}$ and $\mu_B\left(y_{n_{i_j}}\right) = \alpha - \frac{1}{n_{i_j}}$

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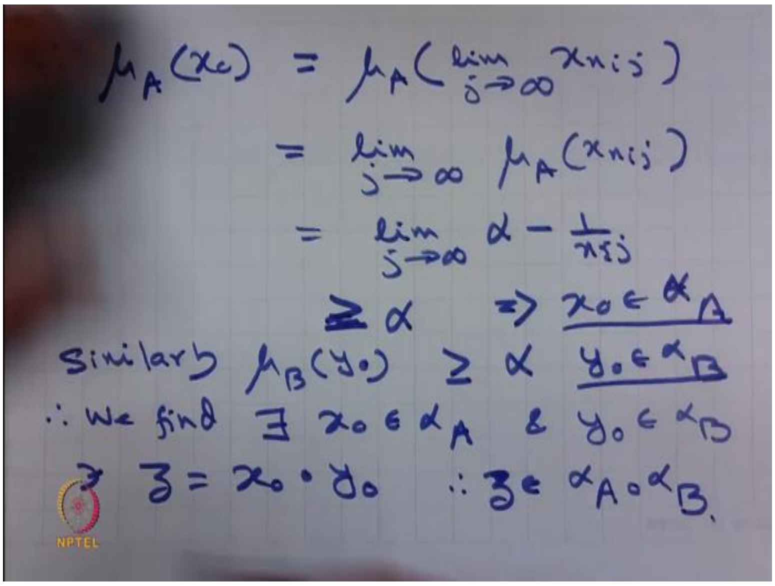

Therefore, $\mu_A(x_0) = \mu_A\left(\lim_{j\to\infty} x_{n_{i_j}}\right) = \lim_{j\to\infty} \mu_A\left(x_{n_{i_j}}\right) = \lim_{j\to\infty} \alpha - \frac{1}{n_{i_j}} \geq \alpha$

Similarly, $\mu_B(y_0) \geq \alpha$

Therefore, we find there exists $x_0 \in\ {}^{\alpha}A$ and $y_0 \in\ {}^{\alpha}B$ such that $z = x_0 \circ y_0$

Therefore, $z \in\ {}^{\alpha}A \circ\ {}^{\alpha}B$.

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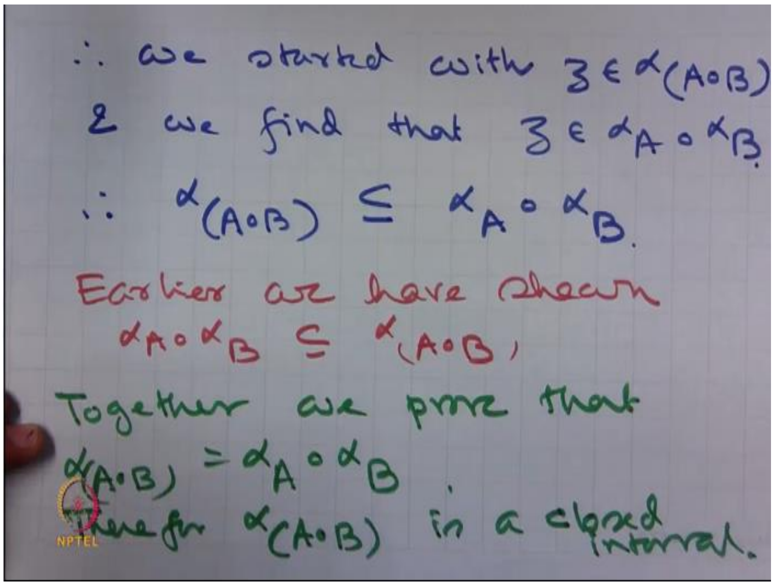

Therefore, we started with $z \in\ {}^{\alpha}(A \circ B)$ and we find that $z \in\ {}^{\alpha}A \circ\ {}^{\alpha}B$

$$\therefore\ {}^{\alpha}(A \circ B) \subseteq {}^{\alpha}A \circ {}^{\alpha}B$$

Earlier we have shown ${}^{\alpha}A \circ {}^{\alpha}B \subseteq {}^{\alpha}(A \circ B)$

So together we proved that ${}^{\alpha}A \circ {}^{\alpha}B = {}^{\alpha}(A \circ B)$

and therefore, ${}^{\alpha}(A \circ B)$, is a closed interval.

In our proof we did not choose any particular α and therefore since it is true for all α therefore ${}^{\alpha}(A \circ B)$ is a closed interval for all α.

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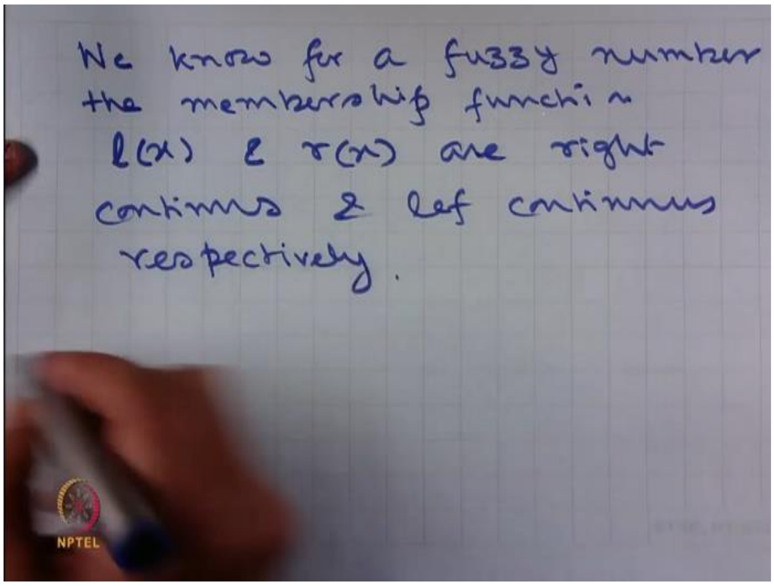

Now what remains to be proved is that $A \circ B$ is continuous.

Suppose $A \circ B$ is not continuous and let z_0 be the point of discontinuity, now we know that for a fuzzy number the membership function $l(x)$ and $r(x)$ are right continuous and left continuous respectively.

If you remember quite a few classes back when we were discussing the membership of a fuzzy number, we have these characteristics.

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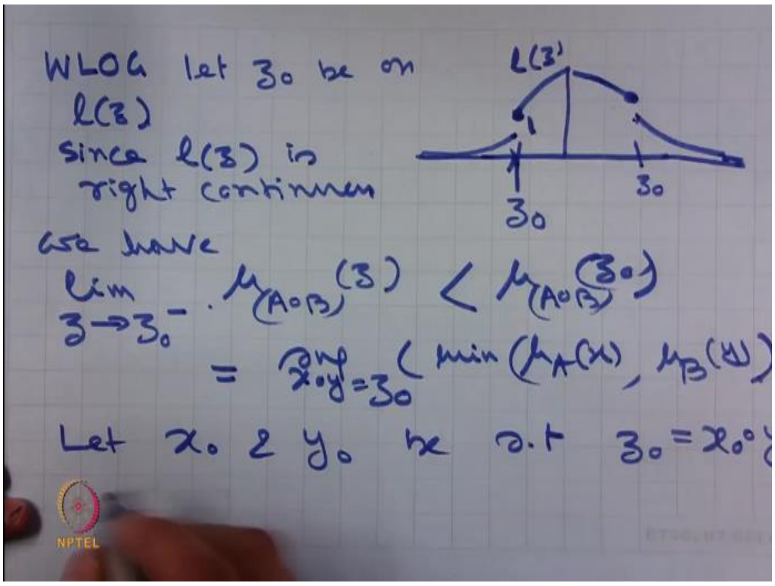

So let us look at the membership function.

So suppose this is the $l(z)$ and suppose this is a point of discontinuity, therefore as we come from this side we will come close to this point but there is going to be a gap between them or a jump between them because this is a point of discontinuity.

So, without loss of generality let z_0 be on this left side; if it is on right side this is the same similar situation if it is a point of discontinuity we will get a very similar structure here.

In either case, it will be the same so, when you proof for $l(z)$ it will also hold good on this side.

Since $l(z)$ is right continuous we have $\lim_{z \to z_0^-} \mu_{A \circ B}(z) < \mu_{A \circ B}(z_0)$ as we can understand there is a gap therefore membership has to be strictly less than $\mu_{A \circ B}(z_0)$

$$\lim_{z \to z_0^-} \mu_{A \circ B}(z) < \mu_{A \circ B}(z_0) = \sup_{x \circ y = z} \min\big(\mu_A(x), \mu_B(y)\big)$$

Let x_0 and y_0 be such that $z_0 = x_0 \circ y_0$

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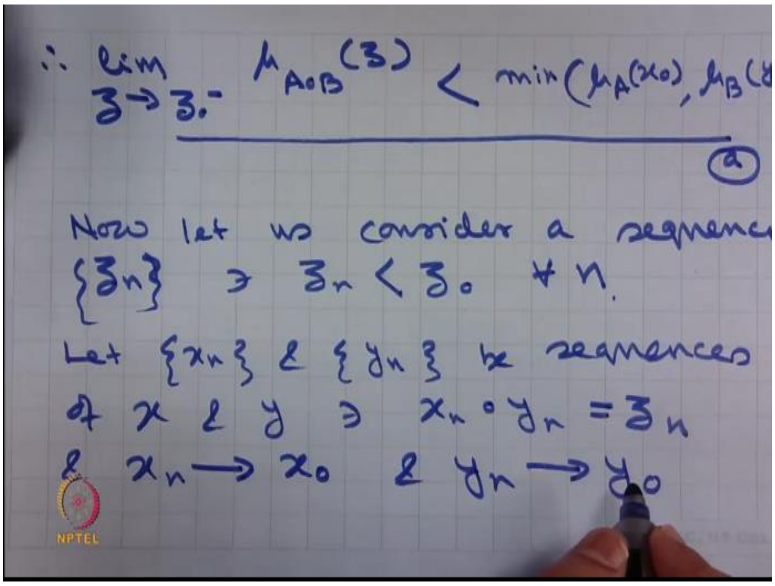

Therefore,

$$\lim_{z \to z_0^-} \mu_{A \circ B}(z) < \min\big(\mu_A(x_0), \mu_B(y_0)\big) \qquad ----(a)$$

So we establish one inequality let us call it (a)

Now let us consider a sequence $\{z_n\}$ such that $z_n < z_0$ for all n.

Let$\{x_n\}$ and $\{y_n\}$ be sequences of x and y such that $x_n \circ y_n = z_n$ and $x_n \to x_0$ and $y_n \to y_0$

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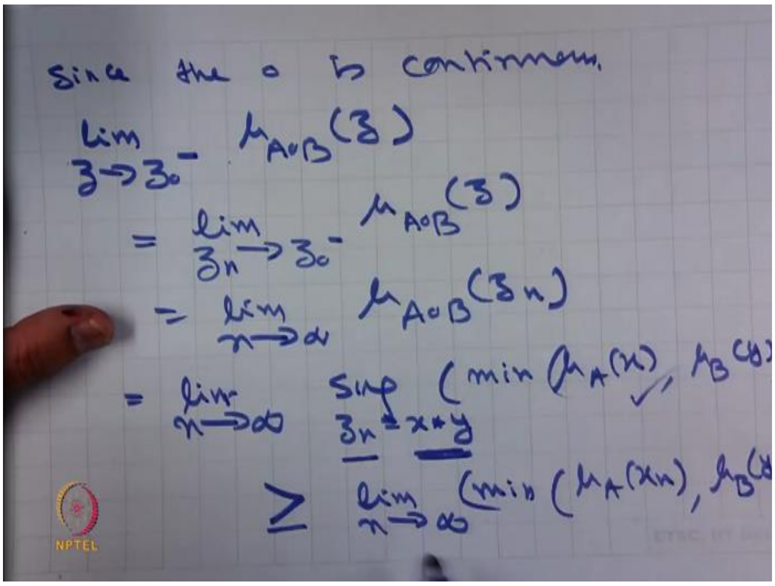

Since the operations are continuous, therefore

$$\lim_{z \to z_0^-} \mu_{A \circ B}(z) = \lim_{z_n \to z_0^-} \mu_{A \circ B}(z) = \lim_{n \to \infty} \mu_{A \circ B}(z_n)$$

$$= \lim_{n \to \infty} \sup_{x \circ y = z_n} \min\big(\mu_A(x), \mu_B(y)\big)$$

$$\geq \lim_{n \to \infty} \min\big(\mu_A(x_n), \mu_B(y_n)\big)$$

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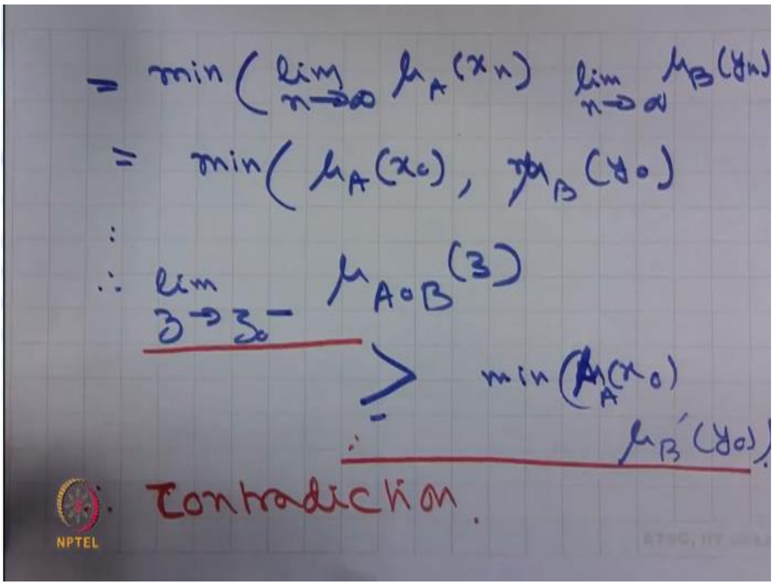

$= \min\left(\lim_{n\to\infty} \mu_A(x_n), \lim_{n\to\infty} \mu_B(y_n)\right)$

$= \min(\mu_A(x_0), \mu_B(y_0))$

Therefore, we find that $\lim_{z\to z_0^-} \mu_{A\circ B}(z) \geq \min(\mu_A(x_0), \mu_B(y_0))$

Earlier if you see we have shown that $\lim_{z\to z_0^-} \mu_{A\circ B}(z) < \min(\mu_A(x_0), \mu_B(y_0))$ and now we have identified that inequality is reversed.

Therefore, there is a contradiction.

This contradiction came because we assume that there exists a point of discontinuity at z_0.

Therefore, what we prove that there cannot be any point of discontinuity in $A \circ B$, therefore, that is a continuous function.

Since we have already found that all intervals are closed intervals and here we find that $A \circ B$ is a continuous therefore, we conclude that $A \circ B$ is a continuous fuzzy number.

And therefore that allows us to work on the ${}^{\alpha}A$ and ${}^{\alpha}B$ to generate the ${}^{\alpha}(A \circ B)$ where $\circ$ is any of addition subtraction multiplication and division.

Okay friends, I stop here today. In the next class I shall do some more arithmetic particularly in solving certain fuzzy equations. Thank you.