Introduction to Fuzzy Sets Arithmetic and Logic Prof. Niladri Chatterjee Department of Mathematics Indian Institute of Technology minus Delhi

Lecture minus 15 Fuzzy Sets Arithmetic and Logic

Welcome students to the MOOCs course on introduction to Fuzzy Set, Arithmetic and Logic.

This is lecture number 15 as I said at the end of the last lecture,

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Extension Principle. This principle allows us to extend the concept of mapping from crisp zets to fuzzy sets. To keep things simple we focus on Fuzzy sets defined on

That today what I will start is called extension principal proposed by none other than Professor Zadeh.

This principle allows us to extend the concept of mapping from crisp sets to fuzzy sets. To keep things simple we focus on fuzzy sets defined on intervals on \mathbb{R} that is the real line.

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So for we have seen mapping. or functions from comparis of IR to R For example consider f(x) = x Let A = [1,3] and let B = f(A) ie B= FXIX= for, xEA]

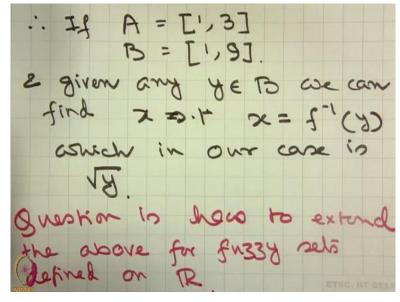
So, far we have seen mappings or functions from subsets of \mathbb{R} to \mathbb{R} ,

For example: Consider $f(x) = x^2$

Let A = [1, 3] and let B = f(A)

 $i.e.B = \{y|y = f(x), x \in A\}$

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Therefore, if A = [1, 3], B = [1, 9]

and given any $y \in B$, we can find x such that $x = f^{-1}(y)$ which in our case is \sqrt{y} Now question is: How to extend the above for fuzzy sets defined on \mathbb{R} ? (**Refer Slide Time: 05:08**)

Suppose A is TFN [123] Then for each y E [9, 9] we need to compute its membership to B. In the above can we can do it ivery simply: MB(S) = MA(VE) gin general ((S'(3) = MA(S'(3)

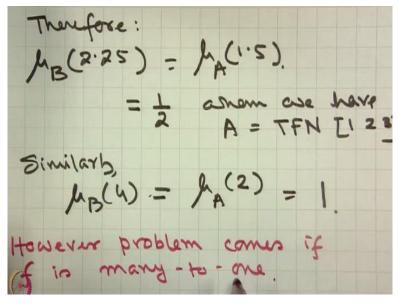
Suppose *A* is the triangular fuzzy number $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, then for each $y \in \begin{bmatrix} 1, 9 \end{bmatrix}$, we need to compute its membership to B.

In the above case we can do it very simply,

$$\mu_B(y) = \mu_A(\sqrt{y})$$

or in general $\mu_B(y) = \mu_A(f^{-1}(y))$

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Therefore, $\mu_B(2.25) = \mu_A(1.5) = \frac{1}{2}$ when we have $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ Similarly, $\mu_B(4) = \mu_A(2) = 1$

However, problem comes if f is many-to-one.

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For example, consider $A = TFN \quad E = 2 \quad 0 \quad 3 \\ and \quad lef \quad f(n) = 2^2.$ f(-1) = f(-1) = 1Question: ashat is MB(1)? Note that here B [0,9]

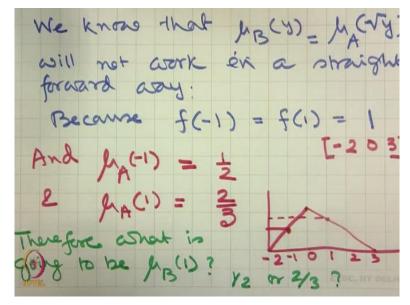
For example: Consider $A = \begin{bmatrix} -2 & 0 & 3 \end{bmatrix}$ and let $f(x) = x^2$

Therefore, f(-1) = f(1) = 1

Question: What is $\mu_B(1)$?

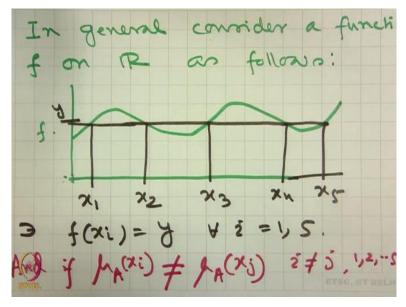
Note that: B = [0, 9] because if my input is from $[-2 \ 0 \ 3]$ then the value of x^2 will lie in the interval [0, 9]

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We know that $\mu_B(y) = \mu_A(\sqrt{y})$ will not work easily in a straight forward way. Because f(-1) = f(1) = 1 and since we have the TFN $\begin{bmatrix} -2 & 0 & 3 \end{bmatrix}$ $\mu_A(-1) = \frac{1}{2}$ and $\mu_A(1) = \frac{2}{3}$ therefore, what is going to be $\mu_B(1)$? It is $\frac{1}{2}$ or $\frac{2}{3}$ that is the question.

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In general consider a function f on \mathbb{R} as follows and

Suppose we consider *y* is like this.

Therefore, we get at least 5 points x_1, x_2, x_3, x_4, x_5 such that $f(x_i) = y$ for all i = 1, ... 5And if $\mu_A(x_i) \neq \mu_A(x_j), i \neq j \in \{1, 2 ... 5\}$

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Then determining $M_B(Y)$ is not straightforward. Another problem is could is 5'(Y) where $f(x_1) - f(x_2) = - = -f(x_1)$

Then determining $\mu_B(y)$ is not straight forward.

Another problem is: What is $f^{-1}(y)$ where $f(x_1) = f(x_2) = \cdots = f(x_5) = y$? (**Refer Slide Time: 15:04**)

Thus are need policies for 1) Determining 5'(3). 2) Determinging MB(3) And both of them will need to consider : to consider: . All X, X2---. > & (Xi) = y. · MA(Xi) + 2

Thus we need policies for:

- 1) determining $f^{-1}(y)$
- 2) determining $\mu_B(y)$

And both of them will need to consider

- All x_1, x_2 such that $f(x_i) = y$
- $\mu_A(x_i) \forall i$

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To take care of such multiple values we extend the domain 2 range of f: X -> Y to P(x) & P(x) ashere P denetro the Power set. To elluptrate let us take a discrete set X = 3-2,-1,0, 1,2,3 P(x) consists of all possible

To take care of such multiple values we extend the domain and range of $f: X \to Y$ to P(X) and

P(Y) where P denotes the power set.

To illustrate:

Let us take a discrete set $X = \{-2, -1, 0, 1, 2, 3\}$

Therefore, P(X) consists of all possible subsets of X

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In particular Let A + PCX). 1) $A = \{2, 2, 1, 0\}$ $f(A) = \{4, 1, 0\}$ 1) $A = \{2, 2, 1, 0\}$ $f(A) = \{4, 1, 0\}$ 1) $A = \{2, 2, 1, 0\}$ $f(A) = \{2, 1, 0\}$ 11) $A = \{2, 2, 1, 0\}$ $f(A) = \{2, 1, 0\}$ 11) $A = \{2, 1, 0\}$ $f(A) = \{2, 1, 0\}$ 11) $A = \{2, 1, 0\}$ $f(A) = \{2, 1, 0\}$ Thus we get prim a subset A A A A X, and a subset B $Y = \{0, 1, 4, 9\}$ $Y = \{0, 1, 4, 9$

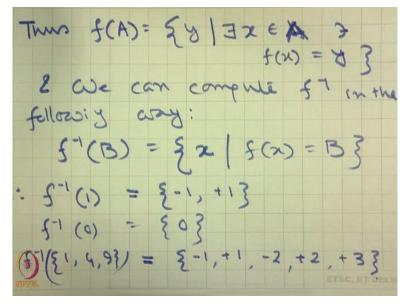
In particular, let $A \in P(X)$

i.
$$A = \{-2, 1, 0\}$$
 then $f(A) = \{4, 1, 0\}$

- ii. $A = \{-2, 2, -1, 1\}$ then $f(A) = \{4, 1\}$
- iii. $A = \{-1, 0, 1, 3\}$ then $f(A) = \{0, 1, 9\}$

Thus we get from a subset A of X, a subset B of $Y = \{0, 1, 4, 9\}$ such that for all $y \in B$ there exists $x \in A$ such that f(x) = y

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Thus $f(A) = \{y | \exists x \in A \text{ such that } f(x) = y\}$ and we can compute f^{-1} in the following way:

$$f^{-1}(B) = \{x | f(x) = B\}$$

Therefore, $f^{-1}(1) = \{-1, 1\}$ $f^{-1}(0) = \{0\}$

$$f^{-1}(\{1,4,9\}) = \{-1,1,-2,2,3\}$$

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The next problem is antignmin of fuggy membership in the event of A is a fuggy set defined or TR Consider for example: A = 2 0 + 12 + 1 + 2/3 + 13 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 Sucstim is have to obtain

The next problem is assignment of fuzzy membership in the event of *A* is a fuzzy set defined on \mathbb{R} .

Consider for example:

$$A = \left\{ \frac{0}{-2} + \frac{1/2}{-1} + \frac{1}{0} + \frac{2/3}{1} + \frac{1/3}{2} + \frac{0}{3} \right\}$$

Question is how to obtain $\mu_B(y)$?

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The formula is:

$$M_B(3) = \max_{Z} \{M_A(2) \mid f(X) = 4\}$$

 $\therefore M_B(1) = \max_{Z} \{M_A(2), M_A(-1)\}$
 $= \max_{Z} \{Z_{3}, M_{2}\}$
 $= 2/3$
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The formula is:

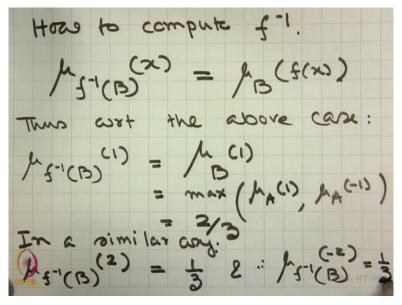
$$\mu_B(y) = \max_{x} \{\mu_A(x) | f(x) = y\}$$

$$\therefore \mu_B(1) = \max\{\mu_A(1), \mu_A(-1)\} = \max\{\frac{2}{3}, \frac{1}{2}\} = \frac{2}{3}$$

and $\mu_B(4) = \max\{\mu_A(2), \mu_A(-2)\} = \max\{\frac{1}{3}, 0\} = \frac{1}{3}$

In this way we can compute $\mu_B(y)$. How to compute f^{-1} ?

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Therefore, $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$

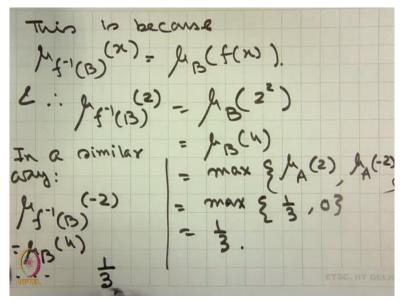
Thus, with respect to the above case:

$$\mu_{f^{-1}(B)}(1) = \mu_B(1) = \max\{\mu_A(1), \mu_A(-1)\} = \frac{2}{3}$$

In a similar way

$$\mu_{f^{-1}(B)}(2) = \frac{1}{3} \text{ and } \mu_{f^{-1}(B)}(-2) = \frac{1}{3}$$

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This is because $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$

And therefore, $\mu_{f^{-1}(B)}(2) = \mu_B(2^2) = \mu_B(4) = \max\{\mu_A(2), \mu_A(-2)\} = \max\{\frac{1}{3}, 0\} = \frac{1}{3}$ In a similar way $\mu_{f^{-1}(B)}(-2) = \mu_B(4) = \frac{1}{3}$

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This allows up to fuzzify a function on cripp sels to a function on FU33y sels. This gives us Extenden Principle:

This allows us to fuzzify a function on crisp sets to a function on fuzzy sets and this gives us Extension Principle,

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which says that: Any given function f: X->Y induces two functions: f: ye(x) -> ye(x) 57: Me(Y) -> Me(X) aner years & yerr) and set of all possible 1333 sets on X 27

Which says that

Any given function $f: X \to Y$ induces two functions:

- $f:\mathfrak{F}(X)\to\mathfrak{F}(Y)$
- f^{-1} : $\mathfrak{F}(Y) \to \mathfrak{F}(X)$

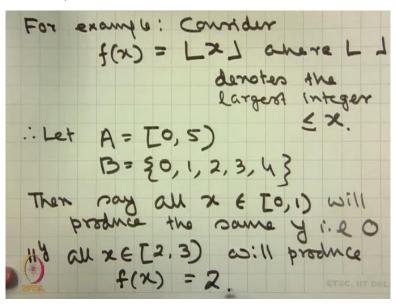
where $\mathfrak{F}(X)$ and $\mathfrak{F}(Y)$ are set of all possible fuzzy sets on X and Y.

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Questim in ashat happens if the number & ashech produce the same y is

Question is: what happens if the number of x which produce the same y is infinite.

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For example:

Consider $f(x) = \lfloor x \rfloor$ where $\lfloor \circ \rfloor$ denotes the largest integer $\leq x$ Therefore, let A = [0, 5] $B = \{0, 1, 2, 3, 4, 5\}$ Then say all $x \in [0, 1)$ will produce the same y i.e. 0

Similarly, all $x \in [2, 3)$ will produce f(x) = 2.

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Question is in such cases head to compute hB(3)? The formula is: $\mu_B(y) = \Omega \psi (x)$ $\chi \in A1$ f(x) = g(x) $\chi \in A1$ f(x) = y $\chi \in L_{2}, y$ $\chi \in L_{2}, y$

Question is: in such cases how to compute $\mu_B(y)$?

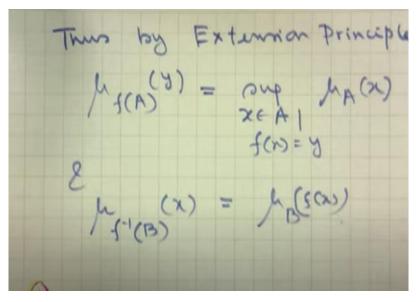
And the formula is:

$$\mu_B(y) = \sup_{x \in A \mid f(x) = y} \mu_A(x)$$
$$\therefore \mu_B(0) = \sup_{x \in [0,1)} \mu_A(x)$$

Thus, the notion of maximum is now replaced with supremum, as there are infinite number of values which produces the same y

We may not be able to identify the maximum therefore, we need to replace it with supremum.

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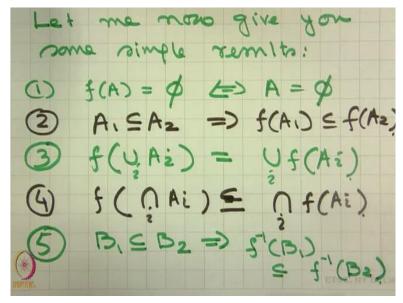
Thus the extension principle

$$\mu_{f(A)}(y) = \sup_{x \in A \mid f(x) = y} \mu_A(x)$$

And

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$$

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Let me now give you some simple results:

- 1) $f(A) = \phi \Leftrightarrow A = \phi$
- 2) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$
- 3) $f(\bigcup_i A_i) = \bigcup_i f(A_i)$
- 4) $f(\cap_i A_i) \subseteq \cap_i f(A_i)$

5)
$$B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

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(a)
$$f^{-1}(\bigcup Bi) = \bigcup f^{-1}(Bi)$$

(b) $f^{-1}(\bigcap Bi) = \bigcap f^{-1}(Bi)$
(c) $f^{-1}(\widehat{Bi}) = \bigcap f^{-1}(\widehat{Bi})$
(c) $f^{-1}(\widehat{Bi}) = \bigcap f^{-1}(\widehat{B$

6) $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$ 7) $f^{-1}(\cap_i B_i) = \cap_i f^{-1}(B_i)$

- 8) $A \subseteq f^{-1}(f(A))$
- 9) $B \supseteq f(f^{-1}(B))$

I leave these an exercise, these are already straight forward.

If you are understood how we are calculating the membership functions and how we are calculating the inverse, all these are absolutely straight forward to prove.

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Important Remlts: which shall that Extension Principle in strong cut worky but Not cut asorthy

So before I stop I give you an important result, which show that extension principle, is strong cut-worthy but not simply cut-worthy.

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Lef f: X > Y be an arbitrary crisp function. Them for any A & Je (X) and for all d E [0,1] the following properties of f fuzzified my the Extension principle hold $\alpha + (f(A)) = f(\alpha + A)$ $\alpha(f(A)) \ge \alpha f(x_A)$

So let $f: X \to Y$ be an arbitrary crisp function. Then for any $A \in \mathfrak{F}(X)$ that is A is a fuzzy set define over X and for all $\alpha \in [0, 1]$ the following properties of f fuzzified by the extension principle holds.

That means we have taken function f from X to Y we have fuzzified them using extension principle then we find the following two properties to hold.

What are these?

- 1) $\alpha^+(f(A)) = f(\alpha^+A)$
- 2) $\alpha(f(A)) \supseteq f(\alpha A)$

So this says that we $\alpha^+(f(A))$ through α^+A and apply f on them and therefore it is strong cut worthy but this containment relationships says that it is not simply cut worthy.

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Prof: dt(f(A)) = f(xtA) Let us consider y ∈ d+(s(A)) => / s(A) > x => sug /ma(x) 7 d. x15(x)= y => $\exists x_0 x_1 f(x_0) = \forall$ $g = f(x_0) f(x_0) = \forall$ $g = f(x_0) f(x_0) = \forall$ $g = f(x_0) = \forall g = f(x_0) = \forall f$

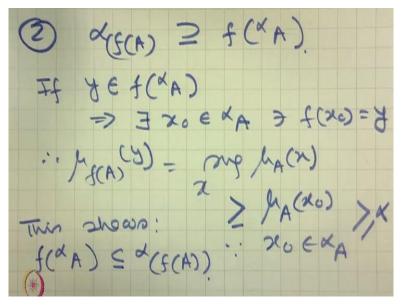
Proof: ${}^{\alpha+}(f(A)) = f({}^{\alpha+}A)$ Let us consider $y \in {}^{\alpha+}(f(A))$ $\Rightarrow \mu_{f(A)}(y) > \alpha$ $\Rightarrow \sup_{x|f(x)=y} \mu_A(x) > \alpha$ $\Rightarrow \exists x_0$ such that $f(x_0) = y$ and $\mu_A(x_0) > \alpha$ $\Rightarrow \exists x_0$ such that $f(x_0) = y$ and $x_0 \in {}^{\alpha+}A$ $\Rightarrow y \in f({}^{\alpha+}A)$ (Refer Slide Time: 46:35)

This shows d'(f(A)) & f(d'A) conversely. support y & f(x+A) => 3 xo 3 forc) = y & xo E K+A => $\exists x_0 \exists f(x_0) = \forall$ $\xi = (ox)f \in oxE \subset \xi$ $\exists M_A C \in OxAA \\ \forall M_A (x_0) > X$ $f(x) = \forall X = \forall E = x + f(A)$

This shows ${}^{\alpha+}(f(A)) \subseteq f({}^{\alpha+}A)$ Conversely, suppose $y \in f({}^{\alpha+}A)$ $\Rightarrow \exists x_0$ such that $f(x_0) = y$ and $x_0 \in {}^{\alpha+}A$ $\Rightarrow \exists x_0$ such that $f(x_0) = y$ and $\mu_A(x_0) > \alpha$ $\Rightarrow \sup_{x|f(x)=y} \mu_A(x) > \alpha$ $\Rightarrow \mu_{f(A)}(y) > \alpha$ $\Rightarrow y \in {}^{\alpha+}(f(A))$

So this proves the first result.

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Now let us prove the second result

 $^{\alpha}(f(A)) \supseteq f(^{\alpha}A)$

If $y \in f({}^{\alpha}A)$ $\Rightarrow \exists x_0 \in {}^{\alpha}A$ such that $f(x_0) = y$ $\therefore \mu_{f(A)}(y) = \sup_{x} \mu_A(x) \ge \mu_A(x_0) \ge \alpha \qquad \because x_0 \in {}^{\alpha}A$ This shows $f({}^{\alpha}A) \subseteq {}^{\alpha}(f(A))$

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Point there may be simaken
ashen
$$y \in \mathcal{A}(f(A)) \neq y \in f(\mathcal{A})$$

consider for example.
 $f(n) = \begin{cases} 10 & 2 \le n \le 100 \\ 20 & n > 100 \end{cases}$
 $\mathcal{B} \quad M_A(n) = 1 - \frac{1}{n} \quad \forall n \ge 2$
 $\mathcal{M}_f(A) \stackrel{(10)}{=} - \frac{max}{100} (1 - \frac{1}{n})$
 $= \frac{99}{100}$

But there may be situation when $y \in \alpha(f(A)) \neq y \in f(\alpha A)$

Consider for example

$$f(n) = \begin{cases} 10 & 2 \le n \le 100 \\ 20 & n > 100 \end{cases}$$

and $\mu_A(n) = 1 - \frac{1}{n} \quad \forall n \ge 2$ Therefore $\mu_{f(A)}(10) = \max_{i=2,\dots,100} 1 - \frac{1}{n} = \frac{99}{100}$

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$$\frac{2}{\sqrt{f(A)}} = \frac{1}{\sqrt{f(A)}} = \frac{1}{\sqrt{f(A)}$$

And $\mu_{f(A)}(20) = \sup_{n>100} \mu_A(x) = \sup_{n>100} \left(1 - \frac{1}{n}\right) = 1$ Therefore, ${}^1(f(A)) = \{20\}$ But $f({}^1A) = \phi$, since there does not exist any *n* in *A* such that $\mu_A(n) = 1$ Because for all of them it is going to be $1 - \frac{1}{n}$ so, it will never get the value 1 therefore, ${}^1A = \phi$ and therefore $f({}^1A) = \phi$

On the other hand because the supremum is 1, ${}^{1}(f(A)) = \{20\}$. This proves the result. Ok students I stop here now I hope you understand the extension principle in the next class I shall work on how extension principal can be used for doing fuzzy arithmetic, thank you.