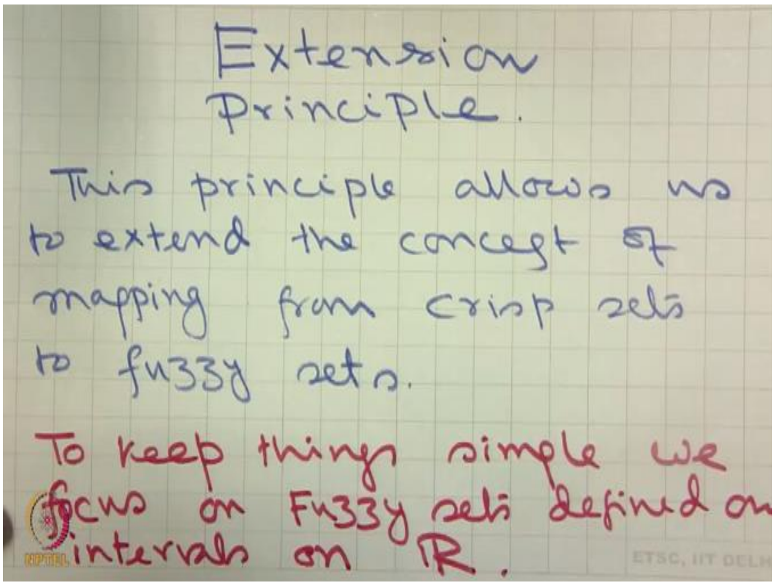
Introduction to Fuzzy Sets Arithmetic and Logic
Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology minus Delhi

Lecture minus 15
Fuzzy Sets Arithmetic and Logic

Welcome students to the MOOCs course on introduction to Fuzzy Set, Arithmetic and Logic. This is lecture number 15 as I said at the end of the last lecture,

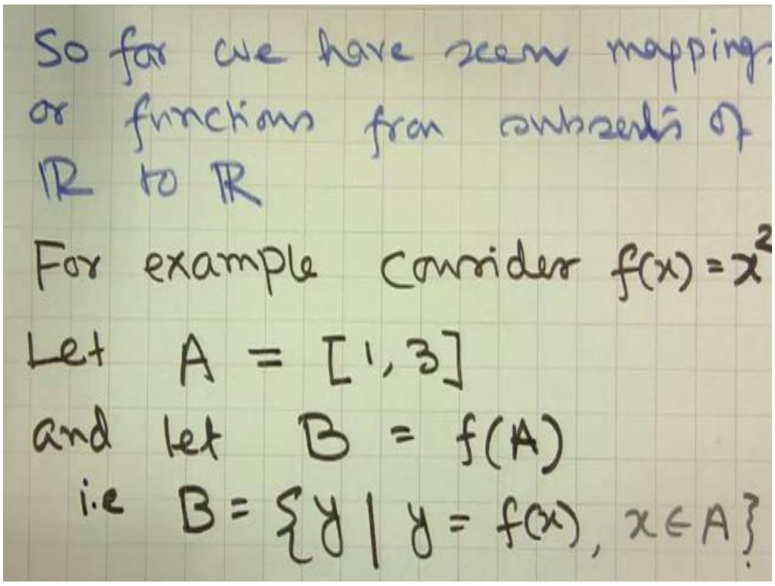
(Refer Slide Time: 00:38)



That today what I will start is called extension principal proposed by none other than Professor Zadeh.

This principle allows us to extend the concept of mapping from crisp sets to fuzzy sets. To keep things simple we focus on fuzzy sets defined on intervals on ℝ that is the real line.

(Refer Slide Time: 02:15)



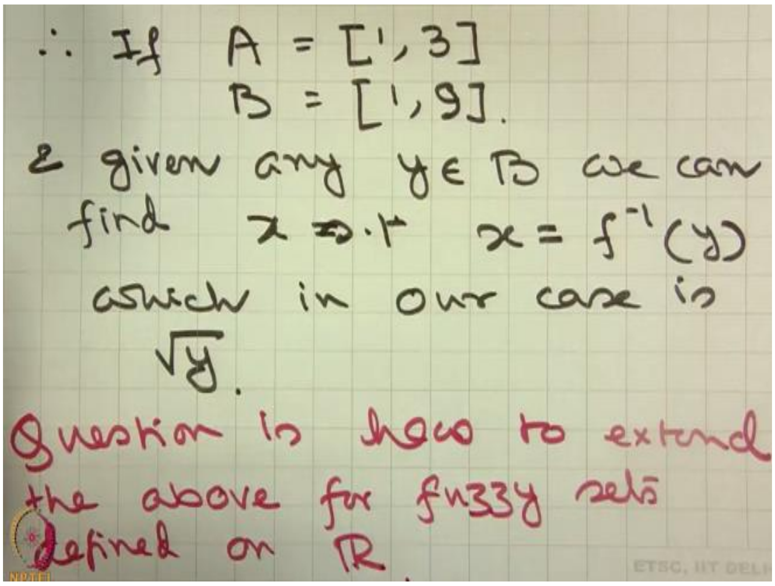
So, far we have seen mappings or functions from subsets of ℝ to ℝ,

For example: Consider $f(x) = x^2$

Let $A = [1, 3]$ and let $B = f(A)$

$i.e. B = \{y | y = f(x), x \in A\}$

(Refer Slide Time: 03:41)

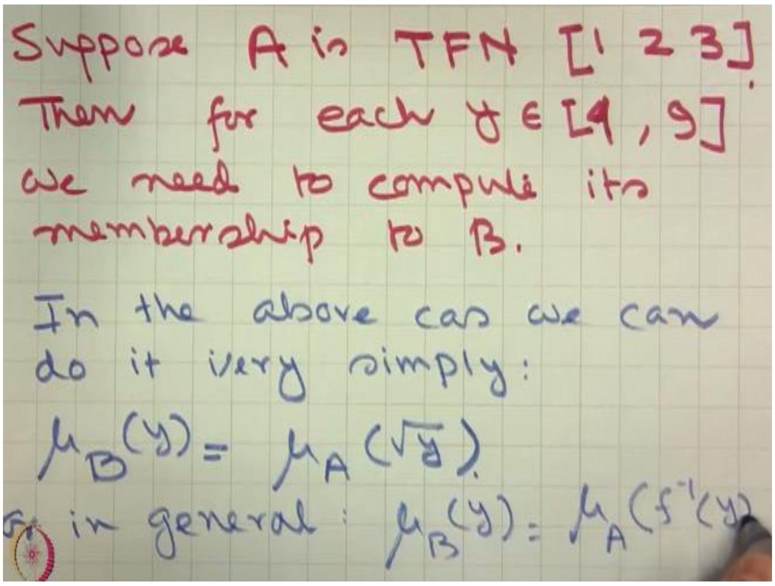


Therefore, if $A = [1, 3]$, $B = [1, 9]$

and given any $y \in B$, we can find x such that $x = f^{-1}(y)$ which in our case is $\sqrt{y}$

Now question is: How to extend the above for fuzzy sets defined on ℝ?

(Refer Slide Time: 05:08)



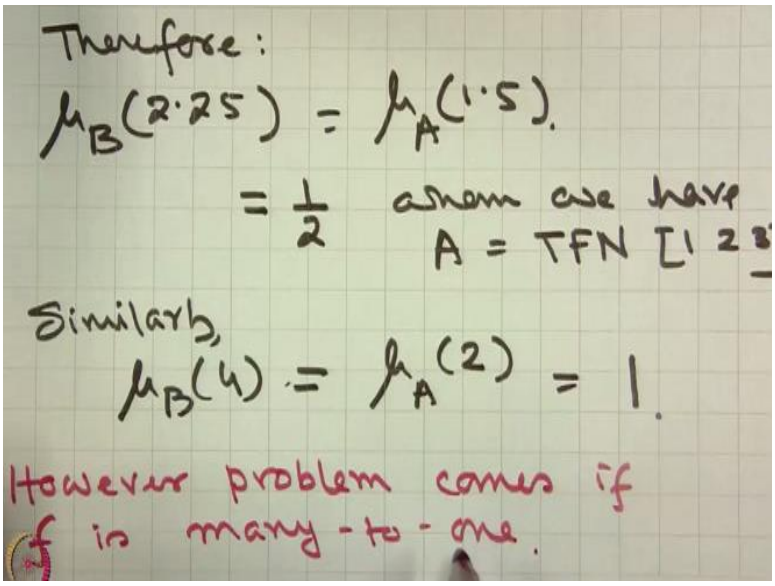
Suppose A is the triangular fuzzy number $[1 \quad 2 \;\; 3]$, then for each $y \in [1, 9]$, we need to compute its membership to B.

In the above case we can do it very simply,

$$\mu_B(y) = \mu_A(\sqrt{y})$$

or in general $\mu_B(y) = \mu_A(f^{-1}(y))$

(Refer Slide Time: 07:02)

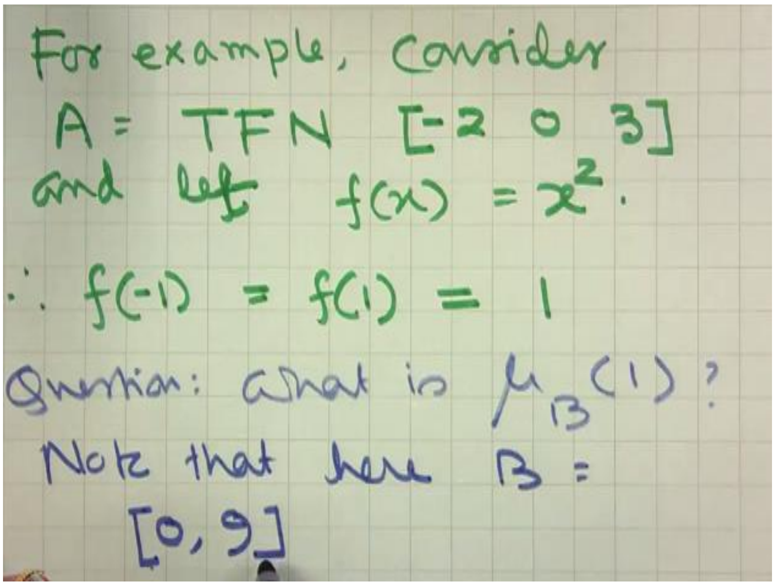


Therefore, $\mu_B(2.25) = \mu_A(1.5) = \frac{1}{2}$ when we have $A = [\,1 \;\; 2 \;\; 3]$

Similarly, $\mu_B(4) = \mu_A(2) = 1$

However, problem comes if f is many-to-one.

(Refer Slide Time: 08:16)



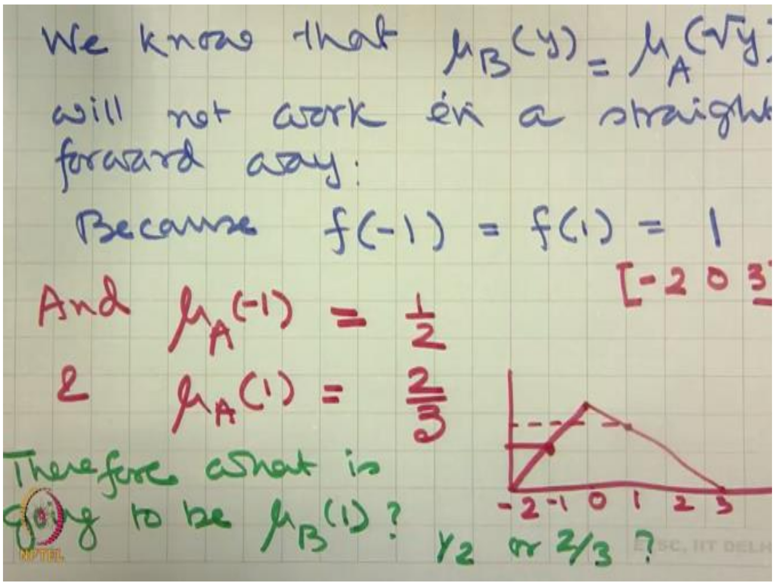
For example: Consider $A = [-2 \quad 0 \quad 3]$ and let $f(x) = x^2$

Therefore, $f(-1) = f(1) = 1$

Question: What is $\mu_B(1)$?

Note that: $B = [0, 9]$ because if my input is from $[-2 \quad 0 \quad 3]$ then the value of x^2 will lie in the interval $[0, 9]$

(Refer Slide Time: 09:53)



We know that $\mu_B(y) = \mu_A(\sqrt{y})$ will not work easily in a straight forward way.

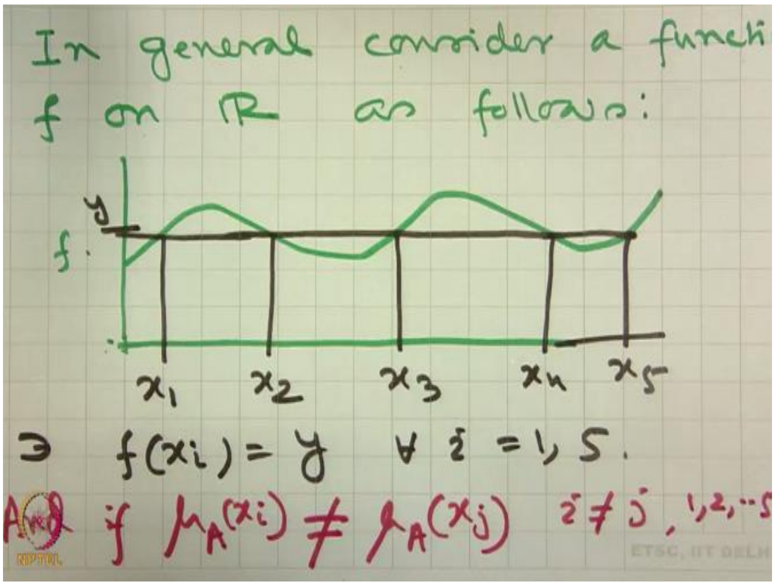
Because $f(-1) = f(1) = 1$ and since we have the TFN $[-2 \quad 0 \quad 3]$

$\mu_A(-1) = \frac{1}{2}$ and $\mu_A(1) = \frac{2}{3}$

therefore, what is going to be $\mu_B(1)$?

It is $\frac{1}{2}$ or $\frac{2}{3}$ that is the question.

(Refer Slide Time: 12:18)



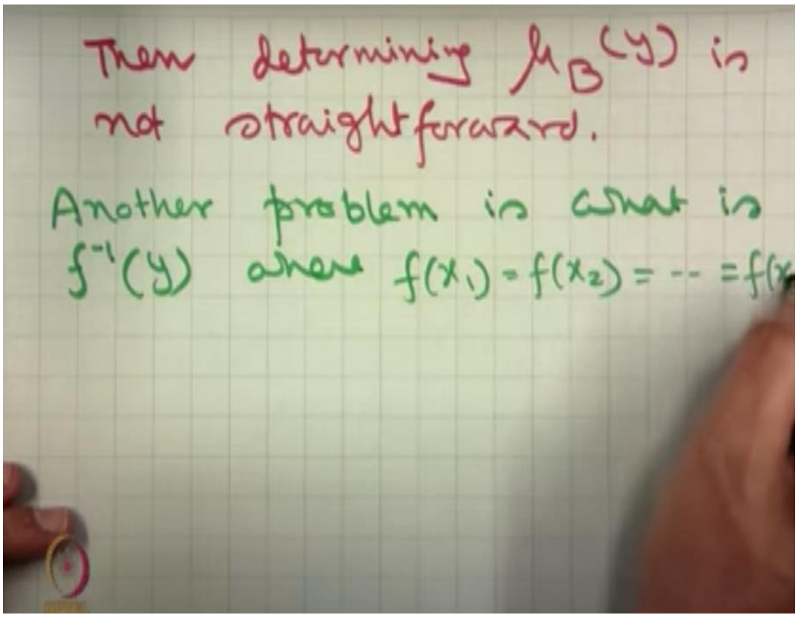
In general consider a function f on $\mathbb{R}$ as follows and

Suppose we consider y is like this.

Therefore, we get at least 5 points x_1, x_2, x_3, x_4, x_5 such that $f(x_i) = y$ for all $i = 1, \dots 5$

And if $\mu_A(x_i) \neq \mu_A(x_j), i \neq j \in \{1, 2 \dots 5\}$

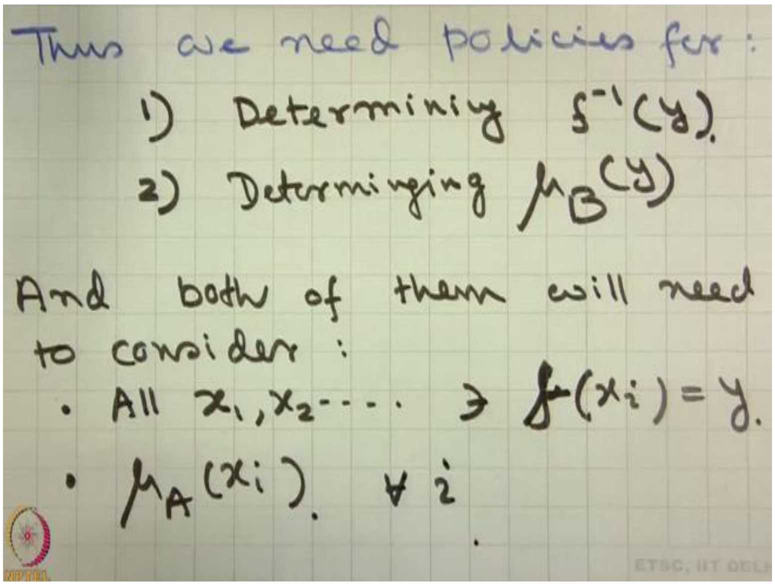
(Refer Slide Time: 14:05)



Then determining $\mu_B(y)$is not straight forward.

Another problem is: What is $f^{-1}(y)$ where $f(x_1) = f(x_2) = \cdots = f(x_5) = y$?

(Refer Slide Time: 15:04)



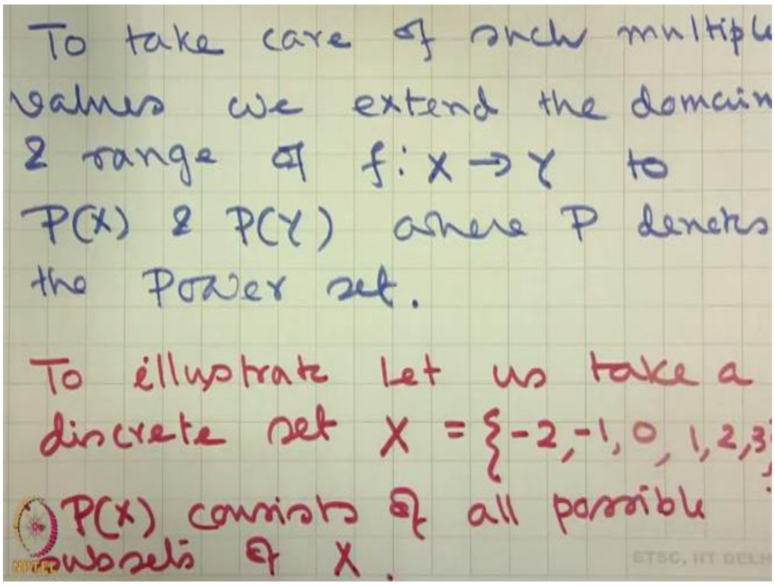
Thus we need policies for:

1) determining $f^{-1}(y)$
2) determining $\mu_B(y)$

And both of them will need to consider

- All x_1, x_2 .... such that $f(x_i) = y$
- $\mu_A(x_i) \quad \forall\ i$

(Refer Slide Time: 16:17)



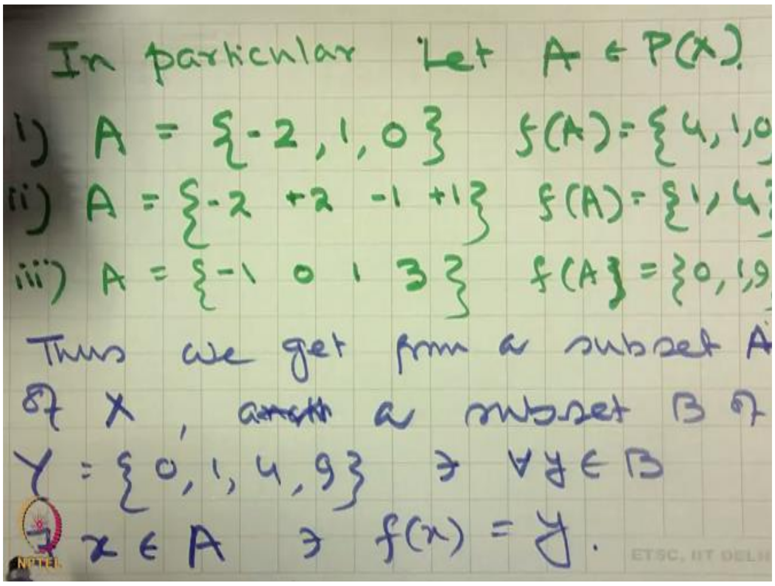
To take care of such multiple values we extend the domain and range of $f: X \to Y$ to $P(X)$ and $P(Y)$ where P denotes the power set.

To illustrate:

Let us take a discrete set $X = \{-2, -1, 0, 1, 2, 3\}$

Therefore, $P(X)$ consists of all possible subsets of X

(Refer Slide Time: 18:26)



In particular, let $A \in P(X)$

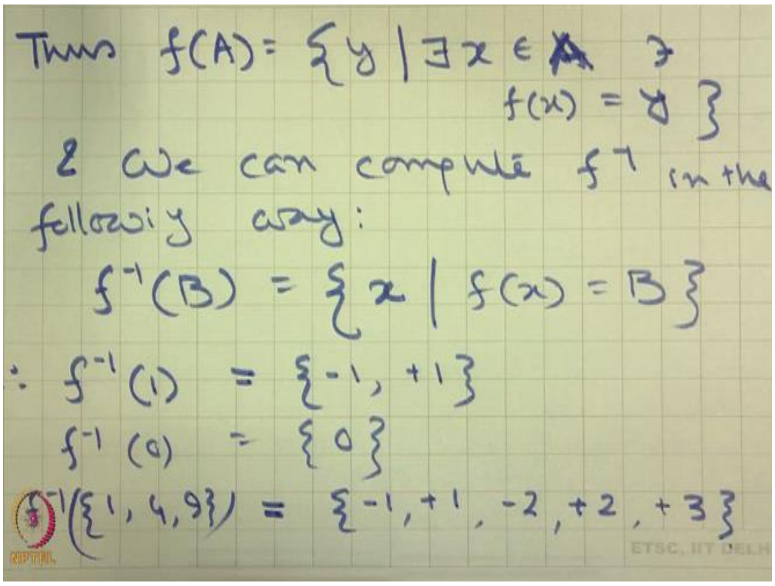
i. $A = \{-2,\ 1, 0\}$ then $f(A) = \{4, 1, 0\}$

ii. $A = \{-2,\ 2,\ -1,\ \ 1\}$ then $f(A) = \{4, 1\}$

iii. $A = \{-1, 0, 1,\ 3\}$ then $f(A) = \{0, 1, 9\}$

Thus we get from a subset A of X, a subset B of $Y = \{0, 1, 4, 9\}$ such that for all $y \in B$ there exists $x \in A$ such that $f(x) = y$

(Refer Slide Time: 20:38)



Thus $f(A) = \{y|\ \exists\ x \in A$ such that $f(x) = y\}$ and we can compute f^{-1} in the following way:

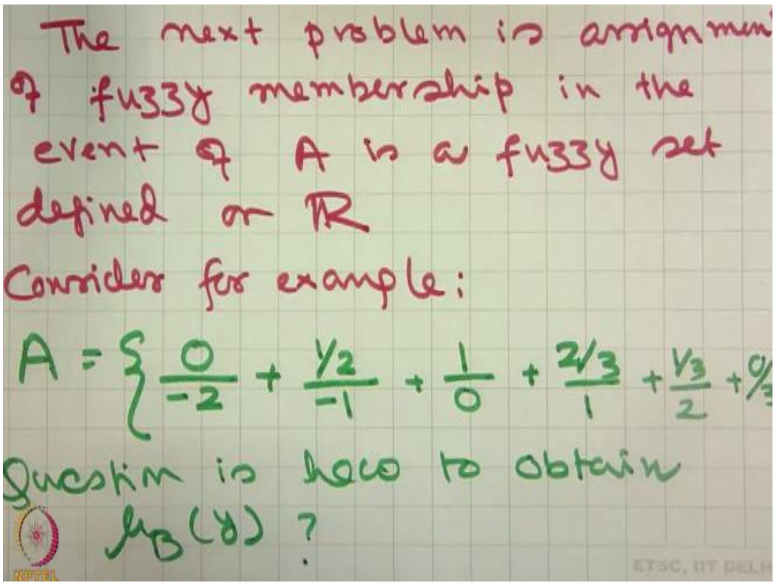
$$f^{-1}(B) = \{x|f(x) = B\}$$

Therefore, $f^{-1}(1) = \{-1,\ 1\}$

$f^{-1}(0) = \{0\}$

$f^{-1}(\{1,4,9\}) = \{-1,1,-2,2,3\}$

(Refer Slide Time: 22:23)



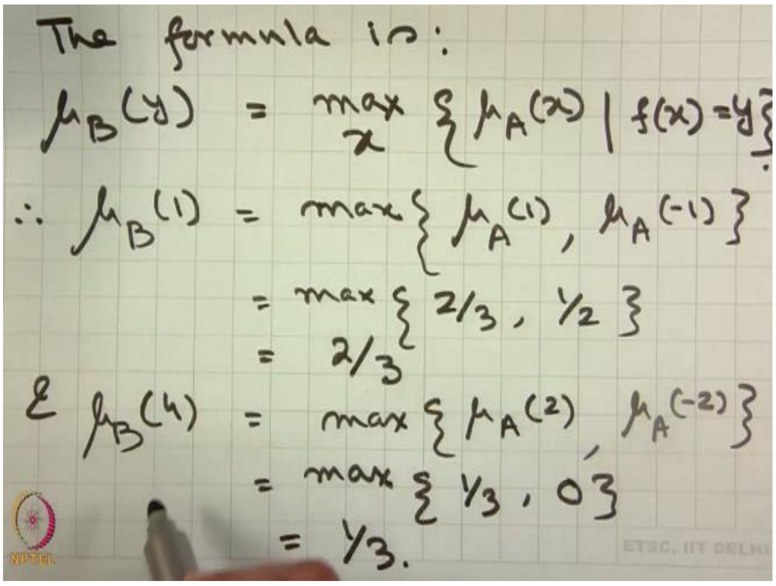
The next problem is assignment of fuzzy membership in the event of A is a fuzzy set defined on $\mathbb{R}$.

Consider for example:

$$A = \left\{\frac{0}{-2} + \frac{1/2}{-1} + \frac{1}{0} + \frac{2/3}{1} + \frac{1/3}{2} + \frac{0}{3}\right\}$$

Question is how to obtain $\mu_B(y)$?

(Refer Slide Time: 24:14)



The formula is:

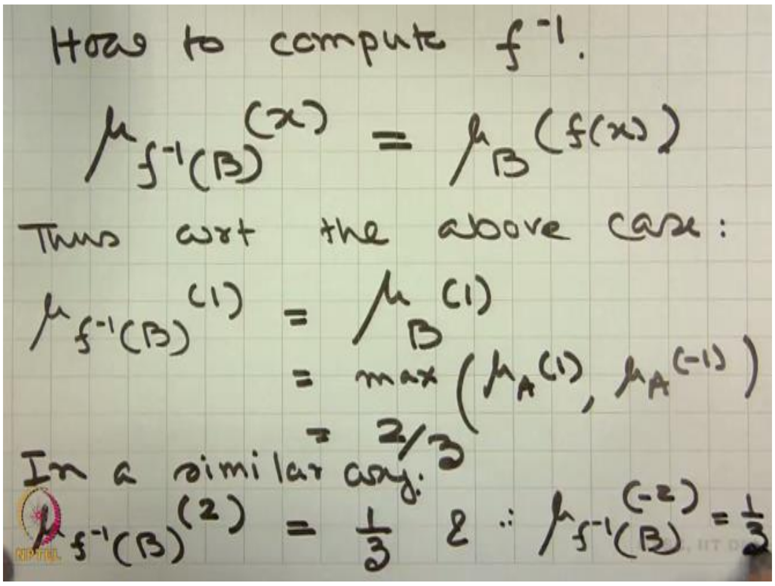
$$\mu_B(y) = \max_{x}\{\mu_A(x) | f(x) = y\}$$

$\therefore \mu_B(1) = \max\{\mu_A(1), \mu_A(-1)\} \; = \max\left\{\frac{2}{3}, \frac{1}{2}\right\} = \frac{2}{3}$

and $\mu_B(4) = \max\{\mu_A(2), \mu_A(-2)\} = \max\left\{\frac{1}{3}, 0\right\} = \frac{1}{3}$

In this way we can compute $\mu_B(y)$. How to compute f^{-1} ?

(Refer Slide Time: 26:02)



Therefore, $\mu_{f^{-1}(B)}(x) = \mu_B\big(f(x)\big)$

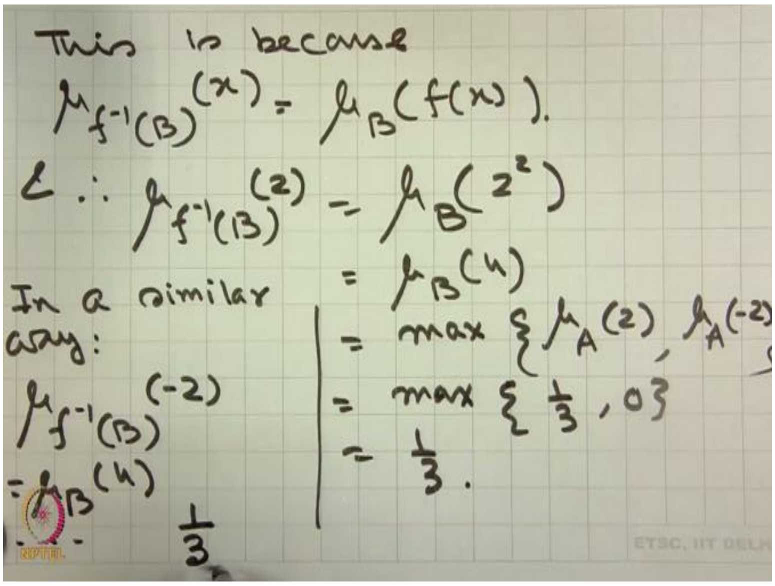
Thus, with respect to the above case:

$\mu_{f^{-1}(B)}(1) = \mu_B(1) = \max\{\mu_A(1), \mu_A(-1)\} = \frac{2}{3}$

In a similar way

$\mu_{f^{-1}(B)}(2) = \frac{1}{3}$ and $\mu_{f^{-1}(B)}(-2) = \frac{1}{3}$

(Refer Slide Time: 28:01)

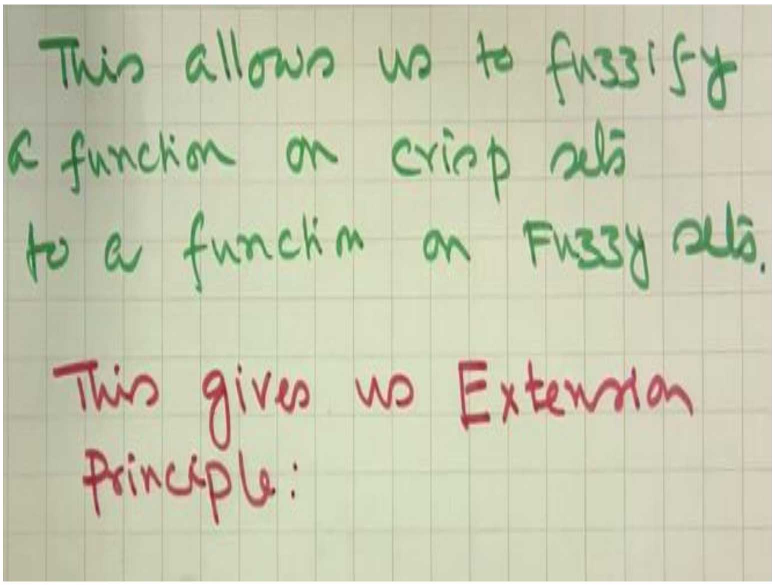


This is because $\mu_{f^{-1}(B)}(x) = \mu_B\big(f(x)\big)$

And therefore, $\mu_{f^{-1}(B)}(2) = \mu_B(2^2) = \mu_B(4) = \max\{\mu_A(2), \mu_A(-2)\} = \max\left\{\frac{1}{3}, 0\right\} = \frac{1}{3}$

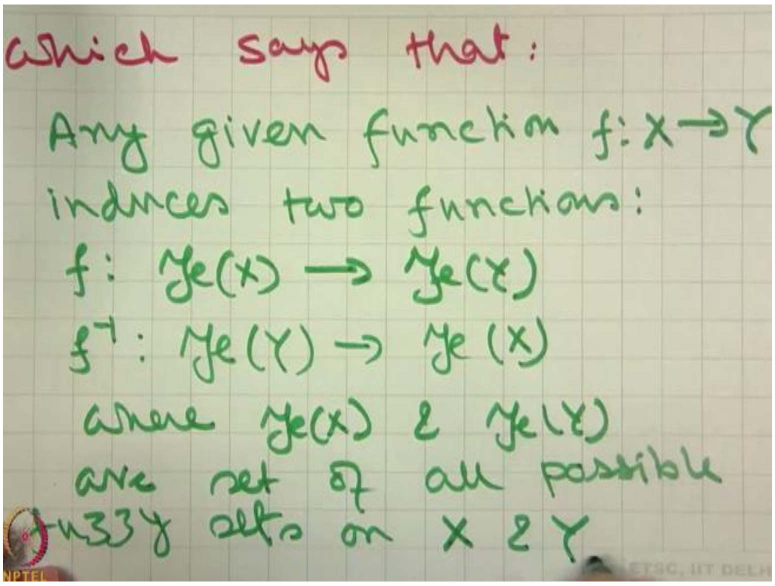
In a similar way $\mu_{f^{-1}(B)}(-2) = \mu_B(4) = \frac{1}{3}$

(Refer Slide Time: 29:33)



This allows us to fuzzify a function on crisp sets to a function on fuzzy sets and this gives us Extension Principle,

(Refer Slide Time: 30:35)



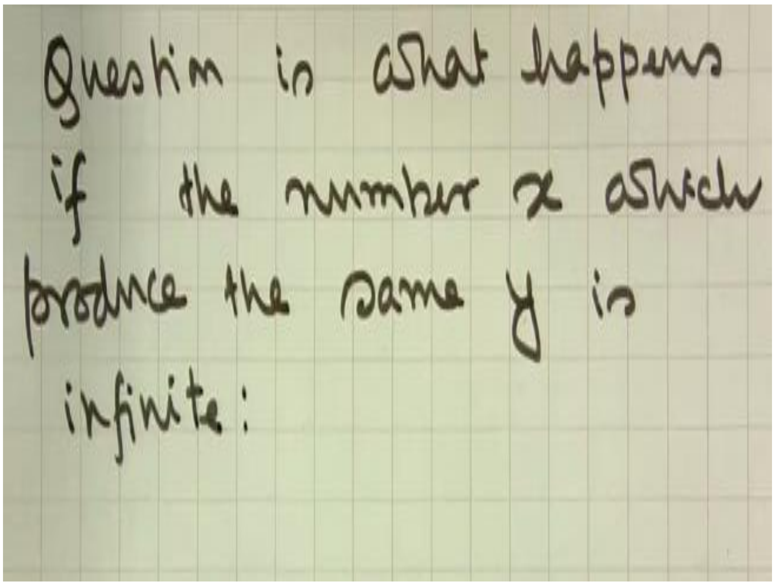
Which says that

Any given function $f: X \to Y$ induces two functions:

- $f: \mathfrak{F}(X) \to \mathfrak{F}(Y)$
- $f^{-1}: \mathfrak{F}(Y) \to \mathfrak{F}(X)$

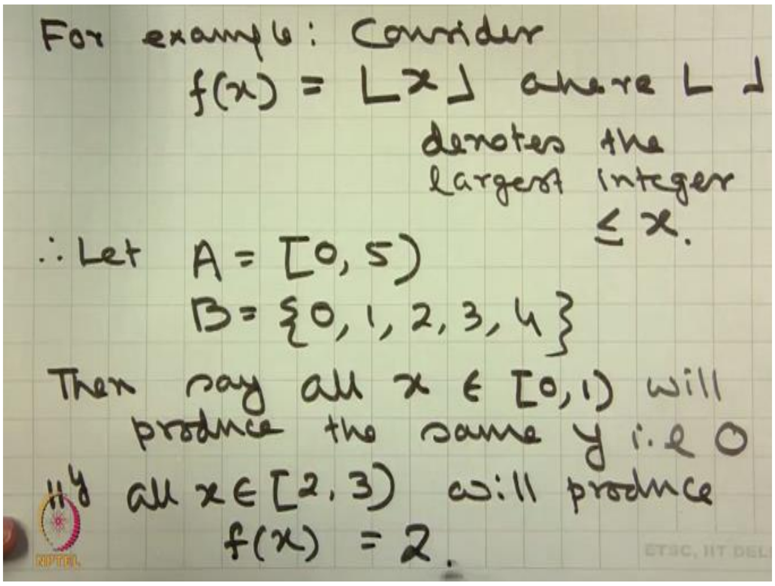
where $\mathfrak{F}(X)$ and $\mathfrak{F}(Y)$ are set of all possible fuzzy sets on X and Y.

(Refer Slide Time: 32:02)



Question is: what happens if the number of x which produce the same y is infinite.

(Refer Slide Time: 32:45)



For example:

Consider $f(x) = \lfloor x \rfloor$ where $\lfloor \circ \rfloor$ denotes the largest integer $\leq x$

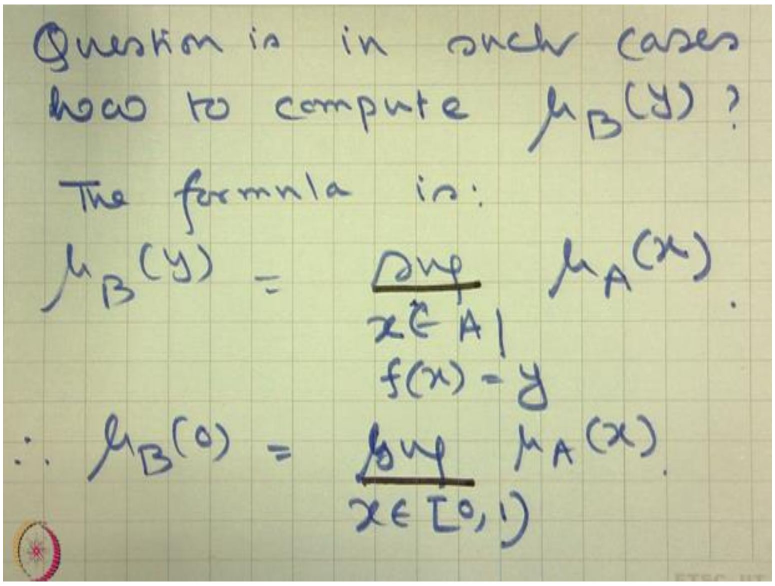
Therefore, let $A = [0, 5]$

$B = \{0, 1, 2, 3, 4, 5\}$

Then say all $x \in [0, 1)$ will produce the same y i.e. 0

Similarly, all $x \in [2, 3)$ will produce $f(x) = 2$.

(Refer Slide Time: 34:31)



Question is: in such cases how to compute $\mu_B(y)$?

And the formula is:

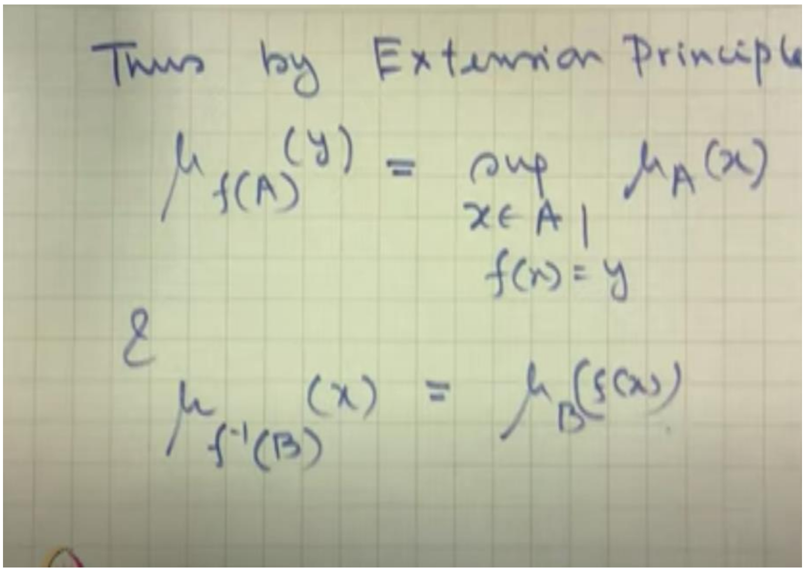
$\mu_B(y) = \sup_{x \in A | f(x) = y} \mu_A(x)$

$\therefore \mu_B(0) = \sup_{x \in [0,1)} \mu_A(x)$

Thus, the notion of maximum is now replaced with supremum , as there are infinite number of values which produces the same y

We may not be able to identify the maximum therefore, we need to replace it with supremum.

(Refer Slide Time: 36:13)



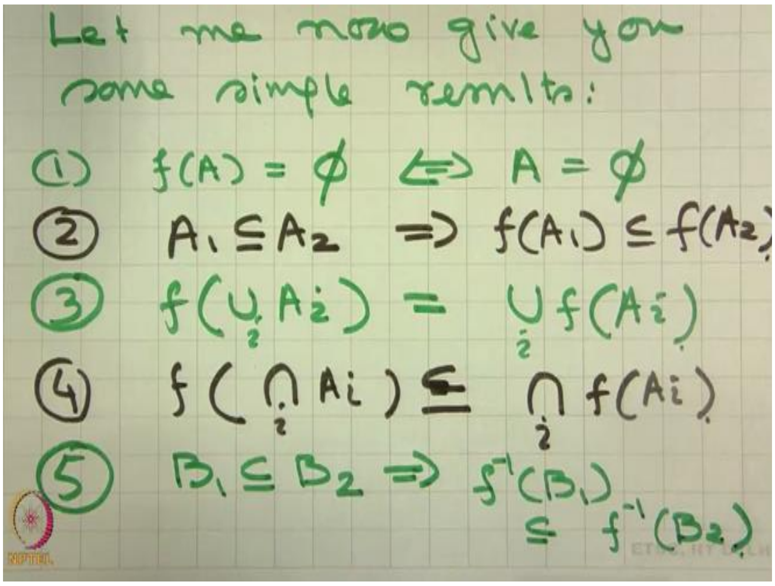
Thus the extension principle

$$\mu_{f(A)}(y) = \sup_{x \in A | f(x) = y} \mu_A(x)$$

And

$$\mu_{f^{-1}(B)}(x) = \mu_B\big(f(x)\big)$$

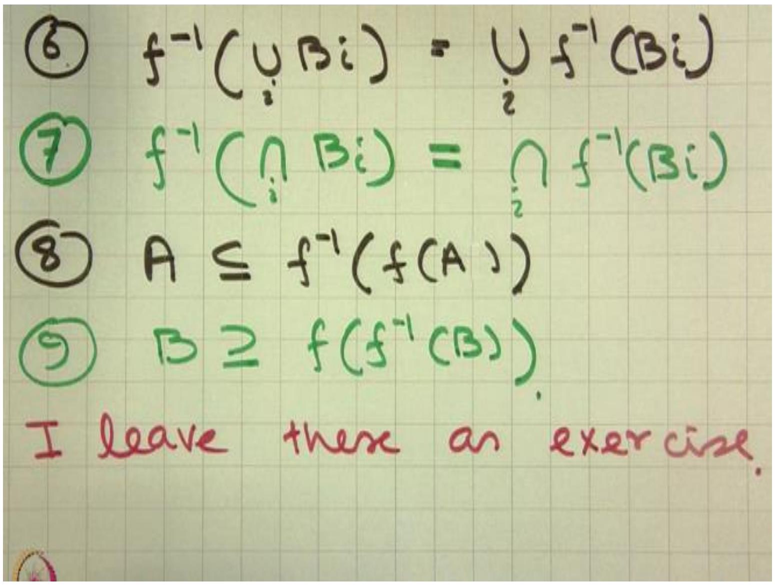
(Refer Slide Time: 37:06)



Let me now give you some simple results:

1) $f(A) = \phi \Leftrightarrow A = \phi$
2) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$
3) $f(\cup_i A_i) = \cup_i f(A_i)$
4) $f(\cap_i A_i) \subseteq \cap_i f(A_i)$
5) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(Refer Slide Time: 39:06)



6) $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$
7) $f^{-1}(\cap_i B_i) = \cap_i f^{-1}(B_i)$

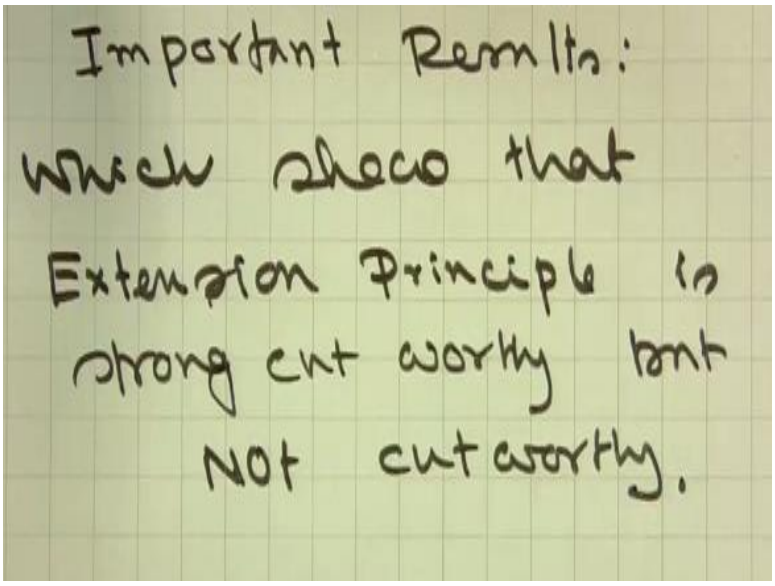
8) $A \subseteq f^{-1}\big(f(A)\big)$

9) $B \supseteq f\big(f^{-1}(B)\big)$

I leave these an exercise, these are already straight forward.

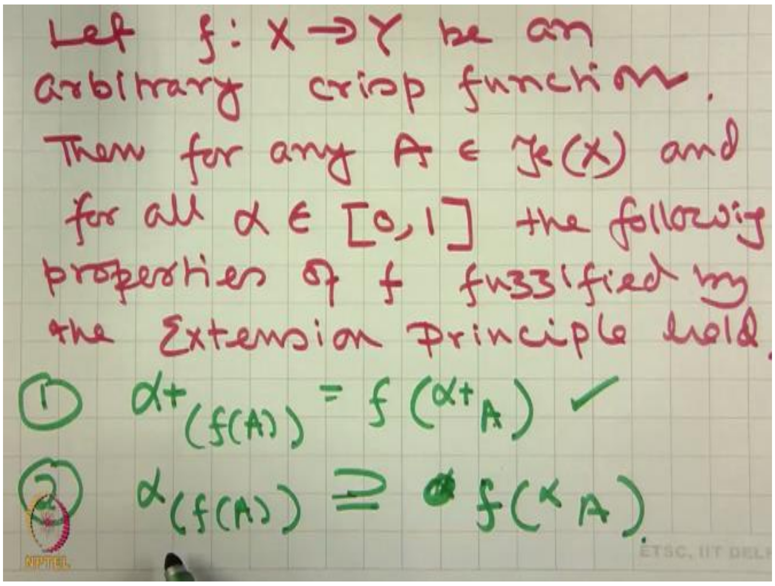
If you are understood how we are calculating the membership functions and how we are calculating the inverse, all these are absolutely straight forward to prove.

(Refer Slide Time: 40:39)



So before I stop I give you an important result, which show that extension principle, is strong cut-worthy but not simply cut-worthy.

(Refer Slide Time: 41:25)



So let $f: X \to Y$ be an arbitrary crisp function. Then for any $A \in \mathfrak{F}(X)$ that is A is a fuzzy set define over X and for all $\alpha \in [0, 1]$ the following properties of f fuzzified by the extension principle holds.

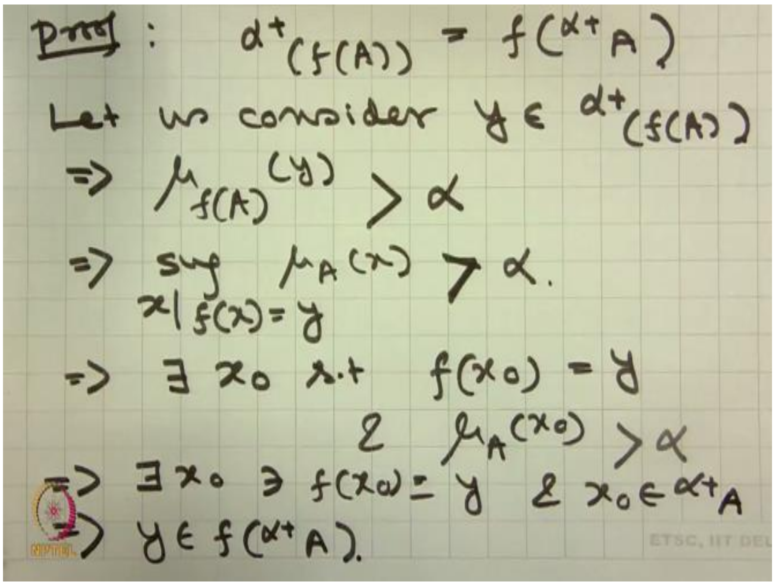
That means we have taken function f from X to Y we have fuzzified them using extension principle then we find the following two properties to hold.

What are these?

1) ${}^{\alpha+}\big(f(A)\big) = f({}^{\alpha+}A)$
2) ${}^{\alpha}\big(f(A)\big) \supseteq f({}^{\alpha}A)$

So this says that we ${}^{\alpha+}\big(f(A)\big)$ through ${}^{\alpha+}A$ and apply f on them and therefore it is strong cut worthy but this containment relationships says that it is not simply cut worthy.

(Refer Slide Time: 44:19)



Proof: ${}^{\alpha+}\big(f(A)\big) = f({}^{\alpha+}A)$

Let us consider $y \in {}^{\alpha+}\big(f(A)\big)$

$\Rightarrow \mu_{f(A)}(y) > \alpha$

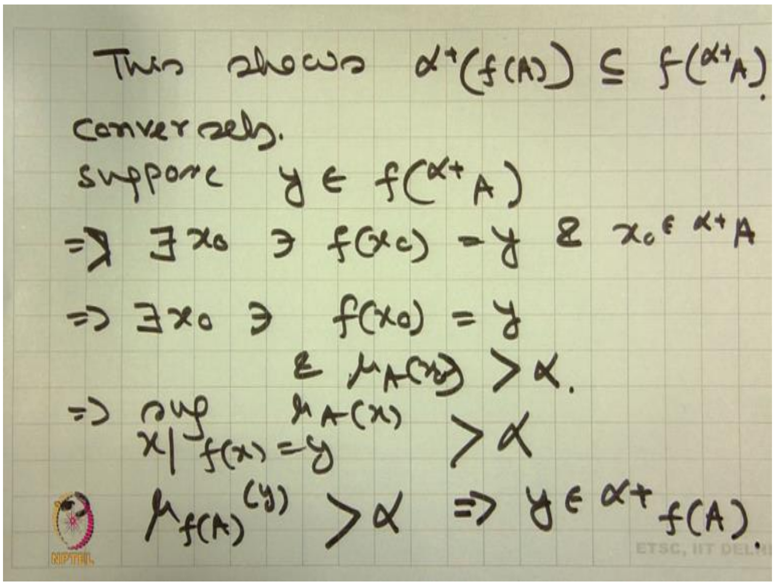
$\Rightarrow \sup\limits_{x|f(x)=y} \mu_A(x) > \alpha$

$\Rightarrow \exists\, x_0$ such that $f(x_0) = y$ and $\mu_A(x_0) > \alpha$

$\Rightarrow \exists\, x_0$ such that $f(x_0) = y$ and $x_0 \in {}^{\alpha+}A$

$\Rightarrow y \in f({}^{\alpha+}A)$

(Refer Slide Time: 46:35)



This shows ${}^{\alpha+}\big(f(A)\big) \subseteq f(\,{}^{\alpha+}A)$

Conversely, suppose $y \in f(\,{}^{\alpha+}A)$

$\Rightarrow \exists\, x_0$ such that $f(x_0) = y$ and $x_0 \in {}^{\alpha+}A$

$\Rightarrow \exists\, x_0$ such that $f(x_0) = y$ and $\mu_A(x_0) > \alpha$

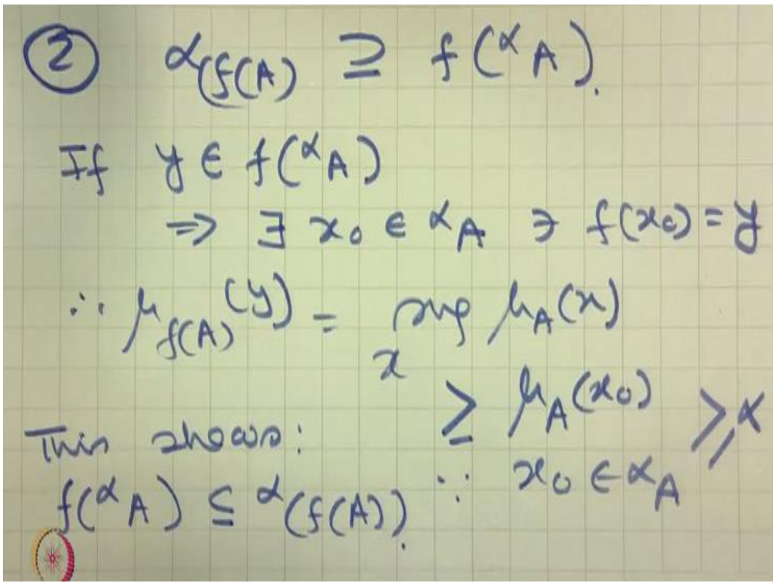
$\Rightarrow \sup\limits_{x|f(x)=y} \mu_A(x) > \alpha$

$\Rightarrow \mu_{f(A)}(y) > \alpha$

$\Rightarrow y \in {}^{\alpha+}\big(f(A)\big)$

So this proves the first result.

(Refer Slide Time: 48:21)



Now let us prove the second result

${}^{\alpha}\big(f(A)\big) \supseteq f(\,{}^{\alpha}A)$

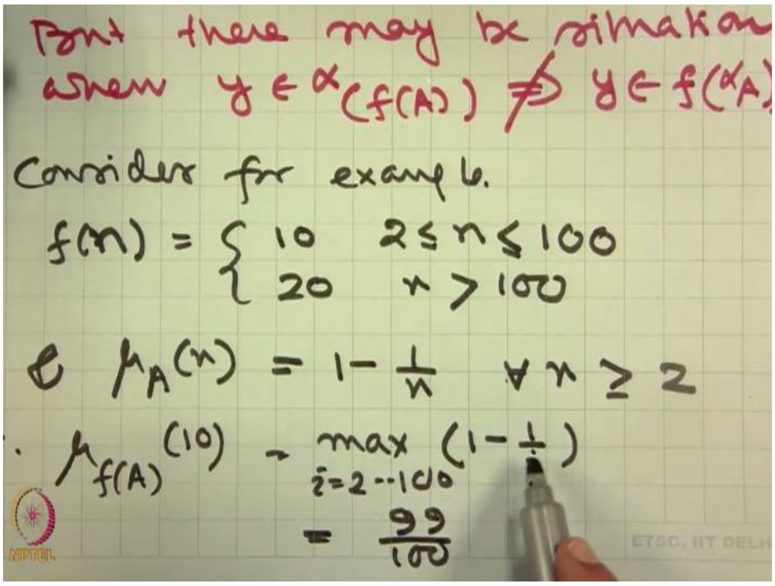
If $y \in f({}^{\alpha}A)$

$\Rightarrow \exists\, x_0 \in {}^{\alpha}A$ such that $f(x_0) = y$

$\therefore\ \mu_{f(A)}(y) = \sup\limits_{x} \mu_A(x) \geq \mu_A(x_0) \geq \alpha \qquad \because x_0 \in {}^{\alpha}A$

This shows $f({}^{\alpha}A) \subseteq {}^{\alpha}\big(f(A)\big)$

(Refer Slide Time: 50:02)



But there may be situation when $y \in {}^{\alpha}\big(f(A)\big) \nRightarrow y \in f({}^{\alpha}A)$

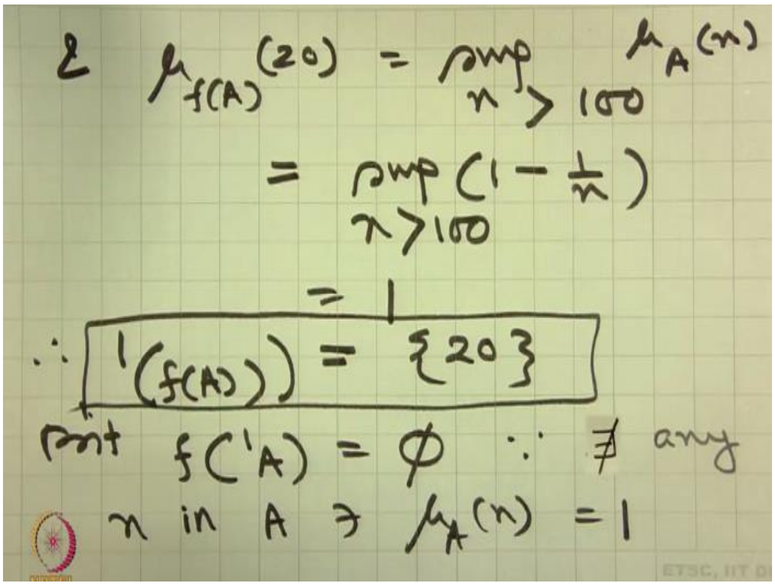
Consider for example

$$f(n) = \begin{cases} 10 & 2 \leq n \leq 100 \\ 20 & n > 100 \end{cases}$$

and $\mu_A(n) = 1 - \frac{1}{n} \quad \forall n \geq 2$

Therefore $\mu_{f(A)}(10) = \max\limits_{i=2,\ldots 100} 1 - \frac{1}{n} = \frac{99}{100}$

(Refer Slide Time: 52:02)



And $\mu_{f(A)}(20) = \sup\limits_{n>100} \mu_A(x) = \sup\limits_{n>100} \left(1 - \frac{1}{n}\right) = 1$

Therefore, $^{1}\big(f(A)\big) = \{20\}$

But $f(\,^{1}A) = \phi$, since there does not exist any n in A such that $\mu_A(n) = 1$

Because for all of them it is going to be $1 - \frac{1}{n}$ so, it will never get the value 1

therefore, $^{1}A = \phi$ and therefore $f(\,^{1}A) = \phi$

On the other hand because the supremum is 1, $^{1}\big(f(A)\big) = \{20\}$. This proves the result.

Ok students I stop here now I hope you understand the extension principle in the next class I shall work on how extension principal can be used for doing fuzzy arithmetic, thank you.