Introduction to Fuzzy Sets Arithmetic and Logic
Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology – Delhi

Lecture - 14
Fuzzy Sets Arithmetic and Logic

Welcome students to the 14th lecture on introduction to fuzzy sets arithmetic and logic.

In the last few lectures, we have been dealing with α-cuts of fuzzy sets.

Now, you may ask why α-cuts are so important with respect to dealing with fuzzy sets. One major advantage of α-cuts is that the sets induced by them are all crisps sets.

Therefore, the advantage of α-cuts is that many properties and operations of crisp sets can be extended to fuzzy sets with the help of α-cuts and any such property of crisp sets that can be extended to fuzzy sets with the help of α-cuts is called a cut-worthy property.

We have seen in our earlier lectures that convexity is a cut- worthy property. Also we have seen the different arithmetic operations can be extended to fuzzy sets with the help of α-cuts.

Now, α-cuts have another advantage and that is that each fuzzy set can be uniquely represented by its α-cuts and this corresponding things are called decomposition of fuzzy sets with the help of α-cuts. So, in today's lecture, we shall focus on the representation of fuzzy sets using α-cuts.

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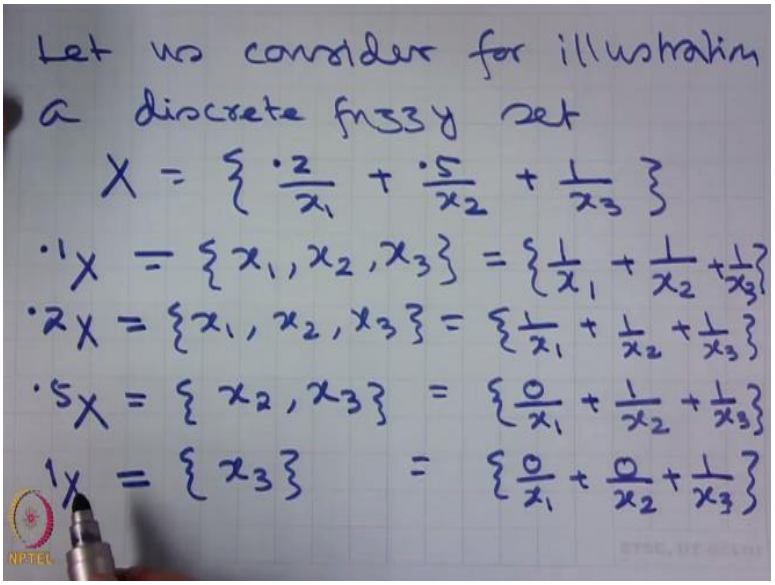

So, let us first considered for illustration a discrete fuzzy set say, we call it X

$$X = \left\{\frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3}\right\}$$

That means, we are looking at one fuzzy set consisting of only three elements $\{x_1, x_2, x_3\}$ and their corresponding membership values.

Now, let us look at different α-cuts of X.

For example, we consider

${}^{0.1}X = \{x_1, x_2, x_3\}$, but we write it using their cardinality.

We can write it as ${}^{0.1}X = \left\{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right\}$

Similarly, ${}^{0.2}X = \{x_1, x_2, x_3\}$ and that also you can write it as ${}^{0.2}X = \left\{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right\}$

What is ${}^{0.5}X$?

This we know that it will comprise only $\{x_2, x_3\}$

Therefore, we can write it as $\left\{\frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right\}$

Let us consider ${}^{1}X$ which consists of the single term $\{x_3\}$ and this we can write it is as $\left\{\frac{0}{x_1} + \frac{0}{x_2} + \frac{1}{x_3}\right\}$

So, I hope you understand the construction.

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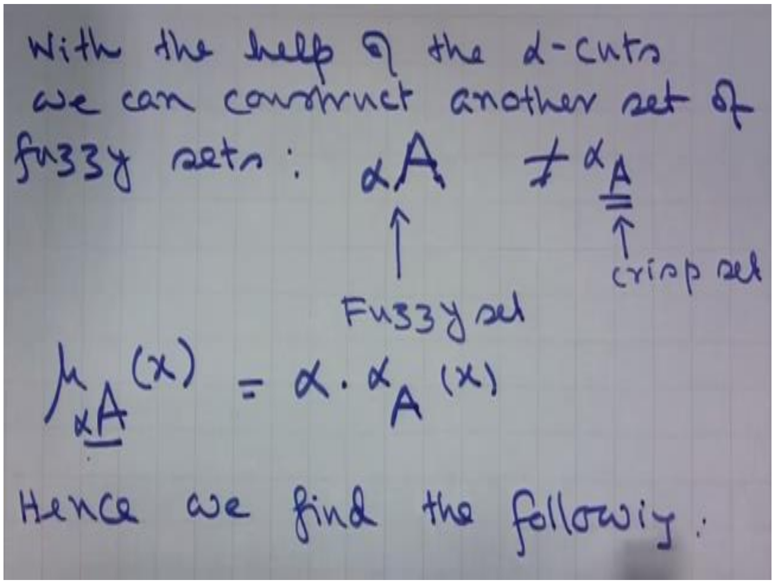

Now, with the help of the α-cuts we can construct another set of fuzzy sets as follows, let them call $_{\alpha}A$, I am putting the α as a subscript on the left hand side. So, you should not confuse this with $^{\alpha}A$ because it is a crisp set but $_{\alpha}A$ is a fuzzy set.

And $\mu_{\,_{\alpha}A}(x) = \alpha \cdot \mu_{\,^{\alpha}A}(x)$

Therefore, we find the following.

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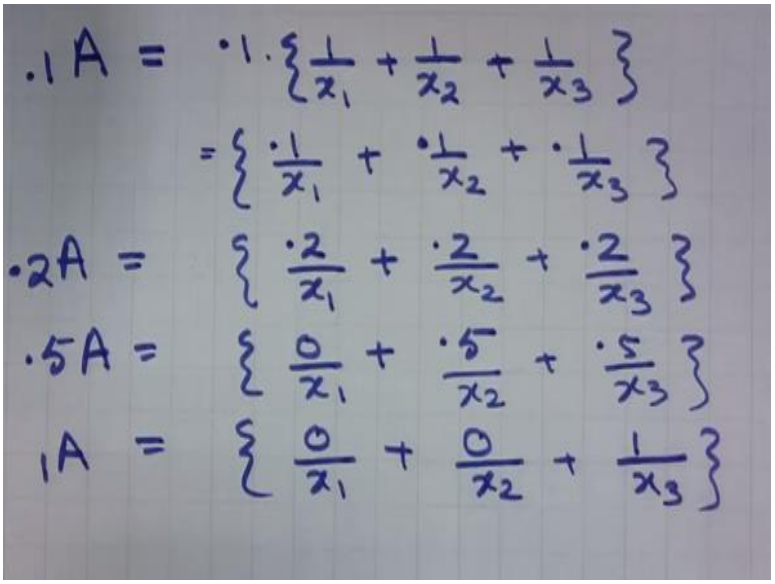

So,

$_{0.1}A = 0.1\left\{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right\} = \left\{\frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3}\right\}$.

In a similar way, therefore $_{0.2}A = \left\{\frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3}\right\}$, $_{0.5}A = \left\{\frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3}\right\}$ and

$_{1}A = \left\{\frac{0}{x_1} + \frac{0}{x_2} + \frac{1}{x_3}\right\}$

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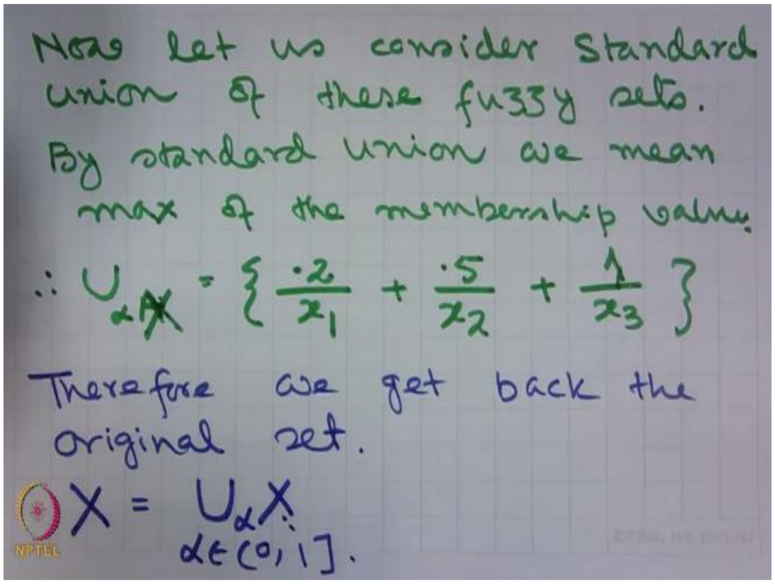

Now, let us consider standard union of these fuzzy sets. And by standard union we mean maximum of the membership values.

Therefore, $\cup_{\alpha}\ {}_{\alpha}X = \left\{\frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3}\right\}$

Therefore, now, if we compare with the original set, we get back the original set or in other words

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X$$

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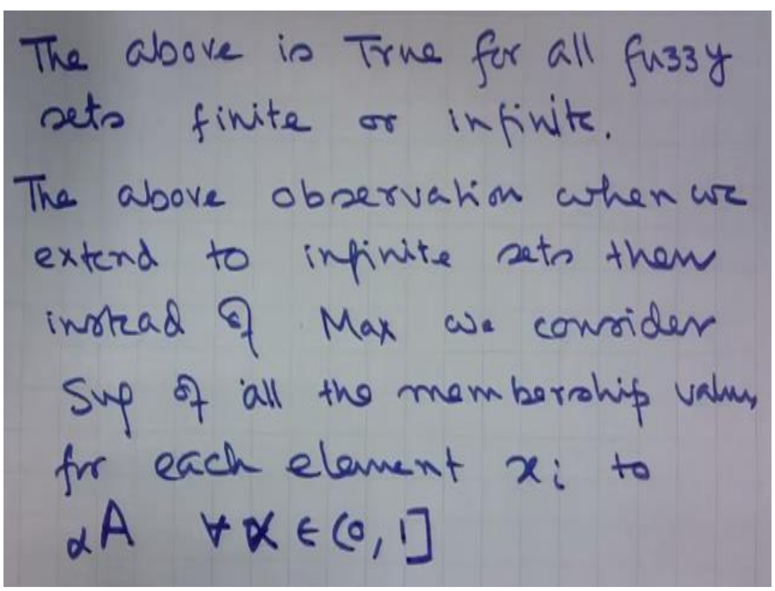

So, the above observation is true for all fuzzy sets finite or infinite.

The above observation when we extend to infinite sets then instead of maximum we consider supremum of all the membership values for each element x_i to ${}_{\alpha}A$ for all $\alpha \in (0,1]$

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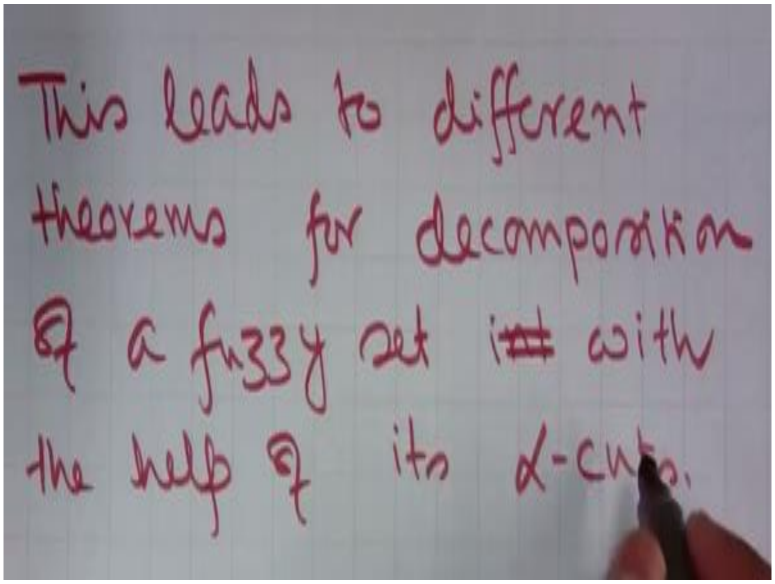

This leads to different theorems for decomposition of a fuzzy set with the help of α-cuts.

So, let me now state the different decomposition theorems and I give proof for them.

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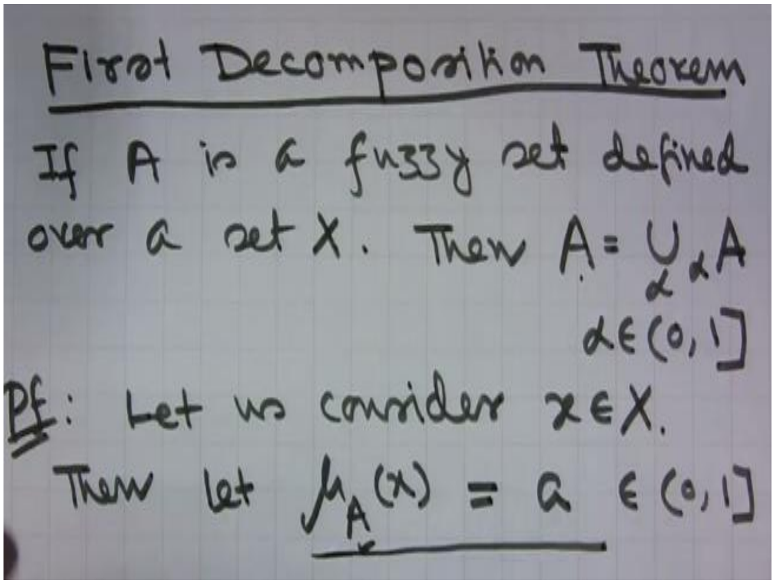

First decomposition theorem:

If A is a fuzzy set defined over a set X. Then

$$A = \bigcup_{\alpha \in (0,1]} {}_{\alpha}A$$

Proof:

Let us consider any $x \in X$.

Then, let $\mu_A(x) = a \in (0, 1]$

So, we have a set X we have defined a fuzzy set A on that on that crisp set X by assigning some membership functions and let us consider any particular element x and let its membership value to the fuzzy set A be small a.

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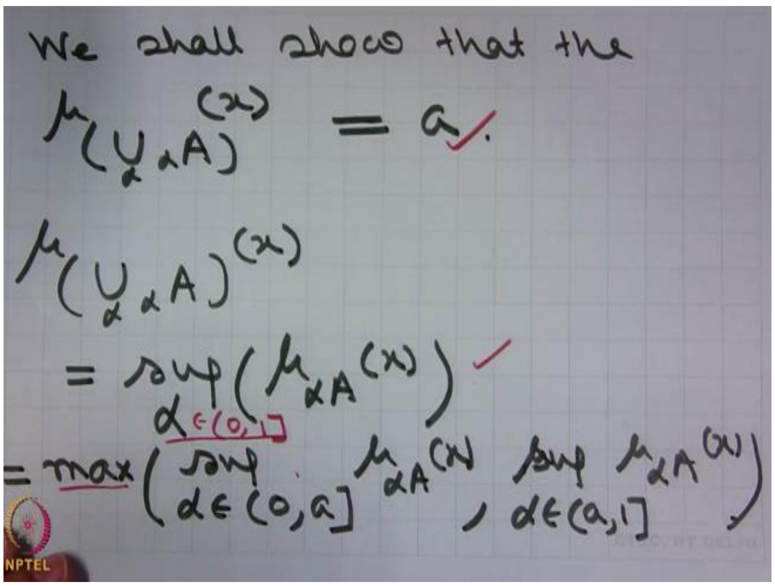

We shall show that $\mu_{(\cup_\alpha\ {}_\alpha A)}(x) = a$ and therefore, what will happen?

Because this x is arbitrary.

Therefore, if we can show that $\mu_{(\cup_\alpha\ {}_\alpha A)}(x) = a$ then we can show that for all x whatever is its membership function to the fuzzy set A will also be same as the membership to $\cup_\alpha\ {}_\alpha A$

Therefore, $\cup_\alpha\ {}_\alpha A$ going to represent the original set A so, that is the motivation. So, we prove as follows.

Now, what is $\mu_{(\cup_\alpha\ {}_\alpha A)}(x) = \sup_{\alpha \in (0,1]} \mu_{\ {}_\alpha A}(x)$

Now, this we divided into two parts.

$$\mu_{(\cup_\alpha\ {}_\alpha A)}(x) = \max\left(\sup_{\alpha \in (0,a]} \mu_{\ {}_\alpha A}(x)\ ,\ \sup_{\alpha \in (a,1]} \mu_{\ {}_\alpha A}(x)\right)$$

So, our aim is to calculate these two different supremum and whichever of them is maximum, we are going to call it the supremum of all $\alpha \in (0, 1]$. So, the logic is clear.

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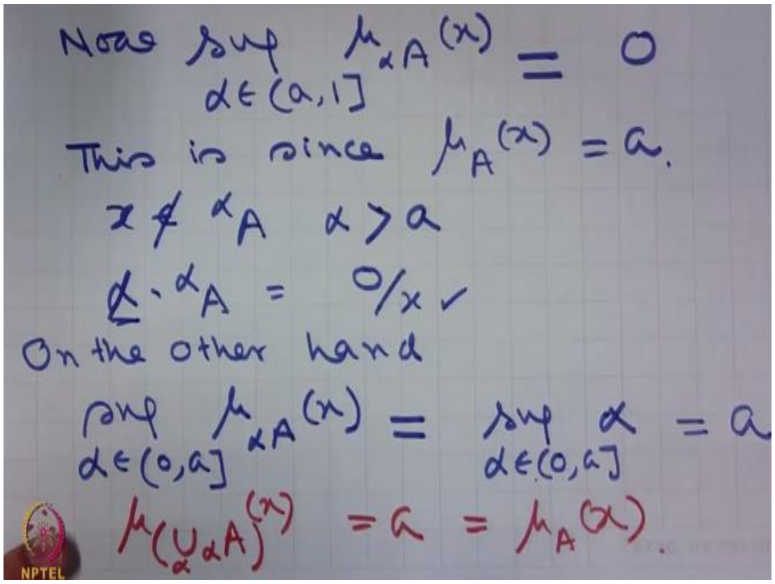

Now, $\sup_{\alpha\in(a,1]} \mu_{\alpha A}(x) = 0$

This is since $\mu_A(x) = a$, $x \notin {}^{\alpha}A$ when $\alpha > a$

Therefore, for all of them the $\alpha \cdot \mu_{{}^{\alpha}A} = 0$ for this x.

On the other hand,

$\sup_{\alpha\in(0,a]} \mu_{\alpha A}(x) = \sup_{\alpha\in(0,a]} \alpha = a$

Because this x will belong to all the α-cuts.

So, when you are multiplying it by the α, the corresponding $\mu_{\alpha A}(x)$ is going to be α and now, I am taking supremum of them from between 0 to a and this is going to be a.

Therefore, $\mu_{(\cup_\alpha\, \alpha A)}(x) = a = \mu_A(x)$

As I said since x is chosen arbitrarily therefore, this proves the theorem.

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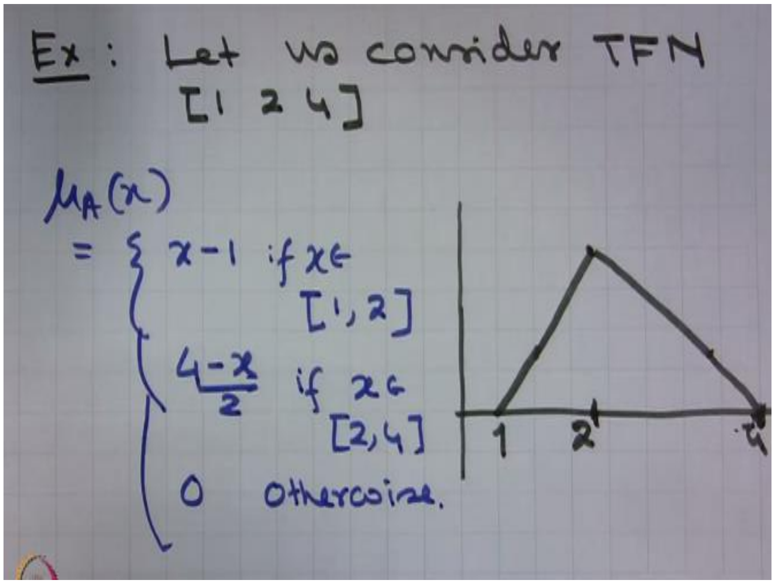

So, example let us consider the TFN the triangular fuzzy number [1   2   4].

Therefore, what is $\mu_A(x)$?

$$\mu_A(x) = \begin{cases} x-1 & x \in [1,2] \\ \dfrac{4-x}{2} & x \in [2,4] \\ 0 & \text{otherwise} \end{cases}$$

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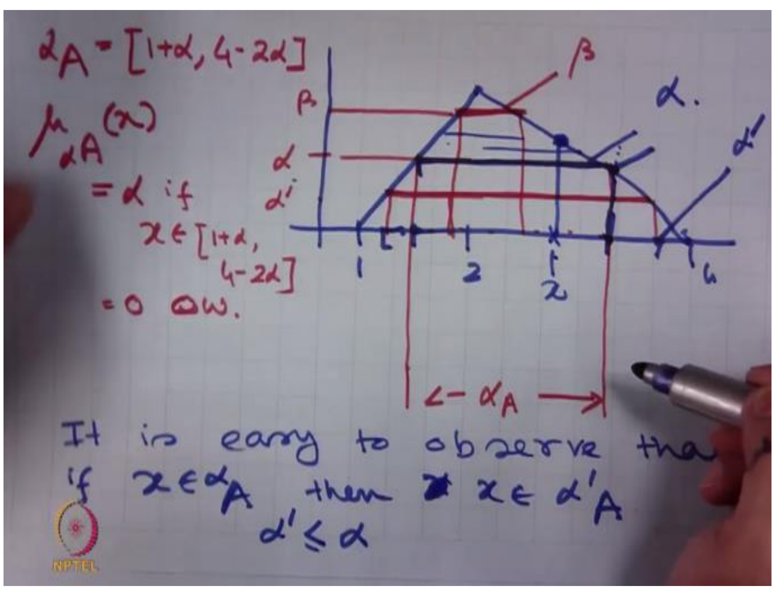

So, what is $^{\alpha}A$?

$$^{\alpha}A = [1+\alpha, 4-2\alpha]$$

$$\therefore \mu_{\,^{\alpha}A}(x) = \begin{cases} \alpha & x \in [1+\alpha, 4-2\alpha] \\ 0 & \text{otherwise} \end{cases}$$

It is easy to observe if $x \in {}^{\alpha}A$ then $x \in {}^{\alpha'}A$ as well when $\alpha' \leq \alpha$

However, if I take $\beta > \alpha$, then we can find that all the elements which belong to ${}^{\beta}A$ and there they will have the membership value equal to β.

Therefore, what is the membership value for each x in the, $\cup_{\alpha}\ {}_{\alpha}A$.

If we consider this x, we can see that as α increases, we are getting the value of the membership to be α and it goes up to this point which is its membership value say α and therefore, above that it is not member of any of these α-cuts. Therefore, it is 0 hence, its membership value is going to be its membership in the original set A. Similarly, if you give me any arbitrary x. I can see that this membership to the union is going to be the supremum of all of them, which is nothing but its membership value to the original set. So, this example illustrates how you can decompose one fuzzy set with the help of its different α-cuts.

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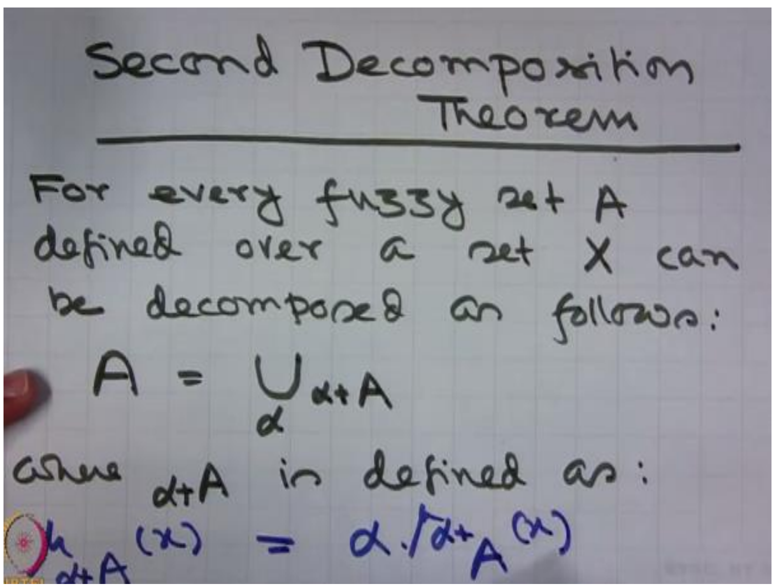

Now, let us talk about the second decomposition theorem.

This theorem suggests that for every fuzzy set A defined over a set X can be decomposed as follows:

$$A = \bigcup_{\alpha} {}_{\alpha+}A$$

Where ${}_{\alpha+}A$ is defined as

$$\mu_{{}_{\alpha+}A}(x) = \alpha \cdot \mu_{{}^{\alpha+}A}(x)$$

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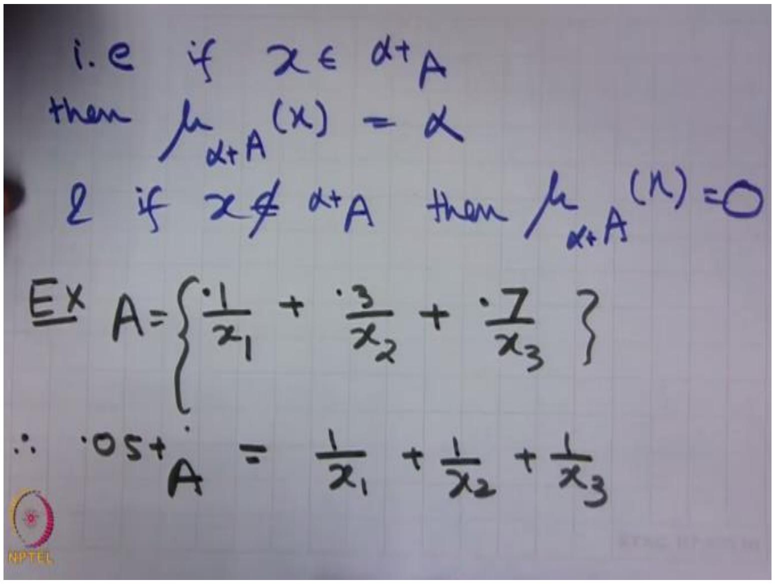

That means, that is if $x \in {}^{\alpha+}A$ then $\mu_{\alpha+A}(x) = \alpha$ and if $x \notin {}^{\alpha+}A$ then $\mu_{\alpha+A}(x) = 0$

Let me illustrate.

Suppose, we have $A = \left\{\frac{0.1}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3}\right\}$

Therefore, ${}^{0.05+}A = \left\{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right\}$

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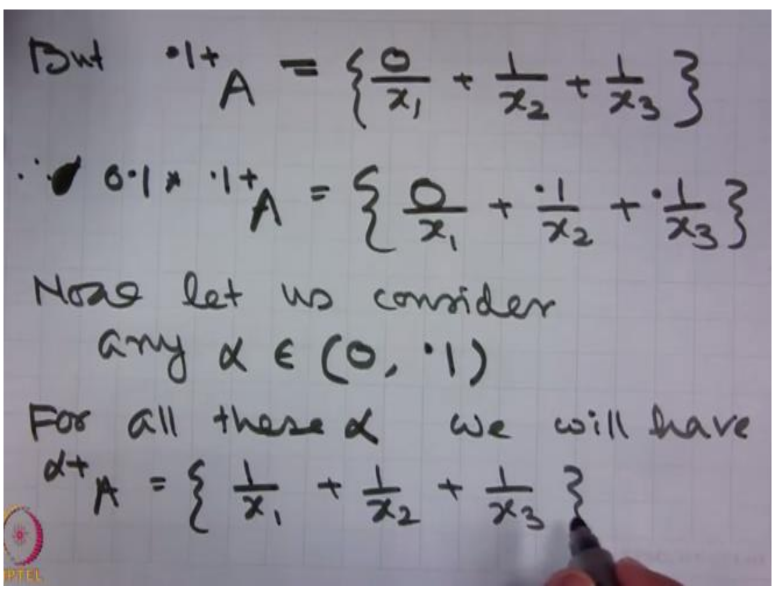

But ${}^{0.1+}A = \left\{\frac{0}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right\}$

$\therefore 0.1 \cdot {}^{0.1+}A = \left\{\frac{0}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3}\right\}$

Now, let us consider any $\alpha \in (0, 0.1)$.

For all these α we will have

$$^{\alpha+}A = \left\{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right\}$$

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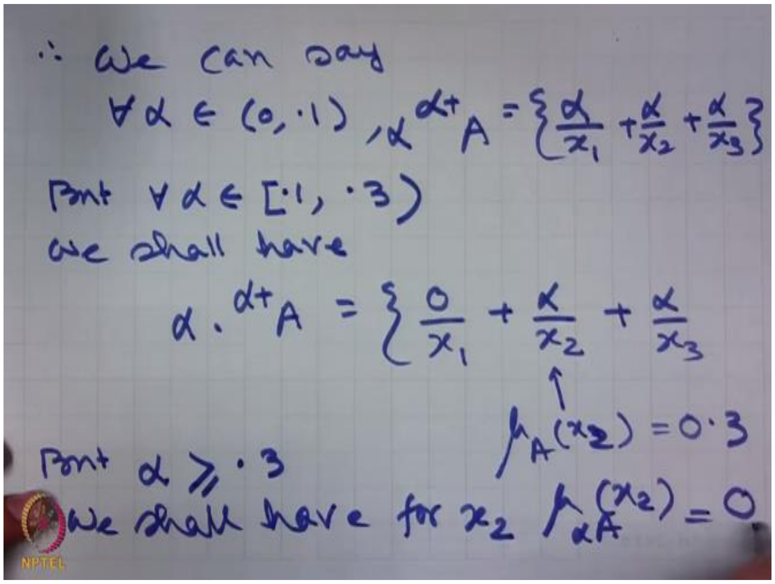

Therefore, we can say, for all $\alpha \in (0, 0.1)$

$$\alpha \cdot {}^{\alpha+}A = \left\{\frac{\alpha}{x_1} + \frac{\alpha}{x_2} + \frac{\alpha}{x_3}\right\}$$

but for all $\alpha \in [0.1, 0.3]$

We shall have

$$\alpha \cdot {}^{\alpha+}A = \left\{\frac{0}{x_1} + \frac{\alpha}{x_2} + \frac{\alpha}{x_3}\right\}$$

This is because $\mu_A(x_2) = 0.3$ but, for $\alpha \geq 0.3$ we shall have for x_2, $\mu_{\alpha A}(x_2) = 0$

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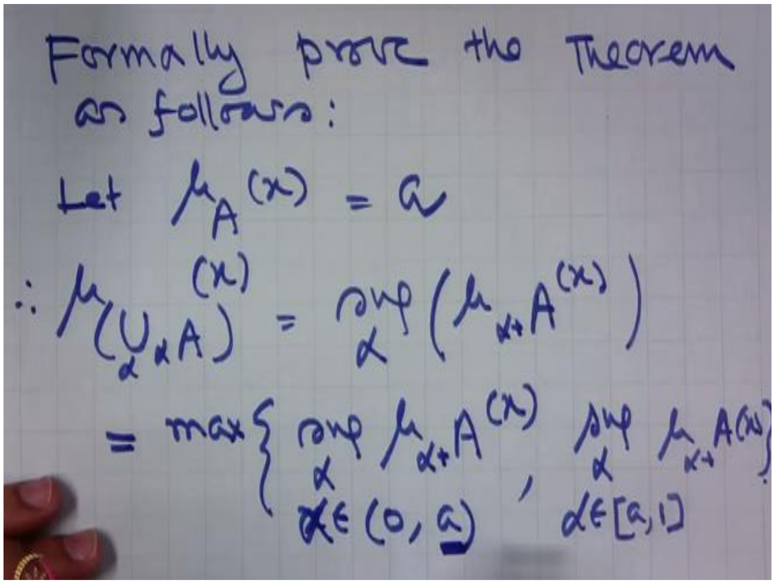

Similarly, we can illustrate for x_3. If we observed that, then we can formally prove the theorem as follows:

Let $\mu_A(x) = a$

$$\therefore \mu_{\cup_{\alpha}\ {}_{\alpha}A}(x) = \sup_{\alpha} \mu_{\ \alpha+A}(x) = \max\left\{ \sup_{\alpha\in(0,a)} \mu_{\ \alpha+A}(x)\ ,\ \sup_{\alpha\in[a,1]} \mu_{\ \alpha+A}(x) \right\}$$

So, notice the difference in the earlier example or in the first decomposition theorem, we had a close bracket here, but since in this case closed bracket a implies that membership is going to be 0 for the corresponding x we put it on this side.

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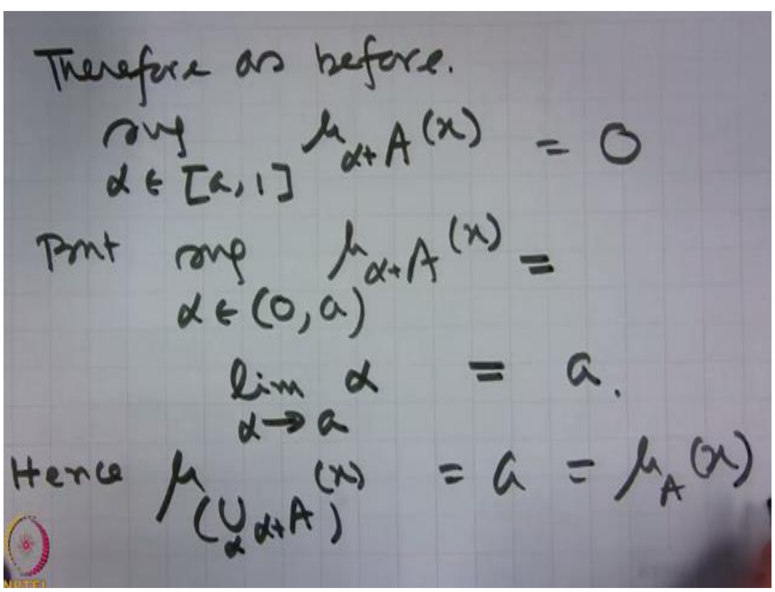

Therefore, as before $\sup\limits_{\alpha\in[a,1]} \mu_{\alpha+A}(x) = 0$ but $\sup\limits_{\alpha\in(0,a)} \mu_{\alpha+A}(x) = \lim\limits_{\alpha\to a}\alpha = a$

Hence, $\mu_{\cup_\alpha\ \alpha+A}(x) = a = \mu_A(x)$ So, this proves the theorem.

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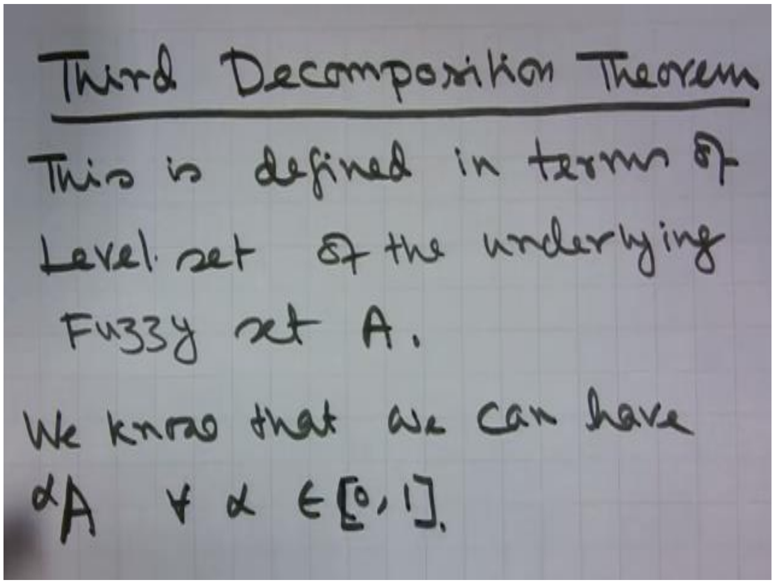

There is a third decomposition theorem:

This is defined in terms of Level set of the underlying Fuzzy set A.

Now, you may ask me what is the level set.

So, let me explain this concept we know that we can have ${}^{\alpha}A \ \ \forall \ \ \alpha \in [0,1]$

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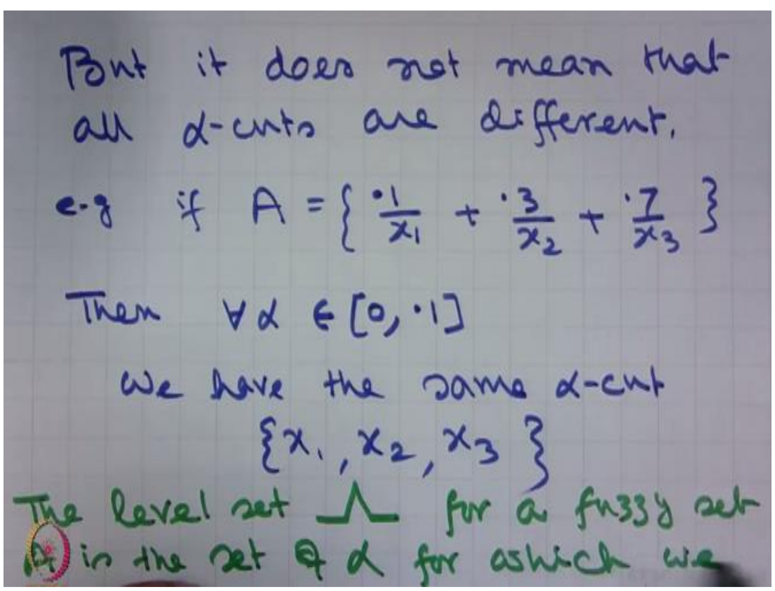

But it does not mean that all α-cuts are different.

For example, if $A = \left\{\frac{0.1}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3}\right\}$ then for all $\alpha \in [0, 0.1]$, we have the same alpha cut namely $\{x_1, x_2, x_3\}$

The level set Λ for a fuzzy set A is the set of α for which we have distinct α-cuts.

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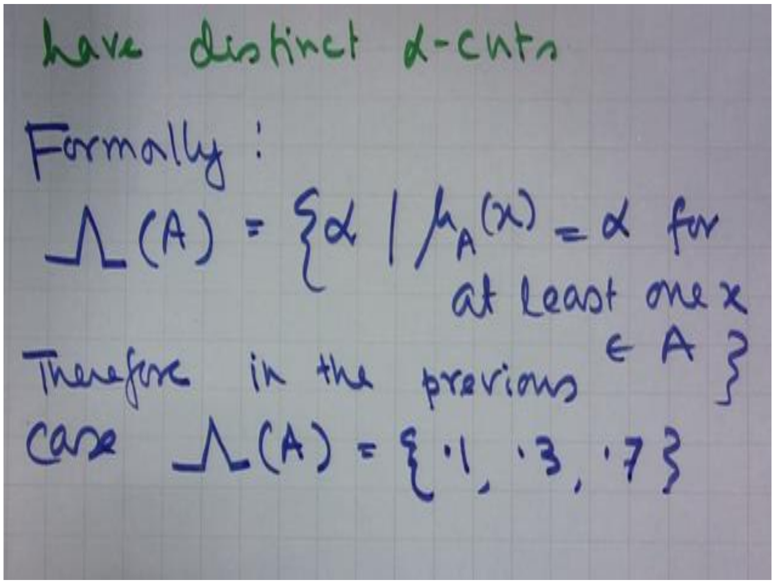

So, formally,

The $\Lambda(A) = \{\alpha | \mu_A(x) = \alpha \text{ for at least one } x \in A\}$.

Therefore, in the previous case $\Lambda(A) = \{0.1, 0.3, 0.7\}$

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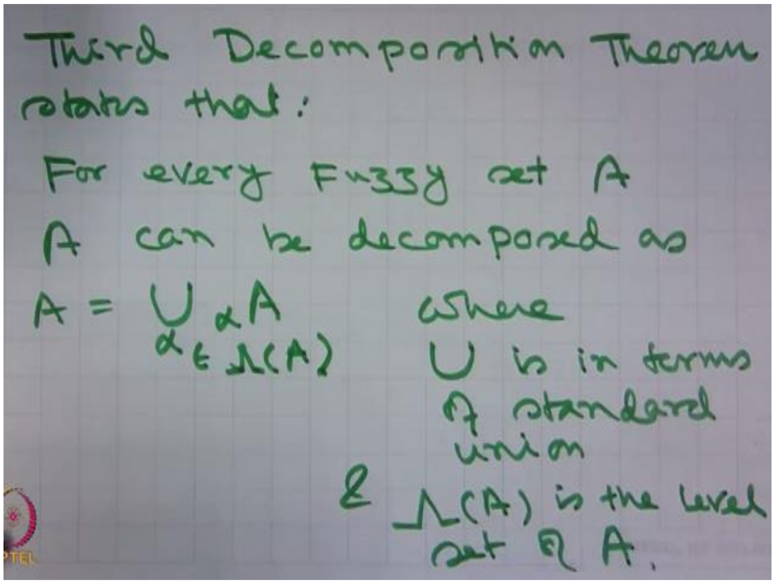

So, the third decomposition theorem states that:

For every fuzzy set A, A can be decomposed as $A = \bigcup_{\alpha\in\Lambda(A)} {}_{\alpha}A$ where $\cup$ is in terms of standard union and $\Lambda(A)$ is the level set of A.

Ok Students, I stop here today, but before I go I give you some results and I ask you to prove yourself.

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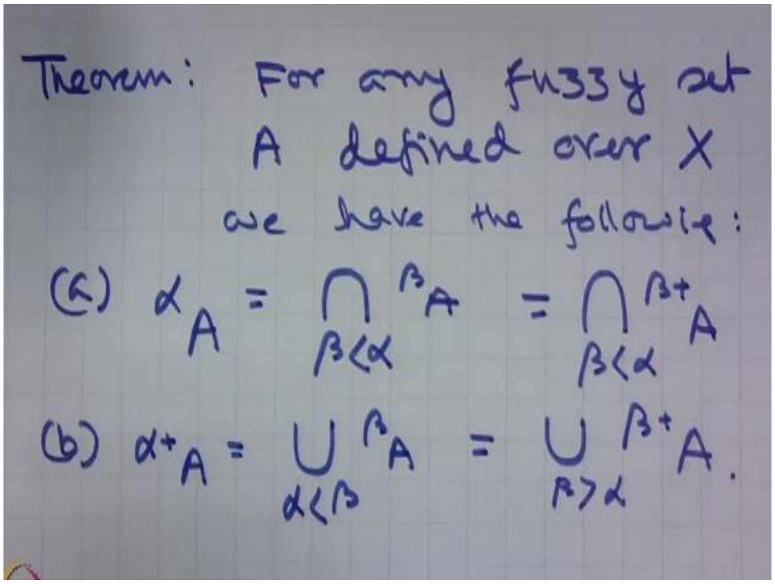

One is that for any fuzzy set A defined over X we have the following:

(a) ${}^{\alpha}A = \cap_{\beta<\alpha} {}^{\beta}A = \cap_{\beta<\alpha} {}^{\beta+}A$

(b) ${}^{\alpha+}A = \cup_{\alpha<\beta} {}^{\beta}A = \cup_{\alpha<\beta} {}^{\beta+}A$

So, as I said I want you to try proving this with the concept of supremum and infimum.

The proof will be very similar that is why I am not doing that in this class.

I stop here today. In the next class, I shall talk about another important concept of fuzzy set which is called extension principle and what is the extension principle?

We have functions defined over crisp sets to crisp set what happens if we impose a fuzzy set on those crisp sets.

Then how are we going to extend the function so, that they can be defined from a fuzzy set defined on the crisp set X to another fuzzy set say B.

Extension principles helps us to define those fuzzy functions or define those fuzzification of those crisp functions when we are thinking in terms of fuzzy sets.

That is for the next class. For the time being let us stop here. Thank you.