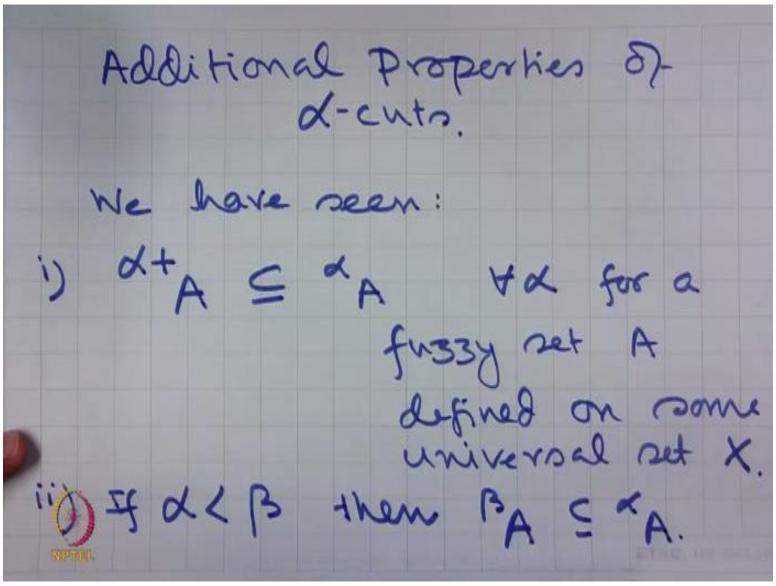
Introduction to Fuzzy Sets Arithmetic and Logic
Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology – Delhi

Lecture - 13
Fuzzy Sets Arithmetic and Logic

Welcome students to the lecture number 13 for the MOOCs on Fuzzy Sets, Arithmetic and logic.

(Refer Slide Time: 00:30)

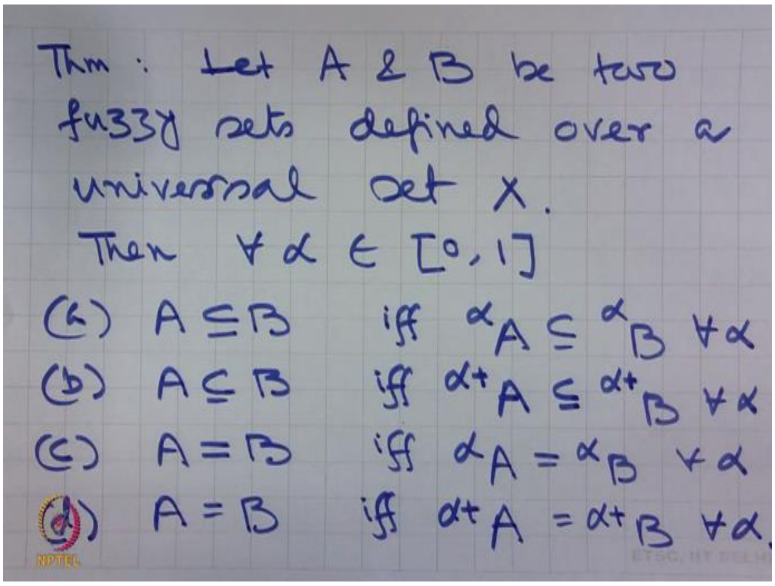


In today's lecture, we shall study some additional properties of alpha cuts, if you remember we have already seen some properties, say for example:

i. $^{\alpha+}A \subseteq {}^{\alpha}A \quad \forall\alpha$ for a fuzzy set A defined on some universal set X.

ii. If $\alpha < \beta$ then, $^{\beta}A \subseteq {}^{\alpha}A$

(Refer Slide Time: 02:01)



Today we examine some more properties of alpha cuts so, let me first state four results:

Theorem:

Let A and B be two fuzzy sets defined over a universal set X. Then for all $\alpha \in [0, 1]$

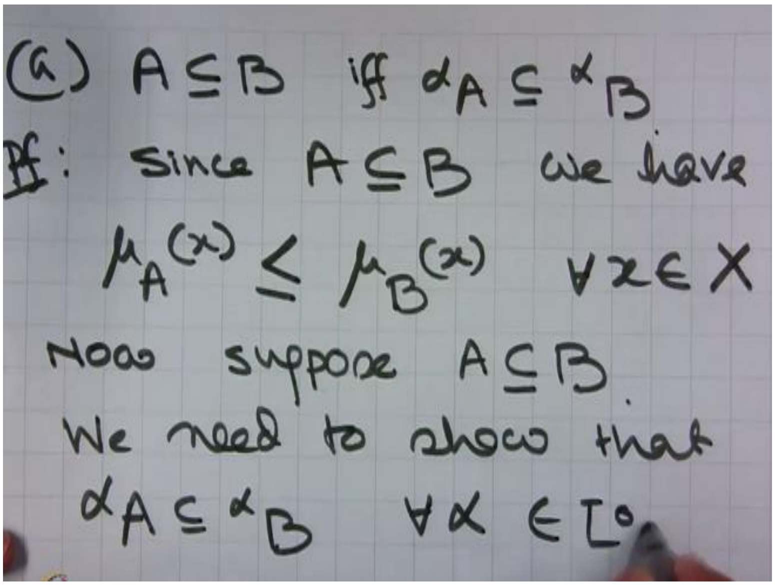
(a) $A \subseteq B$ iff ${}^{\alpha}A \subseteq {}^{\alpha}B \quad \forall\alpha$

(b) $A \subseteq B$ iff ${}^{\alpha+}A \subseteq {}^{\alpha+}B \quad \forall\alpha$

(c) $A = B$ iff ${}^{\alpha}A = {}^{\alpha}B \quad \forall\alpha$

(d) $A = B$ iff ${}^{\alpha+}A = {}^{\alpha+}B \quad \forall\alpha$

(Refer Slide Time: 04:10)



So, let me prove this results:

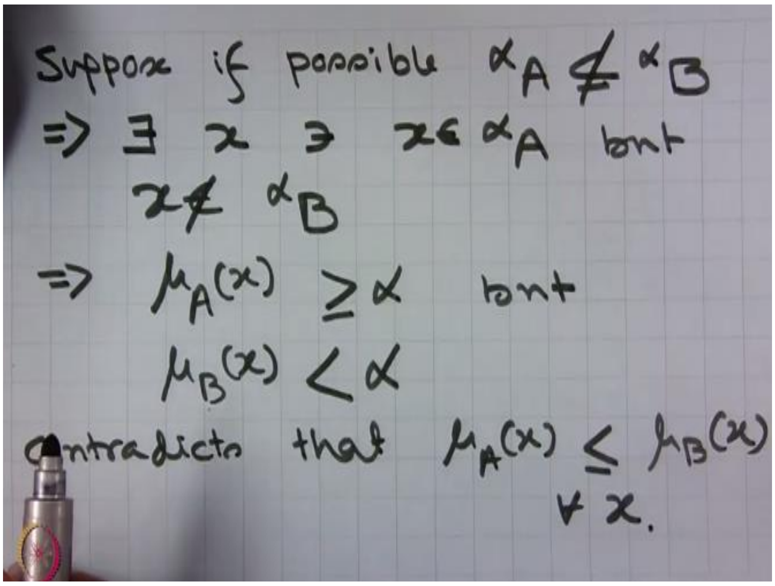
(a) $A \subseteq B$ iff ${}^{\alpha}A \subseteq {}^{\alpha}B \quad \forall\alpha$

Proof: What do you mean by $A \subseteq B$?

Since $A \subseteq B$ we have $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$

Now, suppose $A \subseteq B$, we need to show that ${}^{\alpha}A \subseteq {}^{\alpha}B$ for all $\alpha \in [0,1]$

(Refer Slide Time: 05:38)



Suppose, if possible ${}^{\alpha}A \nsubseteq {}^{\alpha}B$

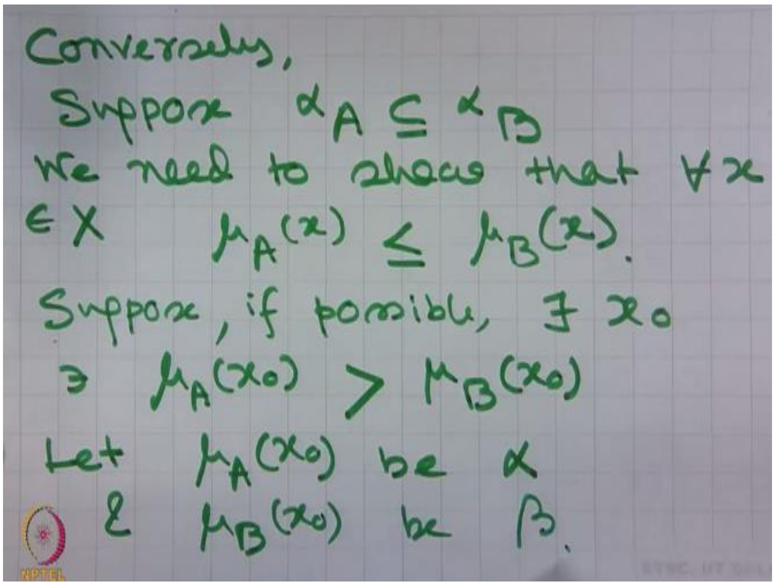
$\Rightarrow$ $\exists\, x$ such that $x \in {}^{\alpha}A$ but $x \notin {}^{\alpha}B$

$\Rightarrow \mu_A(x) \geq \alpha$ but $\mu_B(x) < \alpha$

Contradicts that $\mu_A(x) \leq \mu_B(x)$ for all x

So, this is very obvious from this side that if $A \subseteq B$ then ${}^{\alpha}A \subseteq {}^{\alpha}B$

(Refer Slide Time: 07:08)



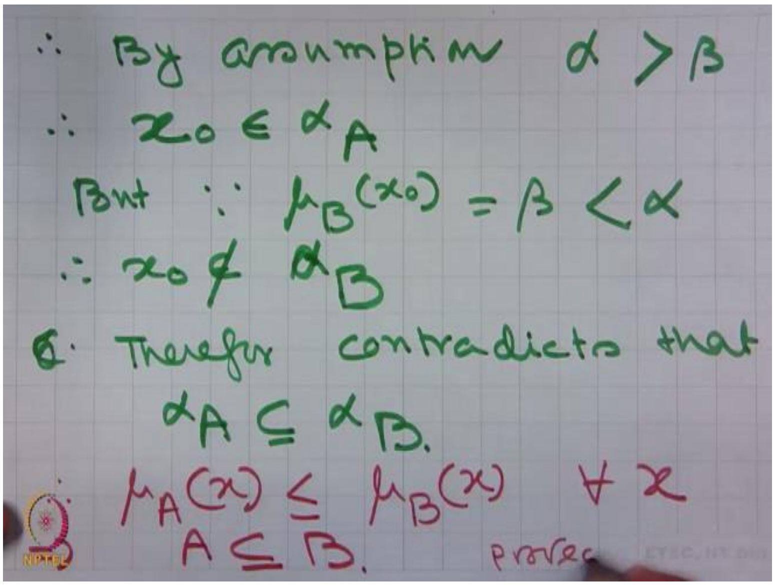
Conversely, suppose ${}^{\alpha}A \subseteq {}^{\alpha}B$

We need to show that for all $x \in X$, $\mu_A(x) \leq \mu_B(x)$

Suppose, if possible there exist x_0 such that $\mu_A(x_0) > \mu_B(x_0)$.

Let $\mu_A(x_0)$ be α and $\mu_B(x_0)$ be β

(Refer Slide Time: 08:39)



Therefore, by assumption $\alpha > \beta$ since, we have a $\mu_A(x_0) > \mu_B(x_0)$

Therefore, $x_0 \in {}^{\alpha}A$ but, since $\mu_B(x_0) = \beta < \alpha$

Therefore, $x_0 \notin {}^{\alpha}B$

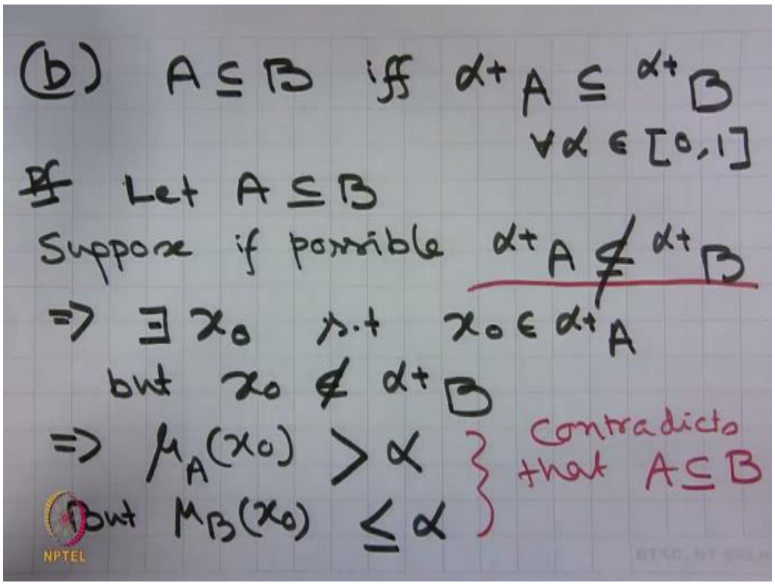
Therefore, contradicts that ${}^{\alpha}A \subseteq {}^{\alpha}B$ and this contradiction arises because of the assumption that there exists such an x_0.

Therefore, we can see that $\mu_A(x) \le \mu_B(x)$, for all x

$\Rightarrow A \subseteq B$

Proved.

(Refer Slide Time: 10:54)



Now, let us prove

(b) $A \subseteq B$ iff ${}^{\alpha+}A \subseteq {}^{\alpha+}B \quad \forall \alpha \in [0, 1]$

Again as before let $A \subseteq B$

Suppose, if possible ${}^{\alpha+}A \nsubseteq {}^{\alpha+}B$

$\Rightarrow$ there exist x_0 such that $x_0 \in {}^{\alpha+}A$ but $x_0 \notin {}^{\alpha+}B$

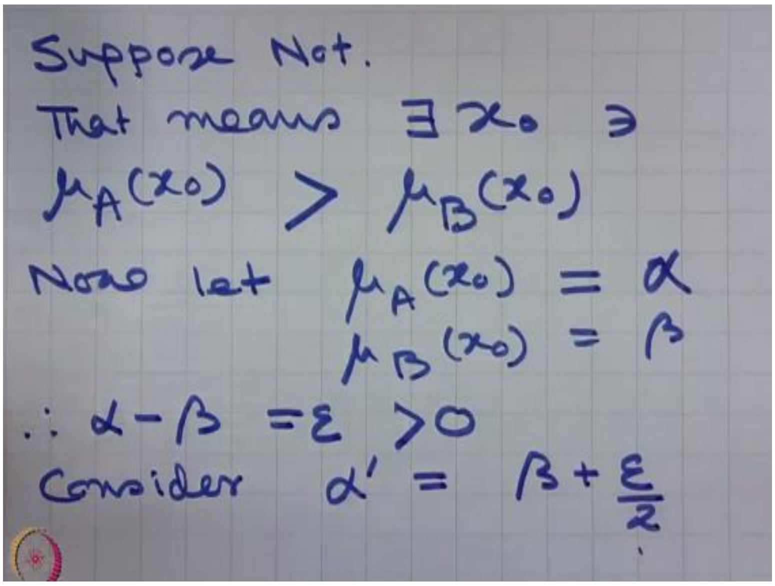
$\Rightarrow \mu_A(x_0) > \alpha$, but $\mu_B(x_0) \leq \alpha$, together contradicts that $A \subseteq B$.

Why is the contradiction?

We assumed that ${}^{\alpha+}A \nsubseteq {}^{\alpha+}B$

Therefore, we prove that ${}^{\alpha+}A \subseteq {}^{\alpha+}B$

(Refer Slide Time: 13:26)



Conversely, suppose ${}^{\alpha+}A \subseteq {}^{\alpha+}B$ for all α, we need to show that $A \subseteq B$.

Suppose not.

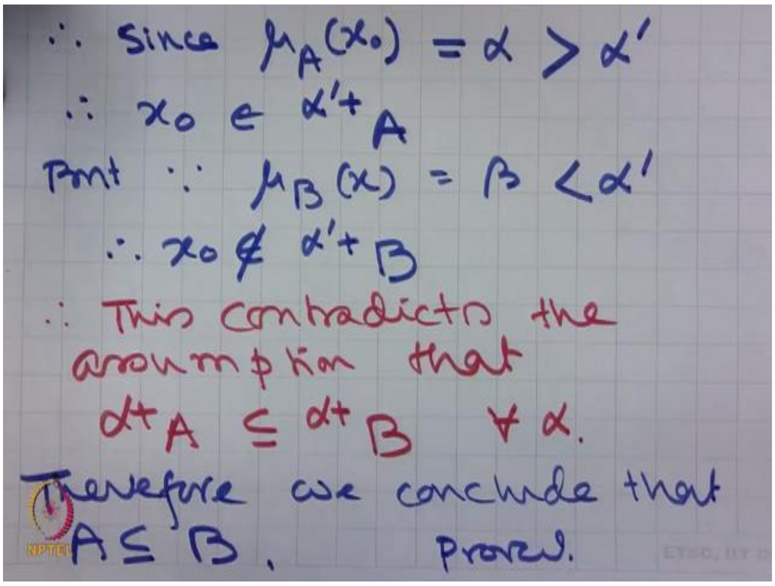
That means, there exist x_0 such that $\mu_A(x_0) > \mu_B(x_0)$ only in that case, we can say that $A \nsubseteq B$

Now, let $\mu_A(x_0) = \alpha$ and $\mu_B(x_0) = \beta$

Therefore, $\alpha - \beta = \epsilon > 0$

Consider, $\alpha' = \beta + \frac{\epsilon}{2}$

(Refer Slide Time: 15:35)



Therefore, since $\mu_A(x_0) = \alpha > \alpha'$

Therefore, $x_0 \in {}^{\alpha'+}A$

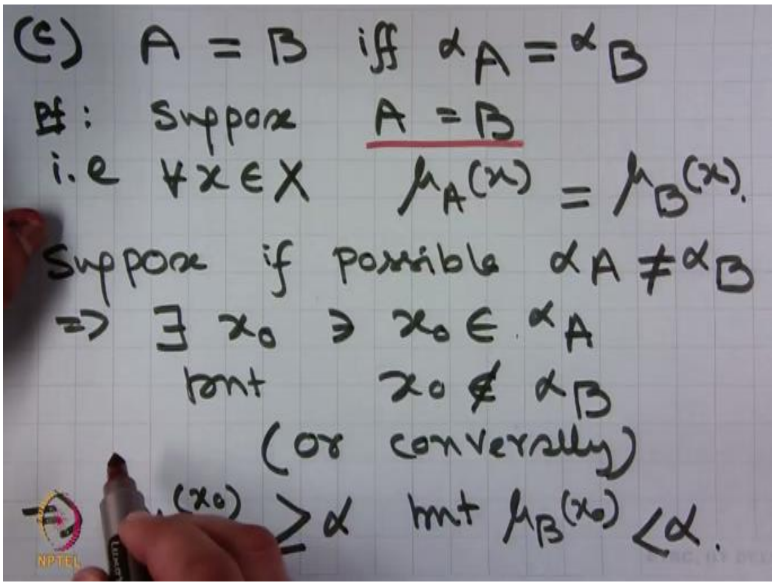
But, since $\mu_B(x_0) = \beta < \alpha'$

Therefore, $x_0 \notin {}^{\alpha'+}B$

So, this contradicts our assumption that ${}^{\alpha+}A \subseteq {}^{\alpha+}B$ for all α

Therefore, we conclude that $A \subseteq B$. Proved.

(Refer Slide time: 17:41)



Let us now look at statement number

(c) $A = B$ iff ${}^{\alpha}A = {}^{\alpha}B \quad \forall \alpha$

Proof: Suppose, $A = B$

that is $\forall\ x \in X\ \mu_A(x) = \mu_B(x)$

Suppose if possible ${}^{\alpha}A \neq {}^{\alpha}B$

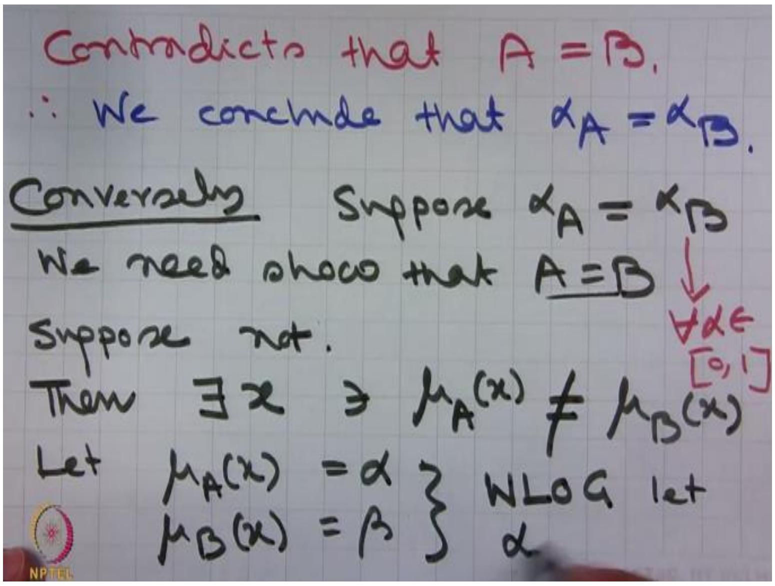
$\Rightarrow$ there exist, x_0 such that $x_0 \in {}^{\alpha}A$ but $x_0 \notin {}^{\alpha}B$ (or conversely).

That is there may be x_0 which belongs to ${}^{\alpha}B$, but that does not belong to ${}^{\alpha}A$, but the line of argument is going to be very similar.

Since $x_0 \in {}^{\alpha}A$, but $x_0 \notin {}^{\alpha}B$, $\Rightarrow \mu_A(x_0) \geq \alpha$ but, $\mu_B(x_0) < \alpha$

This contradicts the assumption that $A = B$

(Refer Slide Time: 19:57)



Therefore, we conclude that ${}^{\alpha}A = {}^{\alpha}B$.

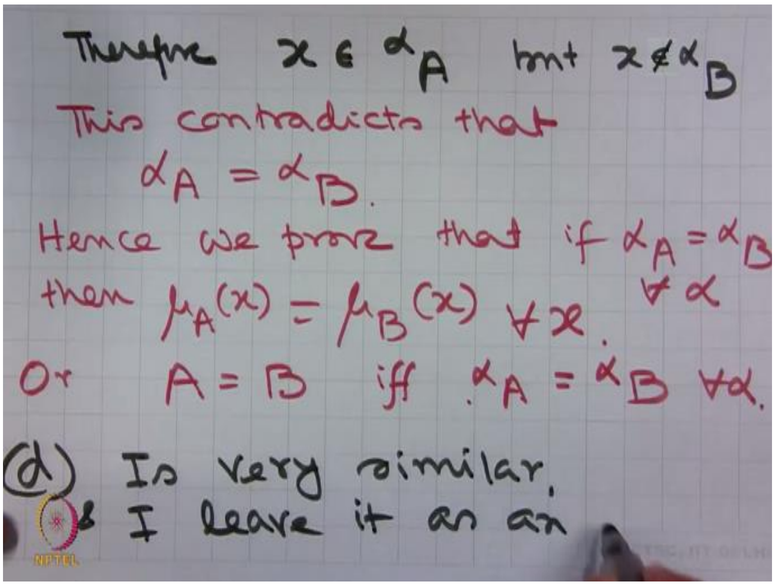
Conversely, suppose ${}^{\alpha}A = {}^{\alpha}B$ for all $\alpha \in [0,1]$

We need to show that $A = B$

Suppose not, then there exist x such that $\mu_A(x) \neq \mu_B(x)$

Let $\mu_A(x) = \alpha$, $\mu_B(x) = \beta$ and without loss of generality let $\alpha > \beta$.

(Refer Slide Time: 22:19)



Therefore, $x \in {}^{\alpha}A$, but $x \notin {}^{\alpha}B$, because $\mu_B(x) = \beta < \alpha$.

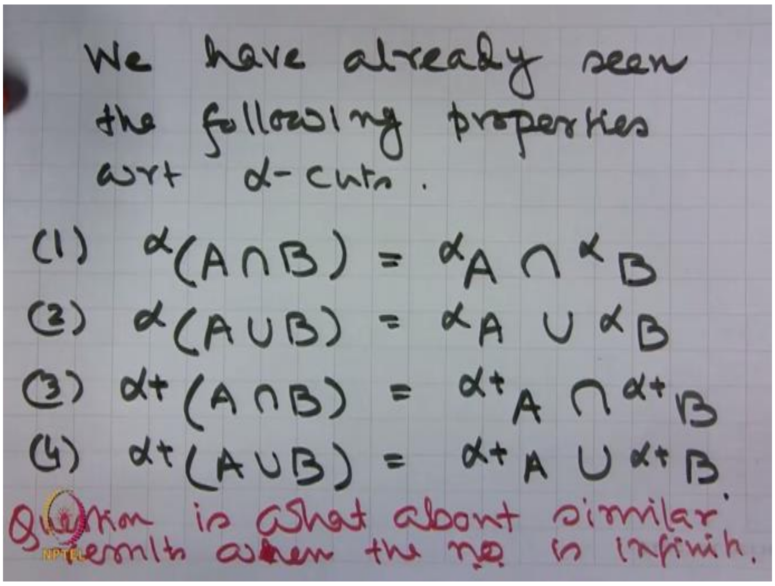
This contradicts that ${}^{\alpha}A = {}^{\alpha}B$

Hence, we prove that if ${}^{\alpha}A = {}^{\alpha}B$ for all α, then $\mu_A(x) = \mu_B(x)$ for all x or

$A = B$, if and only if ${}^{\alpha}A = {}^{\alpha}B$ for all α

Statement (d), is very similar and I leave it as an exercise.

(Refer Slide Time: 24:19)



Ok Students.

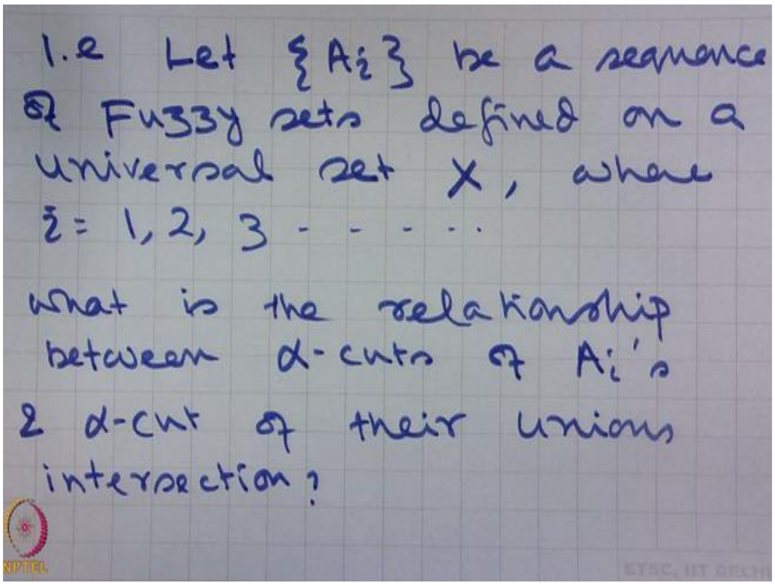
We have already seen the following properties with respect to α-cuts.

1) ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$
2) ${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$
3) ${}^{\alpha+}(A \cap B) = {}^{\alpha+}A \cap {}^{\alpha+}B$
4) ${}^{\alpha+}(A \cup B) = {}^{\alpha+}A \cup {}^{\alpha+}B$

These results we have already seen.

And perhaps it is not difficult to imagine that we can now extend it to finite number of sets as well, question is what about similar results when the number is infinite or in other words,

(Refer Slide Time: 26:45)



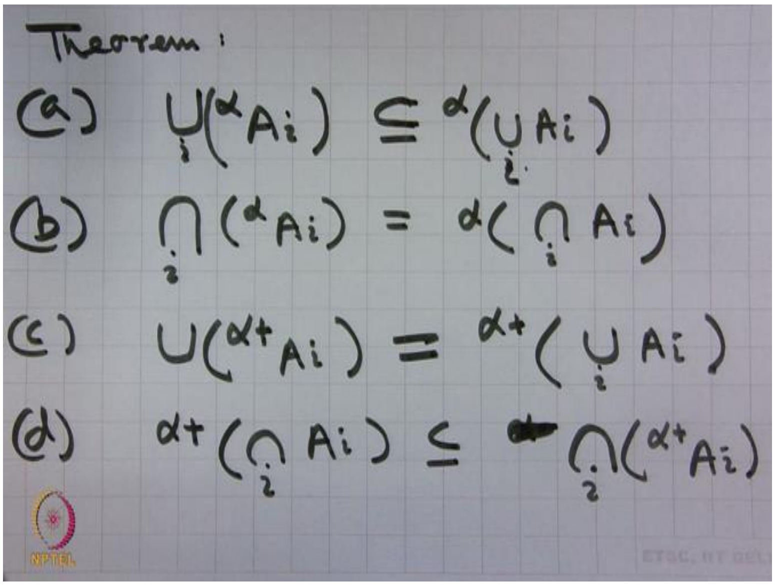
That is, let $\{A_i\}$ be a sequence of fuzzy sets defined on a universal set X, where $i = 1, 2, \ldots \infty$

That means, we are now looking at an infinite collection of fuzzy sets defined over the same universal set X.

Question is what is the relationship between α-cuts of $A_i s$ and α-cut of their unions, intersections?

So, we need to study this when we are looking at an infinite number of sets $A_1, A_2, A_{,3}$ ....etc.

(Refer Slide Time: 28:33)



So the theorem,

(a) $\bigcup_i ({}^{\alpha}A_i) \subseteq {}^{\alpha}(\bigcup_i A_i)$

So, with respect to two fuzzy sets we have seen that they are actually equal but in this case we find that it is containment

(b) $\bigcap_i ({}^{\alpha}A_i) = {}^{\alpha}(\bigcap_i A_i)$

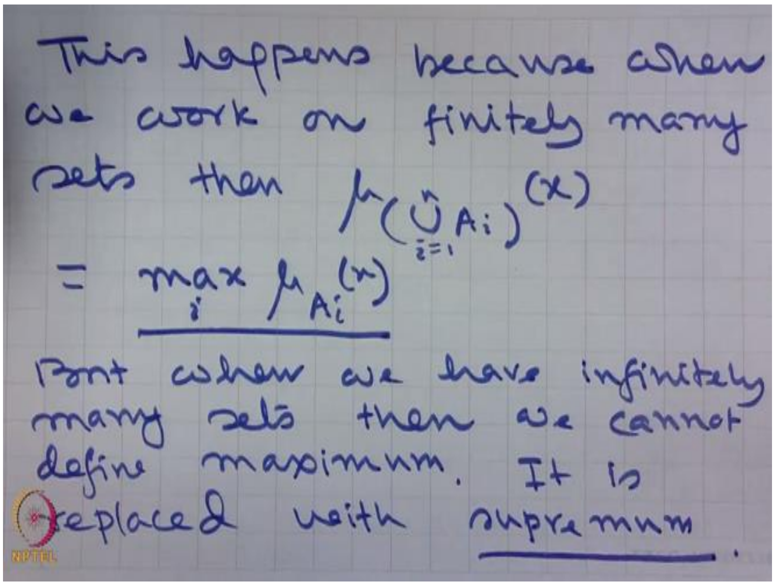
(c) $\cup_i ({}^{\alpha+}A_i) = {}^{\alpha+}(\cup_i A_i)$

(d) ${}^{\alpha+}(\cap_i A_i) \subseteq \cap_i ({}^{\alpha+}A_i)$

Thus, we can see that there is difference when we are looking at only two sets and when we are looking at infinitely many sets.

In these two cases we find that the equality does not hold rather it is containment.

(Refer Slide Time: 31:34)



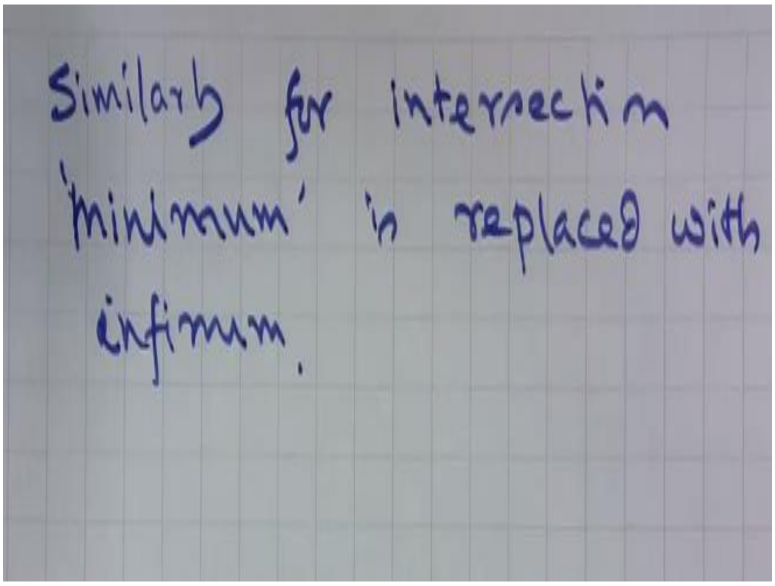
Question is why so?

This happens because when we work on finitely many sets then

$$\mu_{(\cup_{i=1}^{n} A_i)}(x) = \max_{\mathrm{i}} \mu_{A_i}(x)$$

When the number of sets is finite this maximum can be defined but, when we have infinitely many sets then we cannot define maximum. It is replaced with supremum.

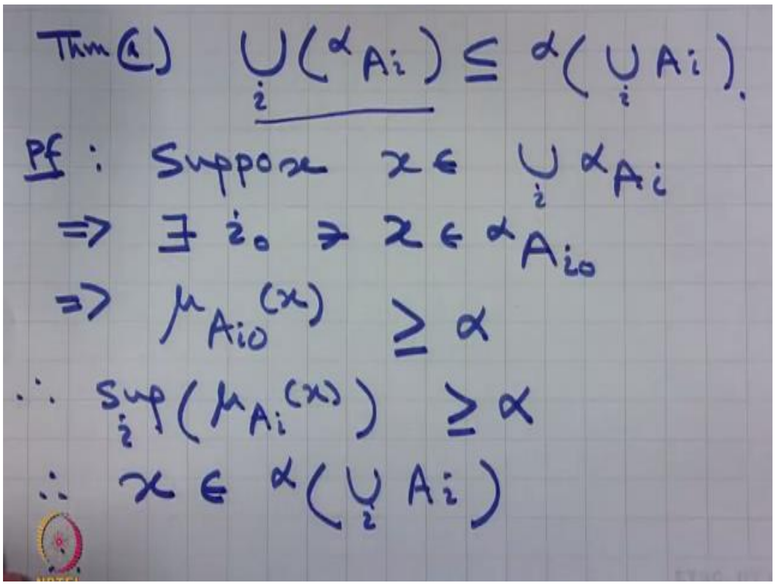
(Refer Slide Time: 33:28)



Similarly, for intersection minimum is replaced with infimum and because of this we see that the equality does not hold for all the cases.

So, with that small insight let us now start proving these results.

(Refer Slide Time: 34:18)



Theorem

(a) $\bigcup_i ({}^\alpha A_i) \subseteq {}^\alpha(\bigcup_i A_i)$

Proof: Suppose $x \in \bigcup_i ({}^\alpha A_i)$

$\Rightarrow$ there exist i_0 such that $x \in {}^\alpha A_{i_0}$

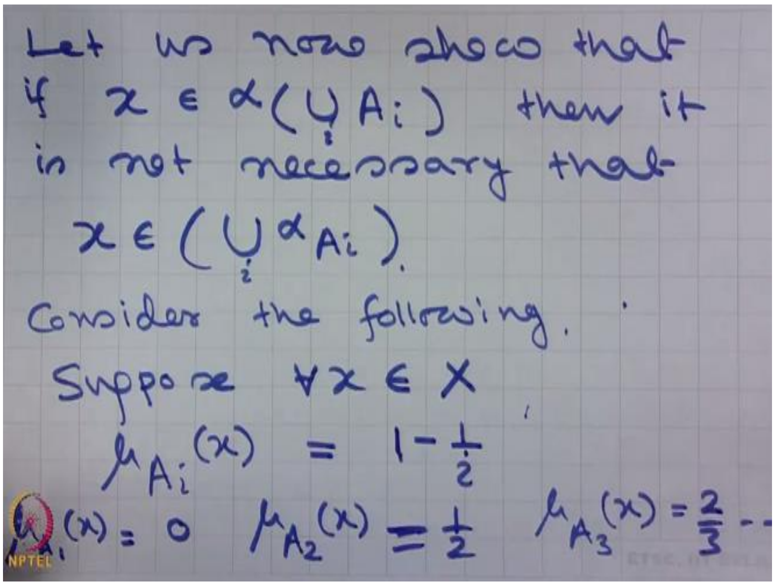
$\Rightarrow \mu_{A_{i_0}}(x) \geq \alpha$

Therefore, $\sup_i \mu_{A_i}(x) \geq \alpha$

Therefore, $x \in {}^\alpha(\bigcup_i A_i)$

We need to show there may exist $x \in {}^\alpha(\bigcup_i A_i)$ which does not belong to $\bigcup_i ({}^\alpha A_i)$

(Refer Slide Time: 36:43)



Let us now show that if $x \in {}^{\alpha}(\cup_i A_i)$, then it is not necessary that $x \in \cup_i({}^{\alpha}A_i)$

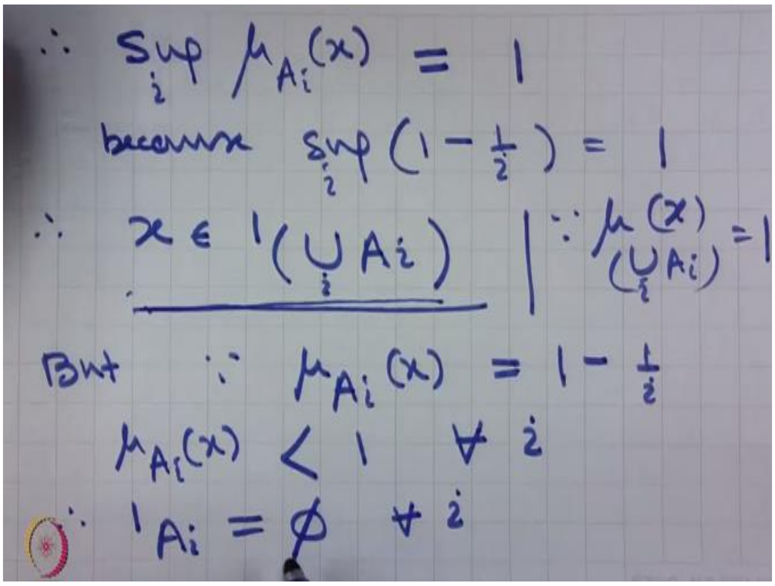
Consider the following, suppose $\forall\ x \in X$

$$\mu_{A_i}(x) = 1 - \frac{1}{i}$$

What does it mean?

That is, $\mu_{A_1}(x) = 0,\ \mu_{A_2}(x) = \frac{1}{2},\ \mu_{A_3}(x) = \frac{2}{3}, \dots \dots$

(Refer Slide Time: 38:30)



Therefore, $\sup\limits_{\mathrm{i}} \mu_{A_i}(x) = 1$, because $\sup\limits_{\mathrm{i}} \left(1 - \frac{1}{i}\right) = 1$

Therefore, $x \in {}^1(\cup_i A_i)$, because the supremum is 1, therefore $\mu_{\cup_i A_i}(x) = 1$

But since $\mu_{A_i}(x) = 1 - \frac{1}{i}$

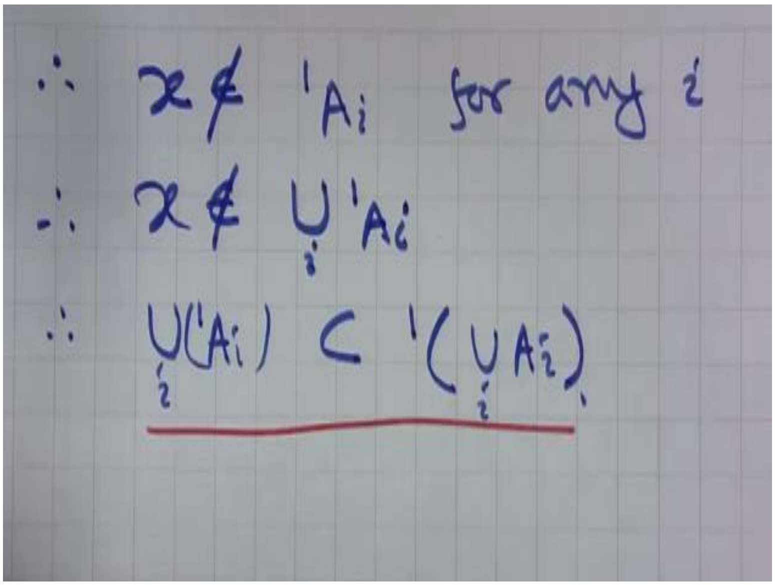
$\mu_{A_i}(x) < 1$ for all i

Therefore, ${}^{1}A_i = \phi$ for all i.

What does it mean?

Which means that here in this case $x \in {}^{1}(\cup_i A_i)$ but ${}^{1}A_i = \phi$ for all of them.

(Refer Slide time: 40:31)



Therefore, $x \notin {}^{1}A_i$ for any i

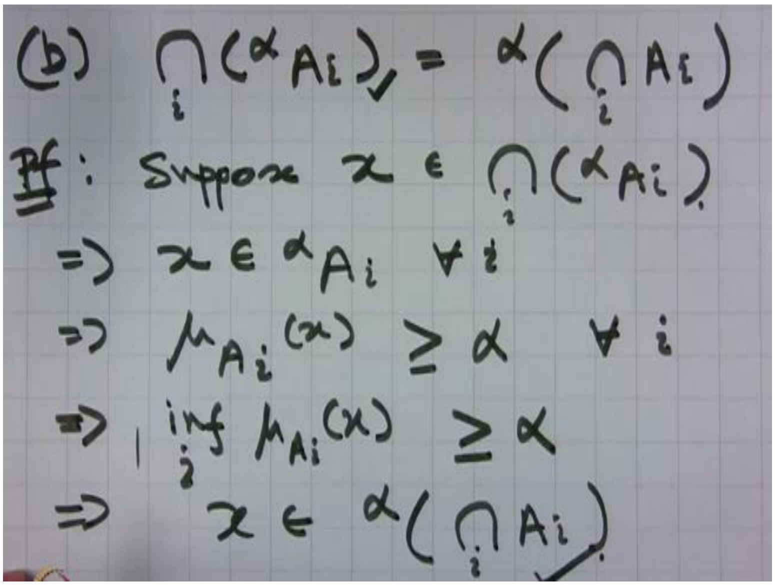
Therefore, $x \notin \cup_i {}^{1}A_i$

Therefore, $\cup_i {}^{1}A_i \subset {}^{1}(\cup_i A_i)$

Proved.

So this is the result that $\cup_i({}^{\alpha}A_i) \subseteq {}^{\alpha}(\cup_i A_i)$

(Refer Slide Time: 41:35)



Let us now prove that second result,

(b) $\bigcap_i({}^{\alpha}A_i) = {}^{\alpha}(\bigcap_i A_i)$

So in this case we can see that there exists equality what do it mean?

It means that if we take any $x \in \bigcap_i({}^{\alpha}A_i)$, we will prove that $x \in {}^{\alpha}(\bigcap_i A_i)$ as well and similarly if I take an $x \in {}^{\alpha}(\bigcap_i A_i)$ will show that $x \in \bigcap_i({}^{\alpha}A_i)$.

Suppose $x \in \bigcap_i({}^{\alpha}A_i)$

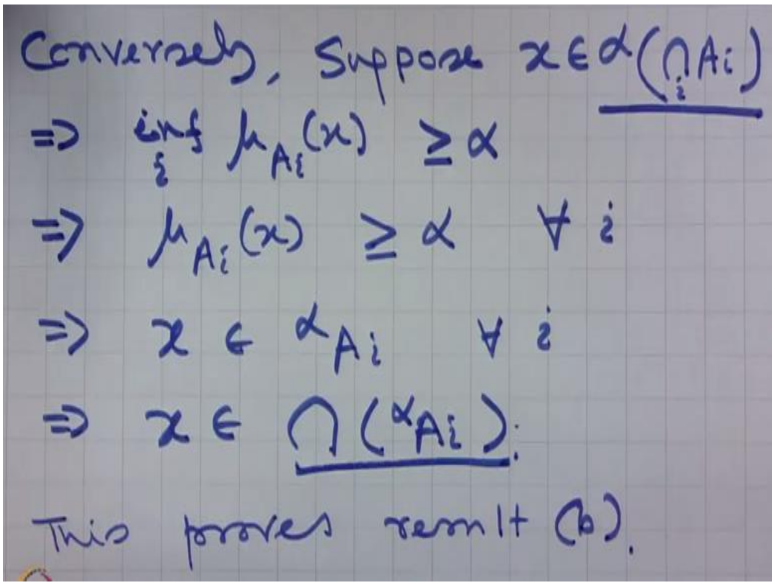
$\Rightarrow x \in {}^{\alpha}A_i$ for all i,

$\Rightarrow \mu_{A_i}(x) \geq \alpha$ for all i

$\Rightarrow \inf_i \mu_{A_i}(x) \geq \alpha$

$\Rightarrow x \in {}^{\alpha}(\bigcap_i A_i)$

(Refer Slide time: 44:09)



Conversely, suppose $x \in {}^{\alpha}(\bigcap_i A_i)$

$\Rightarrow \inf_i \mu_{A_i}(x) \geq \alpha$

$\Rightarrow \mu_{A_i}(x) \geq \alpha$ for all i

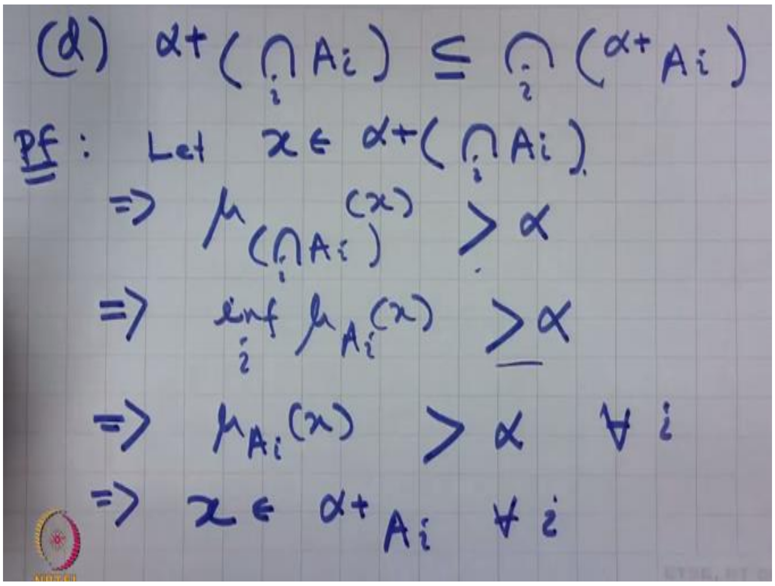
$\Rightarrow x \in {}^{\alpha}A_i$ for all i

$\Rightarrow x \in \bigcap_i({}^{\alpha}A_i)$

So we proved result (b) also, in a similar way one can prove results (c)

I leave as an exercise, but what I prove now is result (d).

(Refer Slide Time: 46:14)



So result (d) is the following, it shows that:

(d) ${}^{\alpha+}(\bigcap_i A_i) \subseteq \bigcap_i({}^{\alpha+}A_i)$

Note that here it is strict containment.

Proof: Let $x \in {}^{\alpha+}(\bigcap_i A_i)$

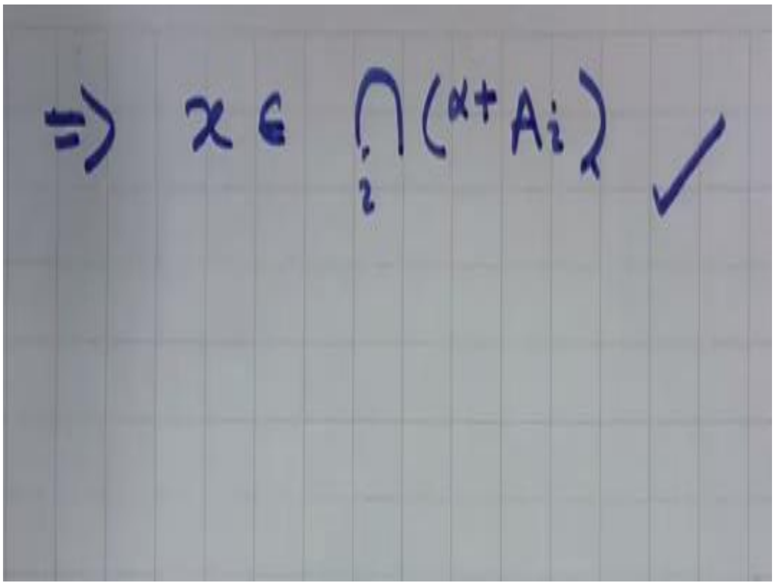
$\Rightarrow \mu_{(\bigcap_i A_i)}(x) > \alpha$

$\Rightarrow \inf_{\mathrm{i}} \mu_{A_i}(x) > \alpha$

$\Rightarrow \mu_{A_i}(x) > \alpha$ for all i

$\Rightarrow x \in {}^{\alpha+}A_i$ for all i

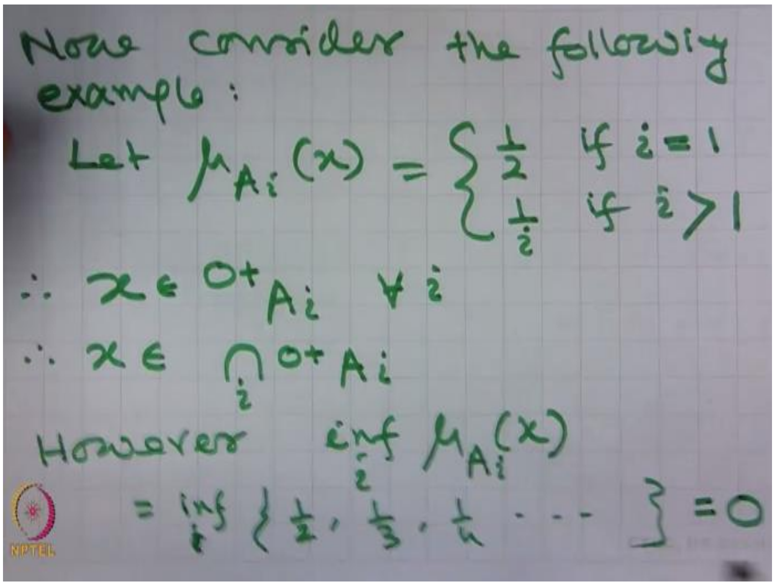
(Refer Slide time: 48:38)



$\Rightarrow x \in \bigcap_i({}^{\alpha+}A_i)$

Now let us look at the other way.

(Refer Slide Time: 49:05)



Now consider the following example:

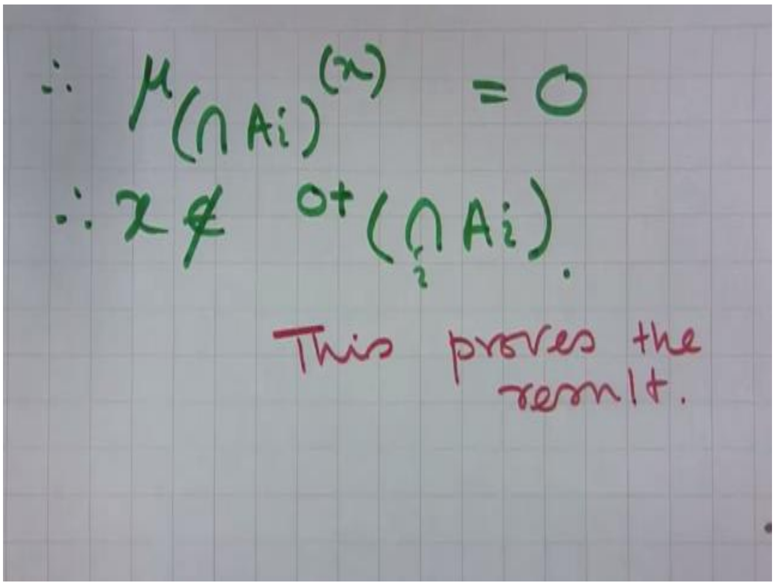
Let $\mu_{A_i}(x) = \begin{cases} \frac{1}{2} & if\ i = 1 \\ \frac{1}{2} & if\ i > 1 \end{cases}$

Therefore, $x \in {}^{0+}A_i$ for all i.

Therefore, $x \in \cap_i {}^{0+}A_i$

However $\inf\limits_{\mathrm{i}} \mu_{A_i}(x) = \inf\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots \dots \dots\right\} = 0.$

(Refer Slide Time: 50:50)



Therefore, $\mu_{(\cap A_i)}(x) = 0$

Therefore, $x \notin {}^{0+}(\cap_i A_i)$

So this proves the result.

I stop here today in the next class I shall look into alpha cuts and strong alpha cuts we will see how these can be used to represent a fuzzy set, Thank you so much.