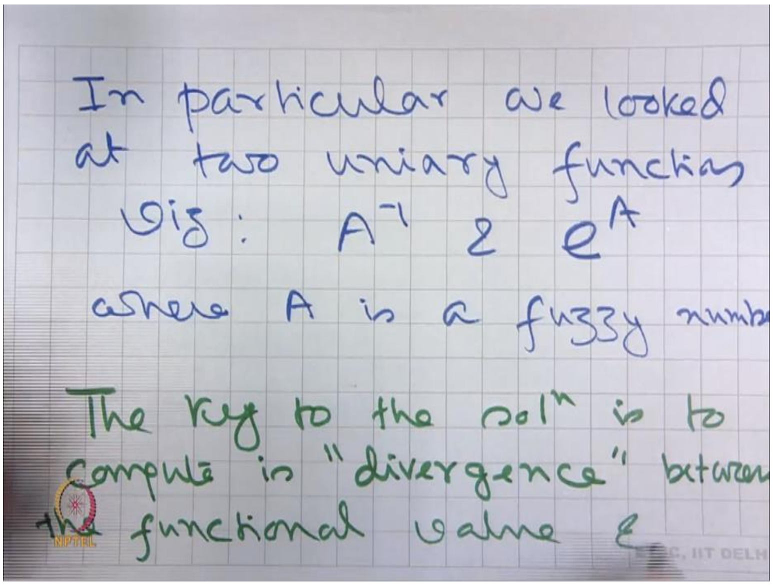
Introduction to Fuzzy Sets Arithmetic and Logic

Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology - Delhi

Module - 4
Lecture - 12
Fuzzy Sets Arithmetic and Logic

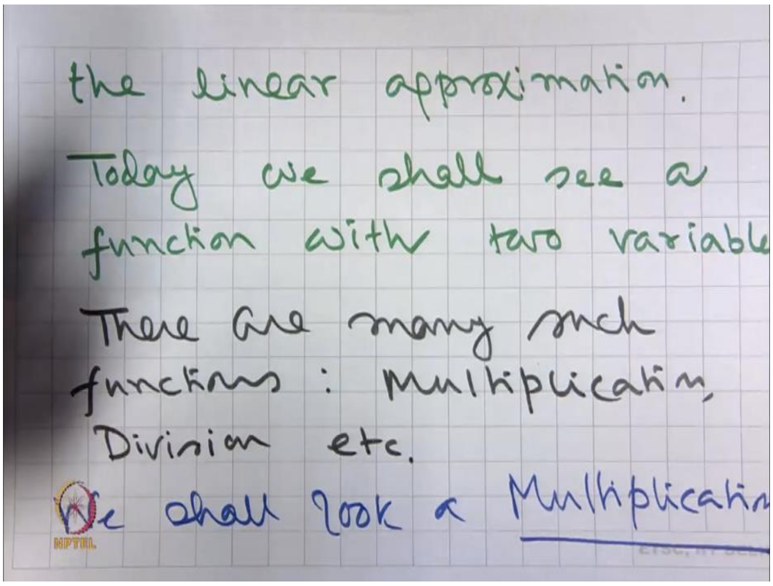
Welcome students to lecture number 12 on fuzzy sets arithmetic and logic. In the last lecture, we started discussing on linear approximation of fuzzy numbers.

(Refer Slide Time: 00:37)



In particular, we looked at two unary functions, namely A^{-1} and e^{A}, where A is a fuzzy number. The key to the solution is to compute the divergence between the functional value and the linear approximation.

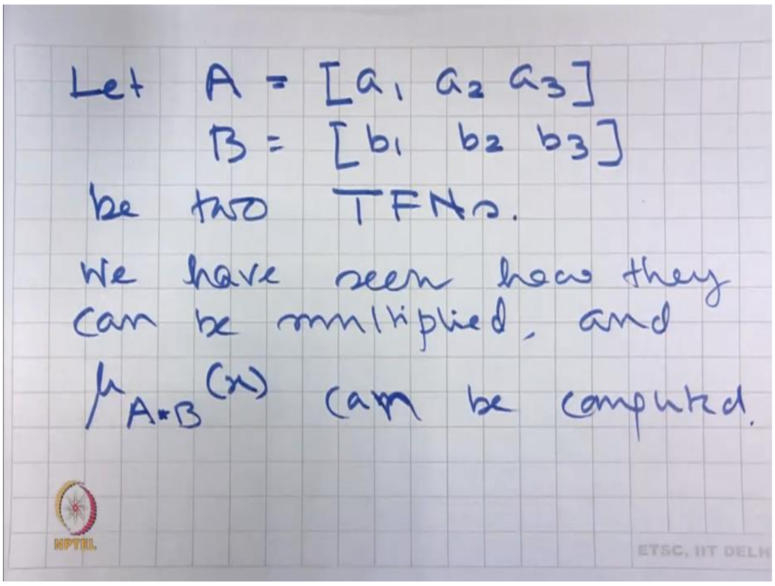
(Refer Slide Time: 01:51)



Today, we shall see a function with two variables.

There are many such functions like the multiplication, division, etcetera. We shall look at multiplication.

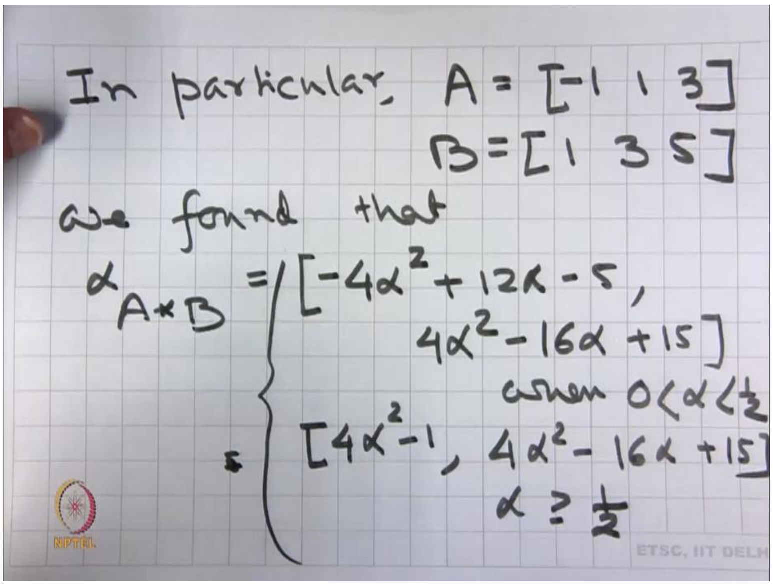
(Refer Slide Time: 03:12)



So, let $A = [a_1 \;\; a_2 \;\; a_3]$ and $B = [b_1 \;\; b_2 \;\; b_3]$ be two triangular fuzzy numbers.

We have seen how they can be multiplied; and $\mu_{A*B}(x)$ can be computed.

(Refer Slide Time: 04:15)



In particular, when $A = [-1\ \ 1\ \ 3]$ and $B = [1\ \ 3\ \ 5]$

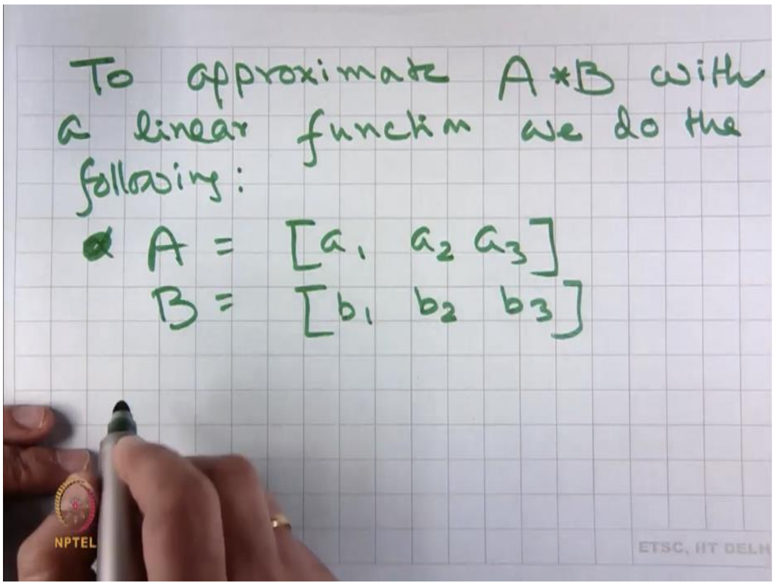
We found that

$$^{\alpha}A * B = \begin{cases} [-4\alpha^2 + 12\alpha - 5, 4\alpha^2 - 16\alpha + 15] & 0 \le \alpha < \dfrac{1}{2} \\ [4\alpha^2 - 1, 4\alpha^2 - 16\alpha + 15] & \dfrac{1}{2} \le \alpha \le 1 \end{cases}$$

So, these results we already have.

Because these are quadratic, we like to use a linear membership function to represent $A * B$.

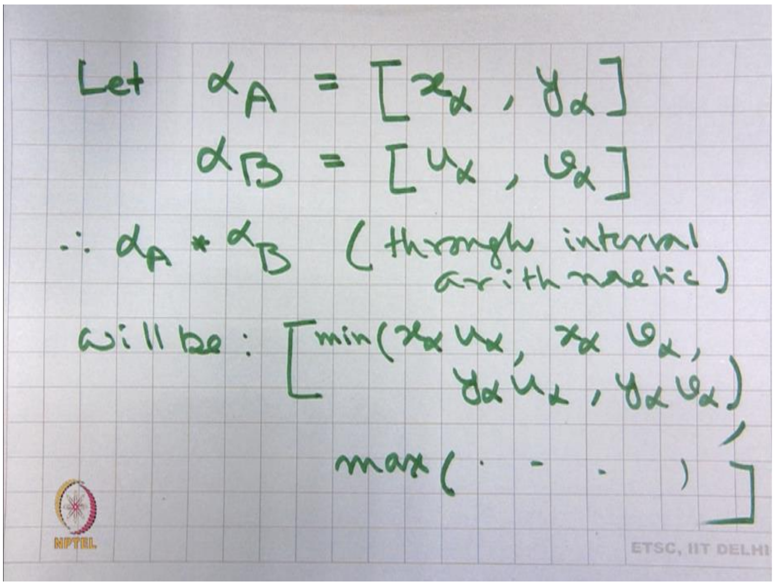
(Refer Slide Time: 06:10)



So, to approximate it with a linear function, we do the following.

so, $A = [a_1\ \ a_2\ \ a_3]$ and $B = [b_1\ \ b_2\ \ b_3]$

(Refer Slide Time: 07:01)



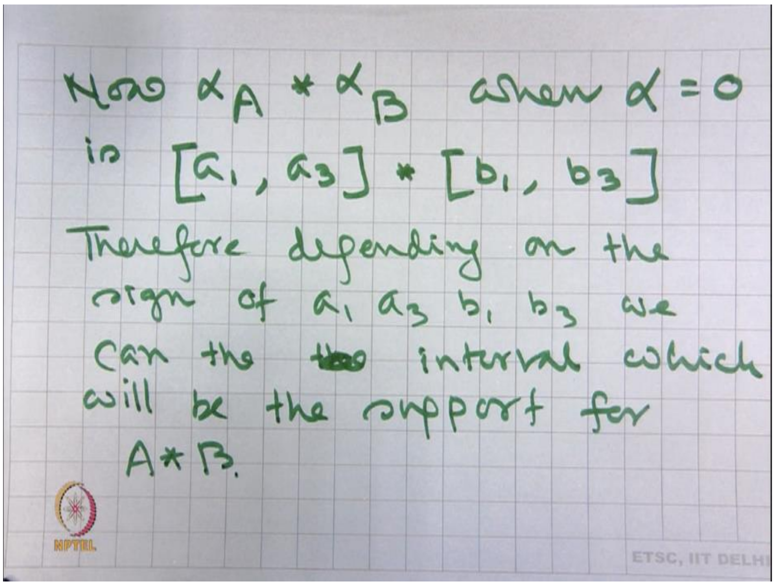
Let ${}^{\alpha}A = [x_\alpha, y_\alpha]$ and ${}^{\alpha}B = [u_\alpha, v_\alpha]$

Therefore, ${}^{\alpha}A * {}^{\alpha}B$ through interval arithmetic will be

$$[\min(x_\alpha * u_\alpha, x_\alpha * v_\alpha, y_\alpha * u_\alpha, y_\alpha * v_\alpha), \max(x_\alpha * u_\alpha, x_\alpha * v_\alpha, y_\alpha * u_\alpha, y_\alpha * v_\alpha)]$$

So, that is what we get by multiplying alpha-cut of A and alpha-cut of B.

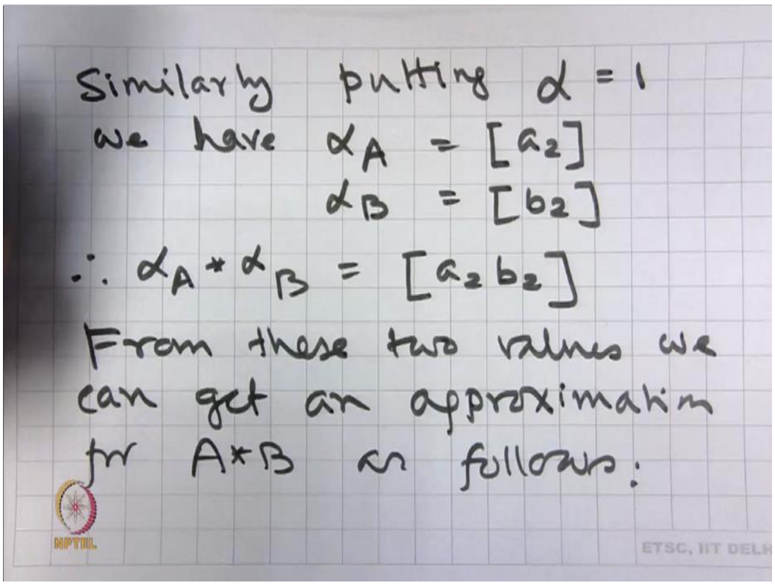
(Refer Slide Time: 08:27)



Now, ${}^{\alpha}A * {}^{\alpha}B$ when $\alpha = 0$ is $[a_1 \;\; a_3] * [b_1 \;\; b_3]$

Therefore, depending on the sign of a_1, a_3, b_1, b_3, we can get the interval which will be the support for $A * B$.

(Refer Slide Time: 09:39)



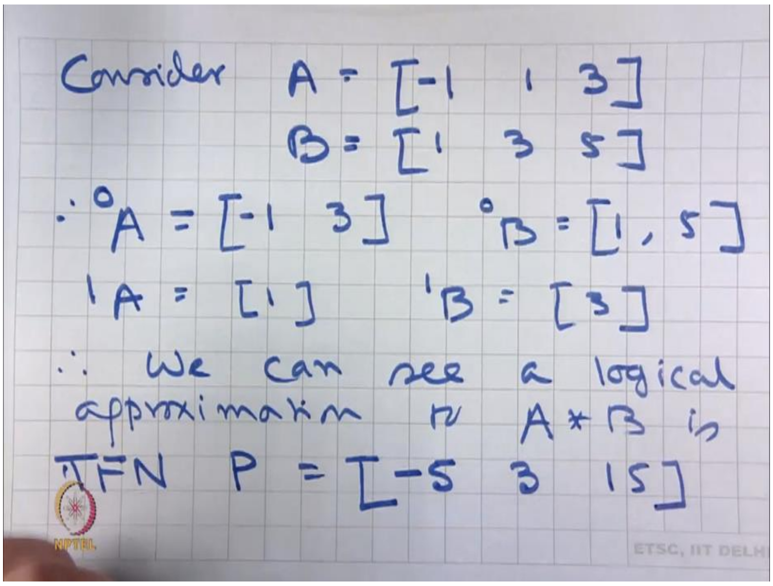
Similarly, putting $\alpha = 1$,

We have $^{\alpha}A = [a_2]$, $^{\alpha}B = [b_2]$.

$\therefore\ ^{\alpha}A * {}^{\alpha}B = [a_2 b_2]$

From these two values, we can get an approximation for $A * B$ as follows.

(Refer Slide Time: 10:50)



So, consider $A = [-1 \;\; 1 \;\; 3]$ and $B = [1 \;\;\; 3 \;\;\; 5]$

Therefore $^{0}A = [-1,\; 3]$ and $^{0}B = [1,\; 5]$

$^{1}A = [1]$ and $^{1}B = [3]$

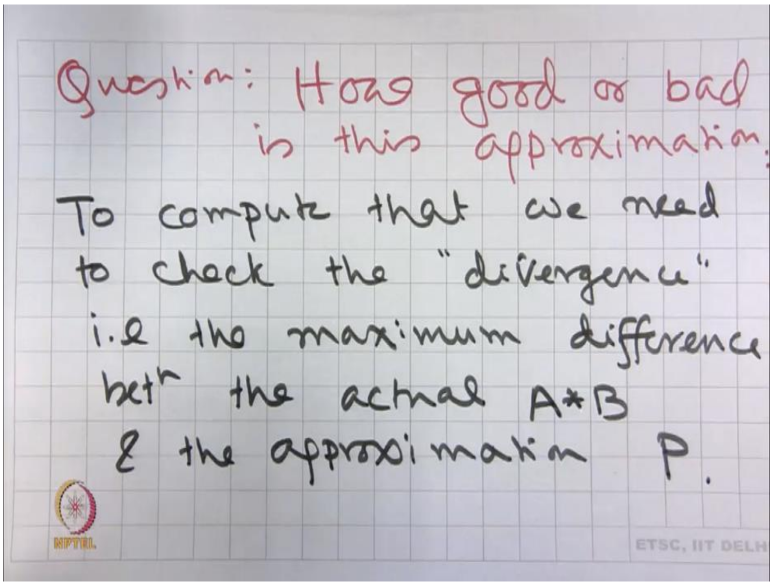
So together, we can see a logical approximation to $A * B$ is TFN $P = [-5 \;\; 3 \;\; 15]$.

If you look at this, we can see that the range goes or the support of $A * B$ is from -5 to 15.

And from ^{1}A and ^{1}B we can get the modal point is 3.

So together, we can see, the logical approximation for $P = [-5 \;\; 3 \;\; 15]$.

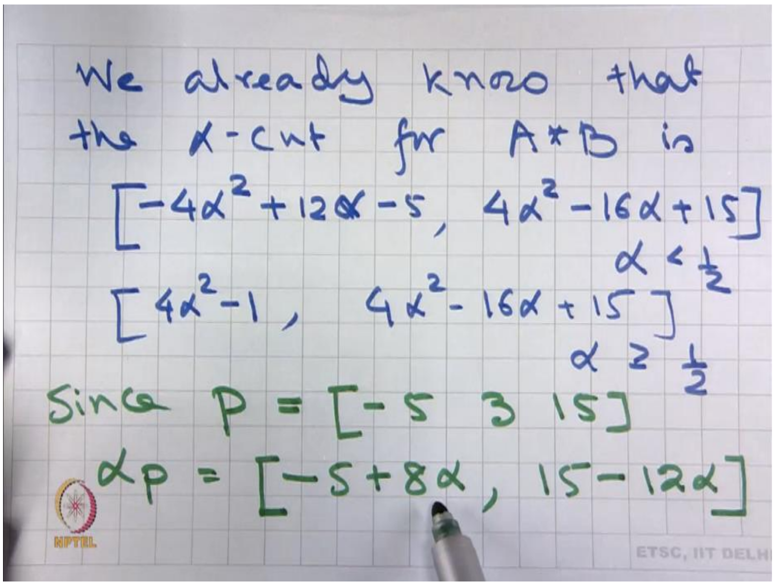
(Refer Slide Time: 12:34)



Question is: How good or bad is this approximation?

So, to compute that, we need to check the divergence. That is, the maximum difference between the actual $A * B$ and the approximation P.

(Refer Slide Time: 13:54)



We already know from lecture 10 that

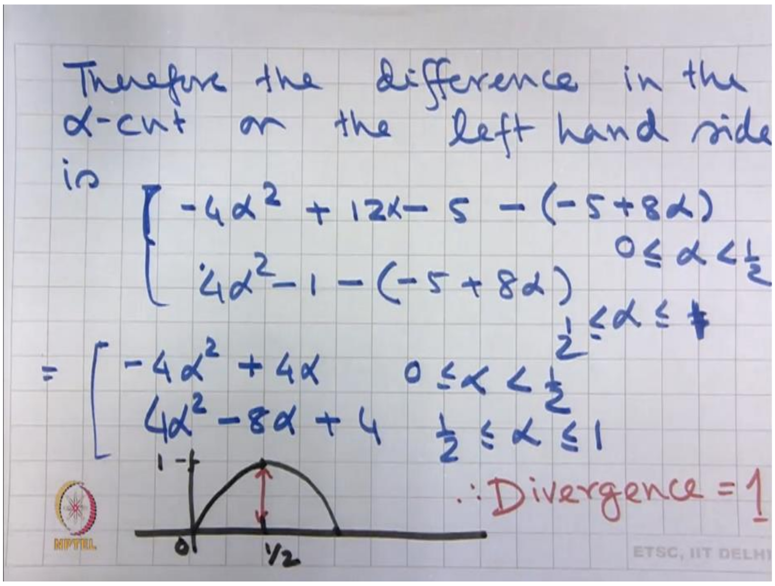
$$
{}^{\alpha}A * B = \begin{cases} [-4\alpha^2 + 12\alpha - 5, 4\alpha^2 - 16\alpha + 15] & 0 \leq \alpha < \dfrac{1}{2} \\ [4\alpha^2 - 1, 4\alpha^2 - 16\alpha + 15] & \dfrac{1}{2} \leq \alpha \leq 1 \end{cases}
$$

Since $P = [-5 \quad 3 \quad 15]$

$$
{}^{\alpha}P = [-5 + 8\alpha, 15 - 12\alpha]
$$

This we know, we have seen many times how to calculate from a given triangular fuzzy number by using the similarity of triangles.

(Refer Slide Time: 15:49)



Therefore, the difference in the alpha-cut on the left-hand side is

$$\begin{cases} -4\alpha^2 + 12\alpha - 5 - (-5 + 8\alpha) & 0 \le \alpha < \frac{1}{2} \\ 4\alpha^2 - 1 - (-5 + 8\alpha) & \frac{1}{2} \le \alpha \le 1 \end{cases}$$

$$= \begin{cases} -4\alpha^2 + 4\alpha & 0 \le \alpha < \frac{1}{2} \\ 4\alpha^2 - 8\alpha + 4 & \frac{1}{2} \le \alpha \le 1 \end{cases}$$

So, if we plot the graph, we get;

At $\alpha = 0$, $-4\alpha^2 + 4\alpha = 0$

At $\alpha = \frac{1}{2}$, $-4\alpha^2 + 4\alpha = 1$

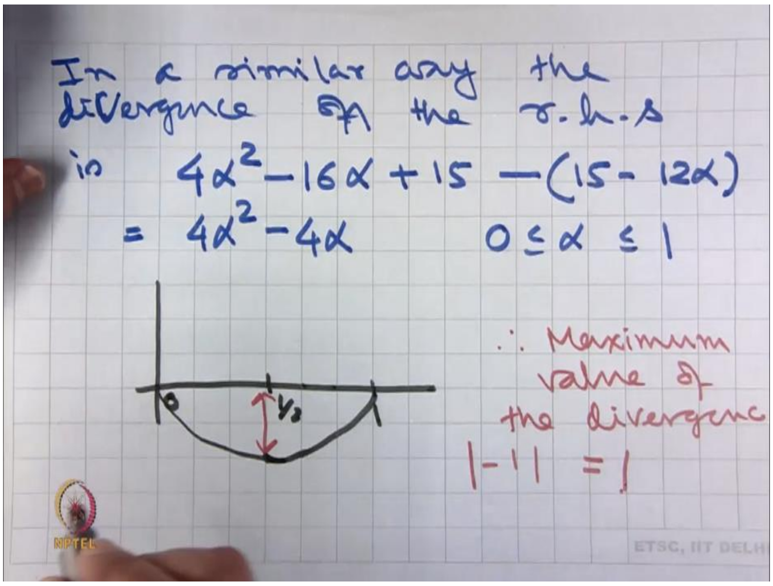
At $\alpha = \frac{1}{2}$, $4\alpha^2 - 8\alpha + 4 = 1$

At $\alpha = 1$, $4\alpha^2 - 8\alpha + 4 = 0$

So, $-4\alpha^2 + 4\alpha$ is an increasing function; $4\alpha^2 - 8\alpha + 4$ is a decreasing function.

And therefore, the maximum divergence that we get is at the point $\alpha = \frac{1}{2}$, which is 1.

(Refer Slide Time: 19:00)



In a similar way, the divergence on the right-hand side is

$4\alpha^2 - 16\alpha + 15 - (15 - 12\alpha) = 4\alpha^2 - 4\alpha$ for $0 \le \alpha \le 1$

So, if we plot this graph for different values of alpha, we get

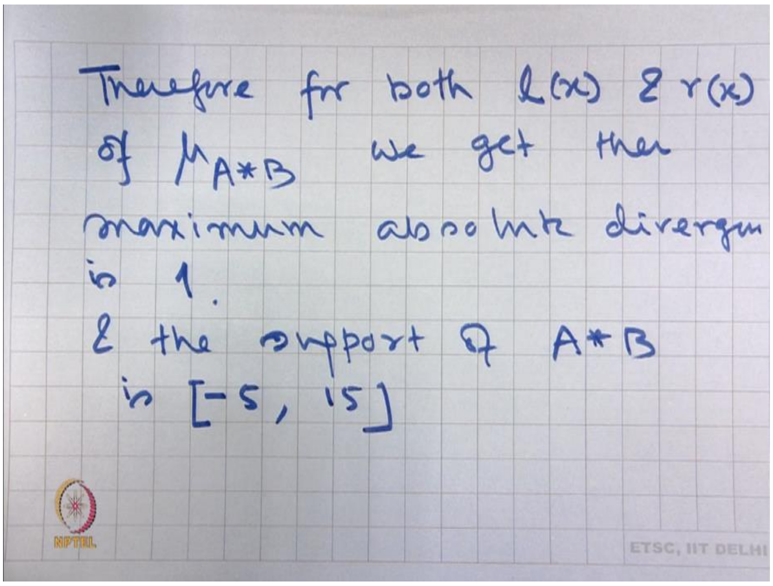
At $\alpha = 0$, $4\alpha^2 - 4\alpha = 0$

At $\alpha = \frac{1}{2}$, $4\alpha^2 - 4\alpha = -1$

At $\alpha = 1$, $4\alpha^2 - 4\alpha = 0$

Therefore, the maximum absolute difference is coming at this point, is $|-1| = 1$

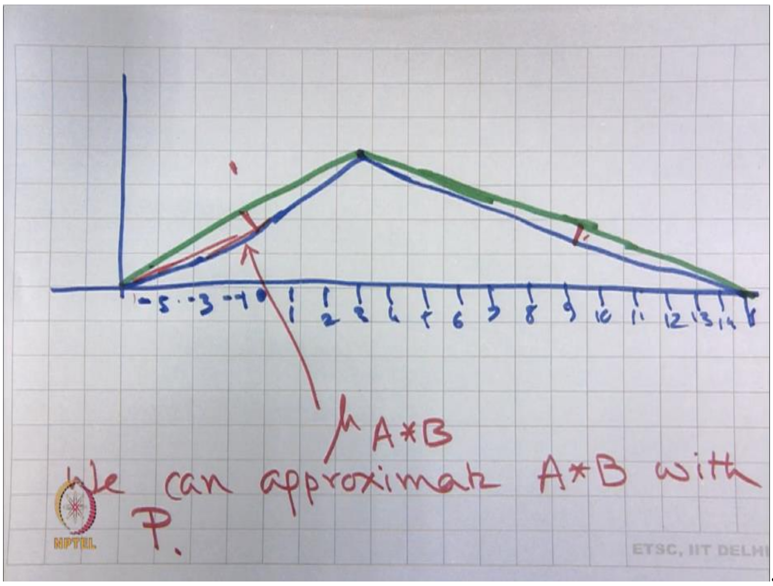
(Refer Slide Time: 21:05)



Therefore, for both left-hand side function of μ_{A*B} and right-hand side function of μ_{A*B}, we get the maximum divergence or maximum absolute divergence is 1.

And the support of $A * B$ is from $[-5, 15]$ So, if we plot it, we get something like this.

(Refer Slide Time: 22:11)



.

And we had this shape. So, this is μ_{A*B}.

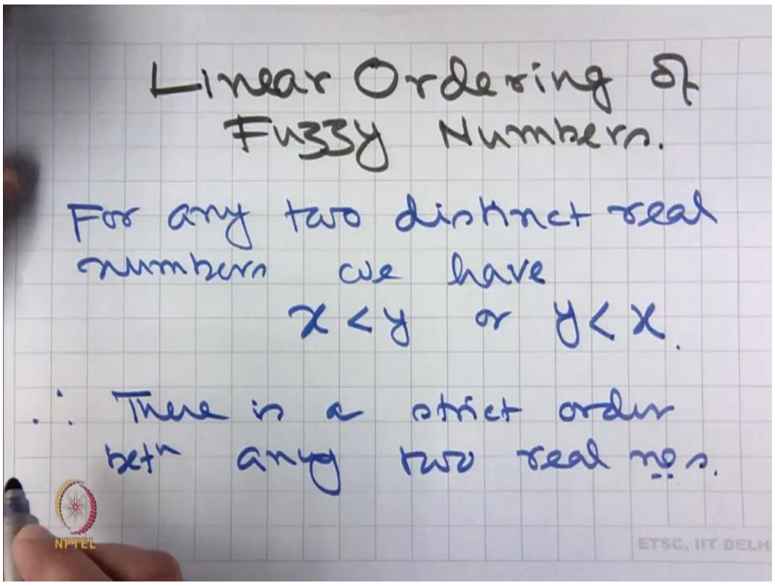
And the linear approximation was something like this. So, this is the amount of divergence where the maximum is coming to be 1 on both sides.

Since the overall difference between the linear approximation and the actual curve is small in comparison with the total support of the fuzzy number, we can approximate $A * B$ with P.

In practical applications, one has to make a policy; what is the maximum allowable divergence for a given fuzzy number. That is very important because, as we keep on combining different fuzzy values to, in our application, more and more error will be accumulated as when we are combining different fuzzy numbers with different operators. So, one has to make a policy beforehand.

And depending upon the application, one can choose to be the threshold; so that, if the difference or the divergence is within that threshold, one can approximate the function with a linear function as indicated in this talk. So, for other fuzzy functions involving two fuzzy numbers may be worked out in a very similar way.

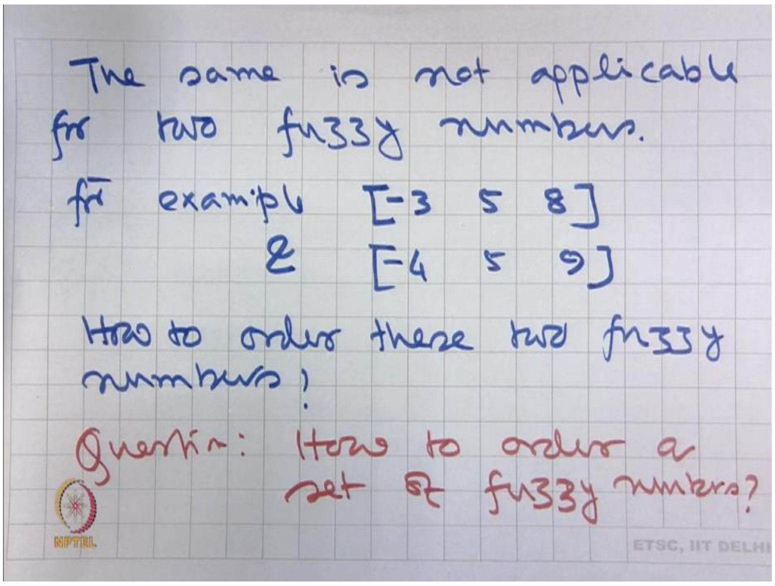
(Refer Slide Time: 25:22)



Let me now change the topic to linear ordering of fuzzy numbers.

For any two real numbers, distinct real numbers, we have either $x < y$ or $y < x$. Since, there is a strict order between any two real numbers.

(Refer Slide Time: 26:27)



The same is not applicable for two fuzzy numbers.

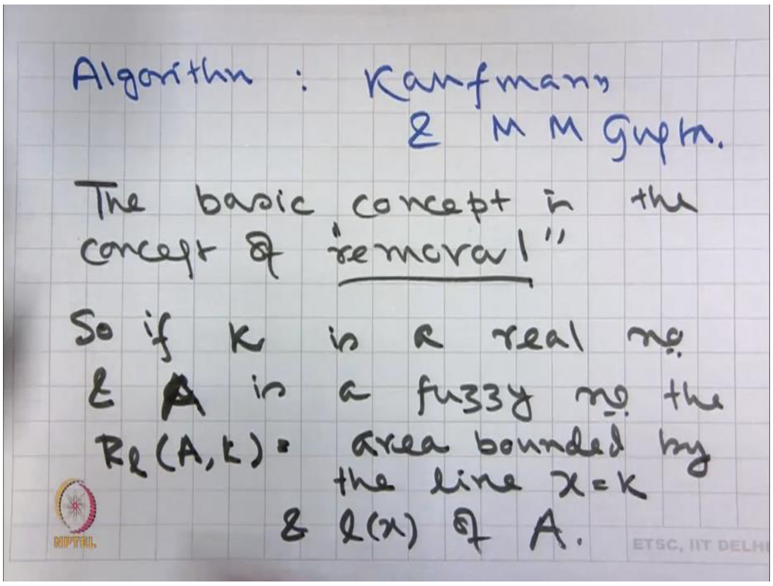
For example, $[-3\ \ 5\ \ 8]$ and $[-4\ \ 5\ \ 9]$

How to order these two fuzzy numbers? It is difficult because, on the left-hand side, this is smaller; on the right-hand side, but this is larger. So, it has a much bigger support compared to this. And both of them are the same modal value. So, it is very difficult to order them as such. For reals, there are many different algorithms for sorting them or putting them in a linear order.

Those who are familiar with computer programming might have heard of bubble sort, merge sort, quick sort, etcetera, for ordering a set of reals.

Question is: How to order a set of fuzzy numbers? So, I give you an algorithm from the book of Kaufmann and M.M. Gupta.

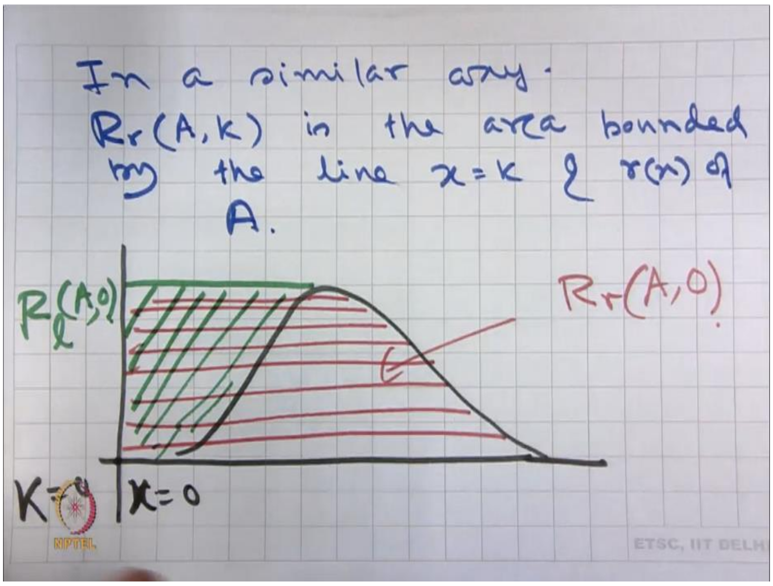
(Refer Slide Time: 28:30)



The basic concept is the concept of removal.

So, if k is a real number and A is a fuzzy number, then removal on left-hand side of A and k $R_l(A, k)$ is area bounded by the line $x = k$ and $l(x)$ of A.

(Refer Slide Time: 30:00)

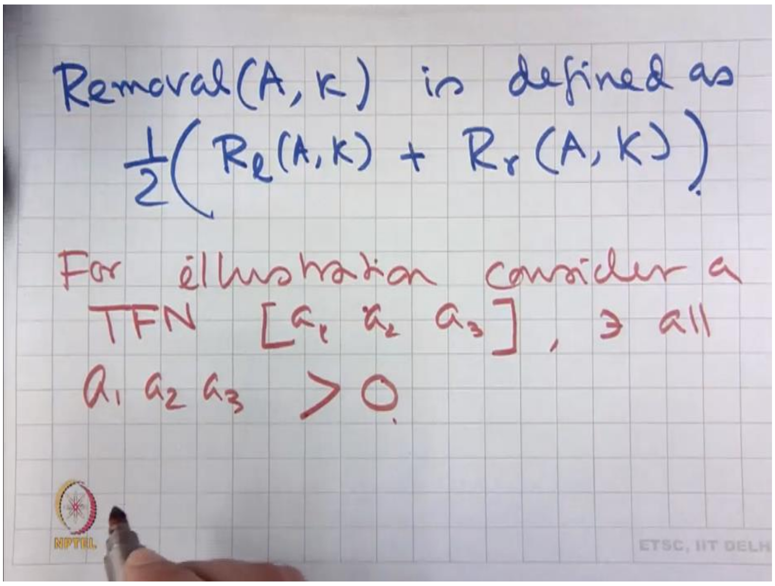


In a similar way, $R_r(A, k)$ is the area bounded by the line $x = k$; and $r(x)$, the right-hand side of the membership for A.

For illustration: Suppose this is the line, $x = 0$ that is, I am taking $k = 0$.

And suppose this is the fuzzy number. Then, this area is $R_l(A,0)$. In a Similar way, this entire area, this is $R_r(A,0)$.

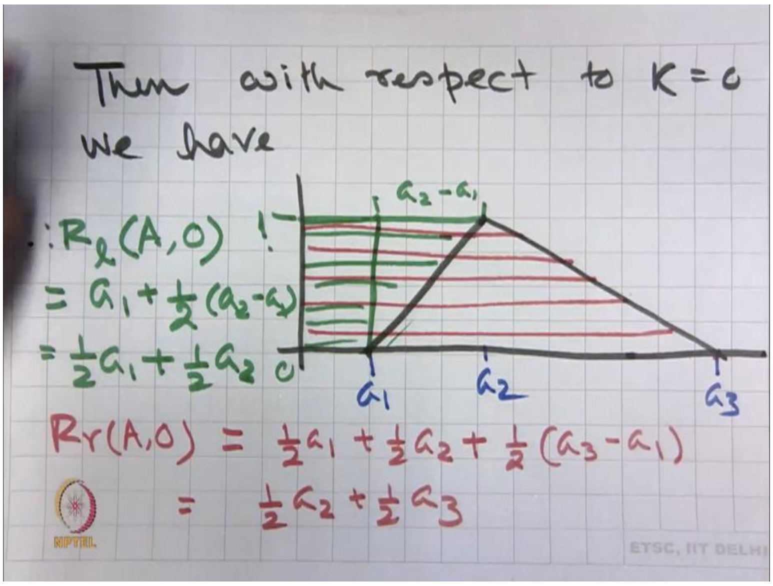
(Refer Slide Time: 31:36)



Removal(A,k) is defined as $\frac{1}{2}\left(R_l(A,k)+R_r(A,k)\right)$

For illustration, consider a TFN $[a_1,a_2,a_3]$ such that all $a_1,a_2,a_3>0$

(Refer Slide Time: 32:40)



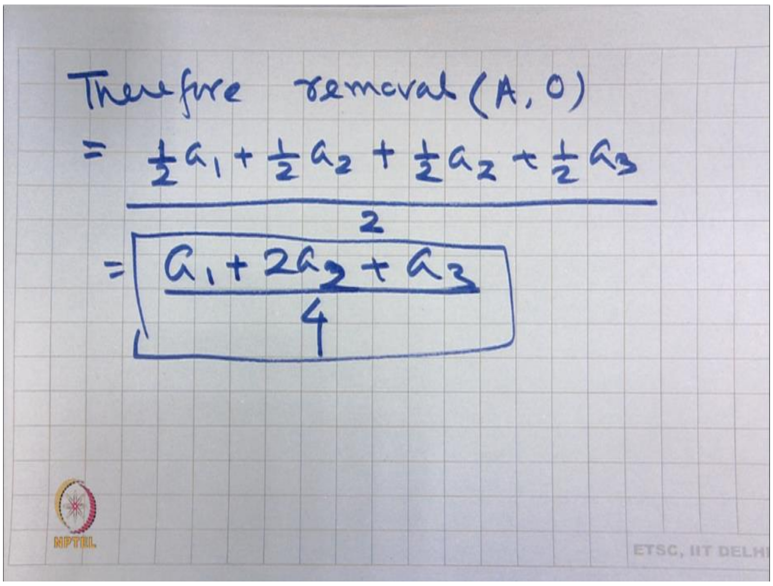
Then, with respect to $k=0$

Therefore,

$$R_l(A,0)=a_1\cdot 1+\frac{1}{2}\cdot(a_2-a_1)=\frac{1}{2}a_1+\frac{1}{2}a_2$$

$$R_r(A,0)=\frac{1}{2}a_1+\frac{1}{2}a_2+\frac{1}{2}(a_3-a_1)=\frac{1}{2}a_2+\frac{1}{2}a_3$$

(Refer Slide Time: 35:05)

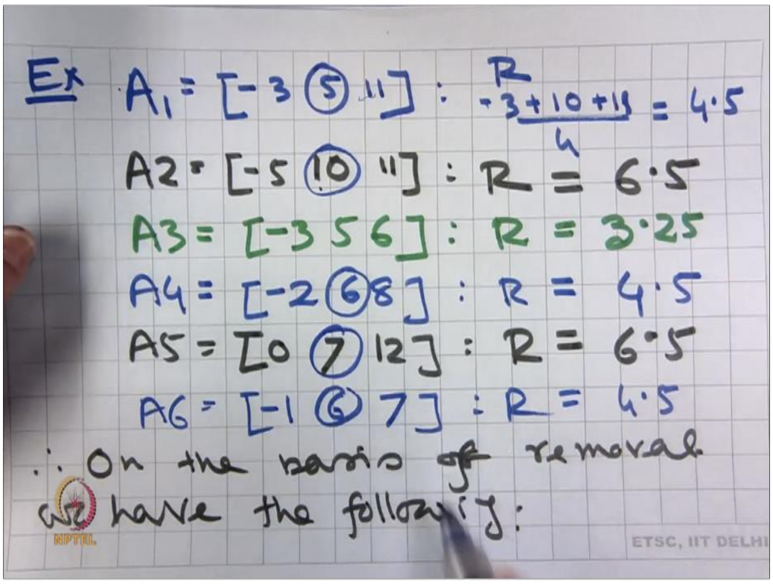


Therefore, removal$(A, 0) = \frac{\frac{1}{2}a_1+\frac{1}{2}a_2+\frac{1}{2}a_2+\frac{1}{2}a_3}{2} = \frac{a_1+2a_2+a_3}{4}$

So, this is the quantity that one calculates first to group the set of fuzzy numbers we want to sort into some equivalent numbers.

All those triangular fuzzy numbers for which this quantity is same, can be put into the same group. So, let me take an example.

(Refer Slide Time: 36:12)



Consider

$A_1 = [-3 \;\; 5 \;\; 11]$ $\qquad R = \dfrac{-3+10+11}{4} = 4.5$

$A_2 = [-5 \;\; 10 \;\; 11]$ $\qquad R = 6.5$

$A_3 = [-3 \;\; 5 \;\; 6]$ $\qquad R = 3.25$

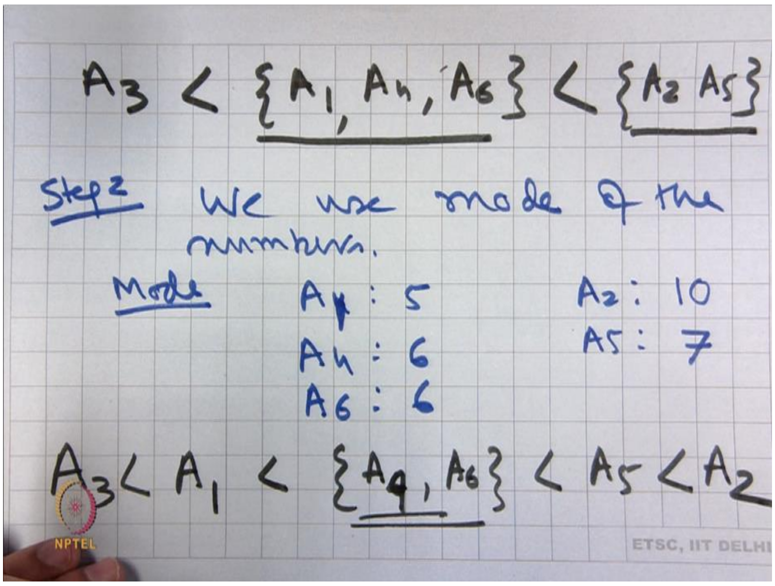
$A_4 = [-2 \;\; 6 \;\; 8]$ $\qquad R = 4.5$

$A_5 = [0 \;\; 7 \;\; 12]$ $\qquad R = 6.5$

$A_6 = [-1 \;\; 6 \;\; 7]$ $\qquad R = 4.5$

Therefore, on the basis of removal, we have the following.

(Refer Slide Time: 38:52)



$$A_3 < \{A_1, A_4, A_6\} < \{A_2, A_5\}$$

So, we got some partial order. Although, we find these are tied at the same place, because of the same value of removal.

Step 2: We use mode of the numbers

What is a mode?

The mode is the point where the fuzzy number gives the highest membership value.

A_1 : 5

A_2 : 10

A_4 : 6

A_5 : 7

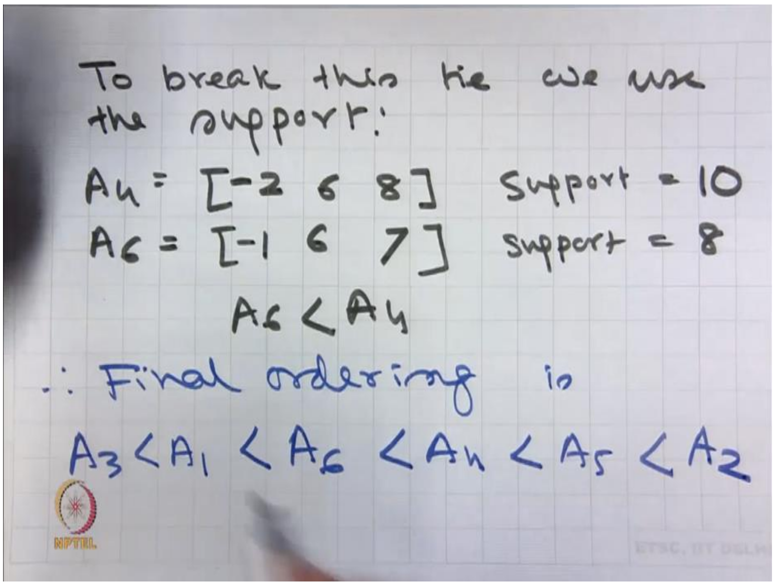
A_6 : 6

Therefore, at this stage, we have the order

$$A_3 < A_1 < \{A_4, A_6\} < A_5 < A_2$$

Therefore, we could impose an order with these two steps. But still we have a tie between A_4 and A_6

(Refer Slide Time: 41:24)



To break this tie, we use the support:

$A_4 = [-2 \;\; 6 \;\; 8]$ Support$= 10$

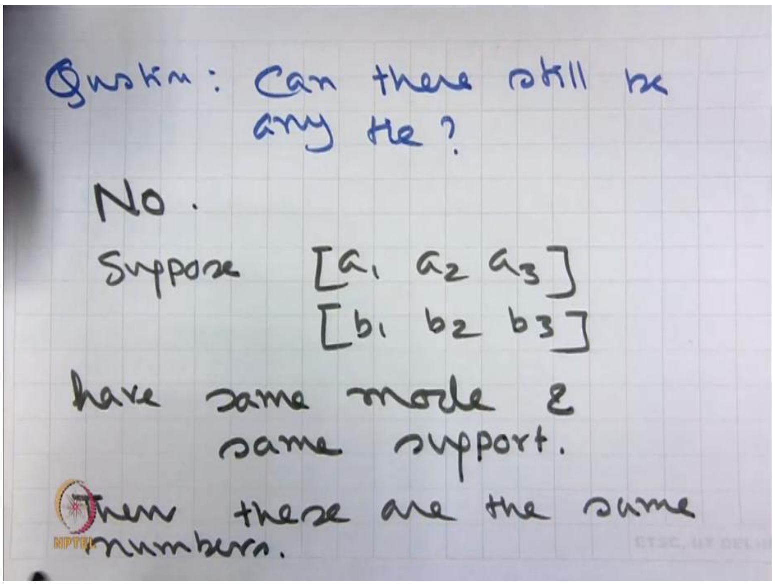
$A_6 = [-1 \;\; 6 \;\; 7]$ Support$= 8$

And therefore, we can write $A_6 < A_4$.

Therefore, final ordering is

$$A_3 < A_1 < A_6 < A_4 < A_5 < A_2$$

(Refer Slide Time: 42:39)

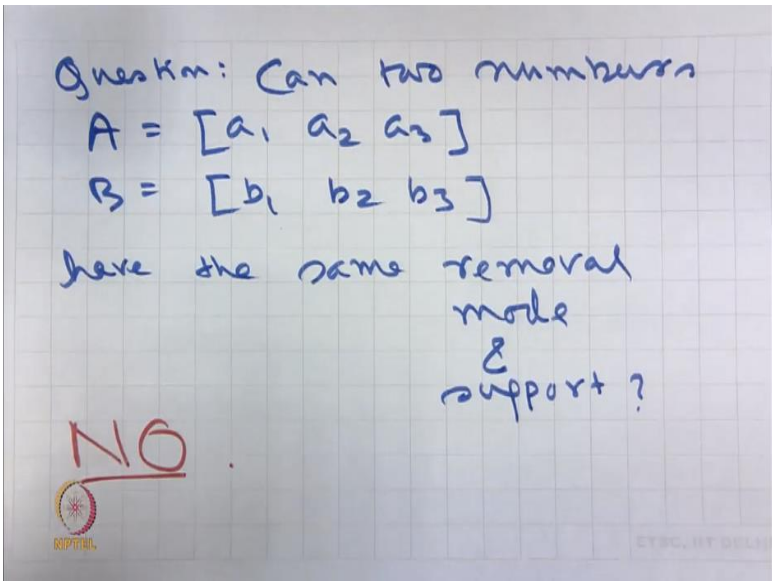


Question is: Can there still be any tie? Answer is no.

Suppose $[a_1, a_2, a_3]$ and $[b_1, b_2, b_3]$ have same mode and same support, then these are the same numbers.

Or in other words, two TFNs have the same mode and same support, they are actually the same number.

(Refer Slide Time: 43:54)

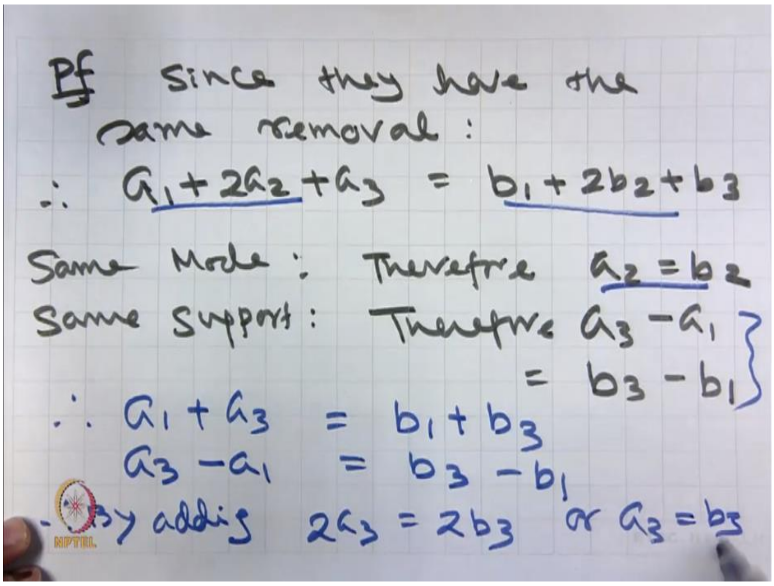


Question is:

Can two numbers $A = [a_1, a_2, a_3]; B = [b_1, b_2, b_3]$ have the same removal, mode and support?

The answer is no.

(Refer Slide Time: 44:44)



Proof:

Since they have the same removal, $\therefore a_1 + 2a_2 + a_3 = b_1 + 2b_2 + b_3$

Same mode: $\therefore a_2 = b_2$

Same support: $\therefore a_3 - a_1 = b_3 - b_1$

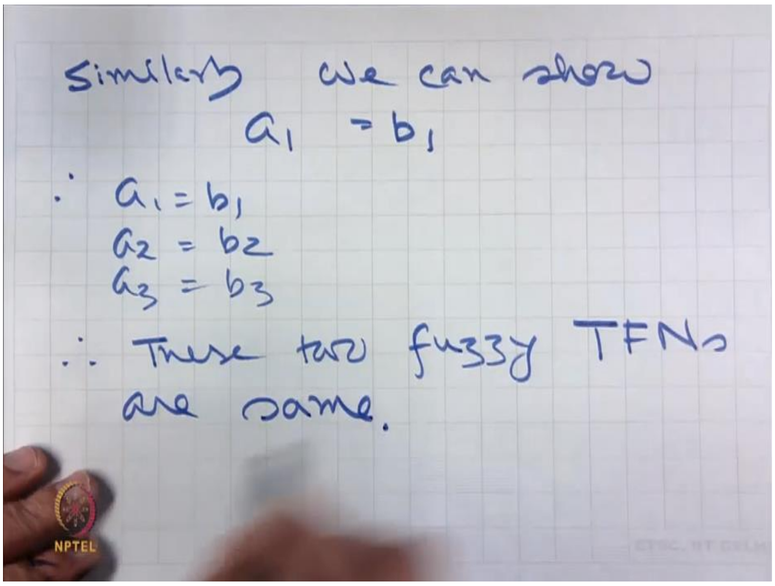
Now, from these, we get

$$a_1 + a_3 = b_1 + b_3$$

$$a_3 - a_1 = b_3 - b_1$$

Therefore, we can say, by adding the two $2a_3 = 2b_3$ or $a_3 = b_3$

(Refer Slide Time: 46:30)



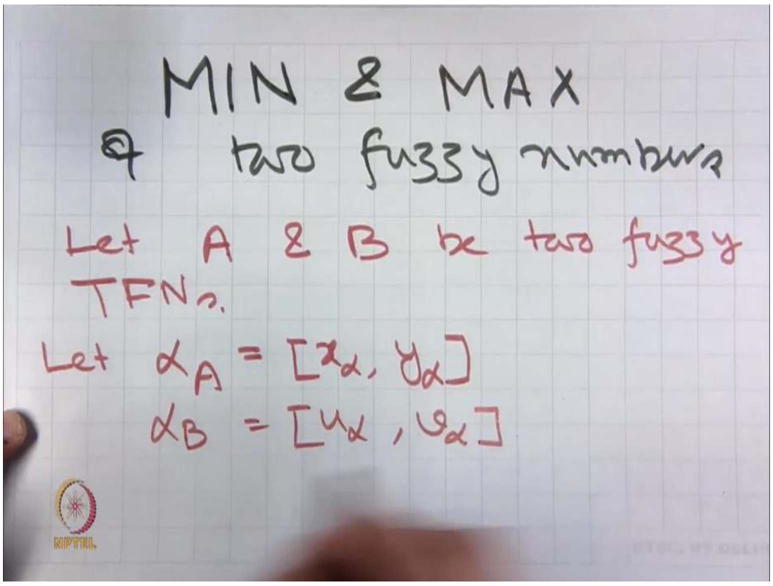
And similarly, we can show $a_1 = b_1$a 1 is equal to b 1.

Therefore, $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$

Therefore, these two fuzzy TFNs are same.

Let me now introduce you to two more functions, which are called MIN and MAX of two fuzzy numbers.

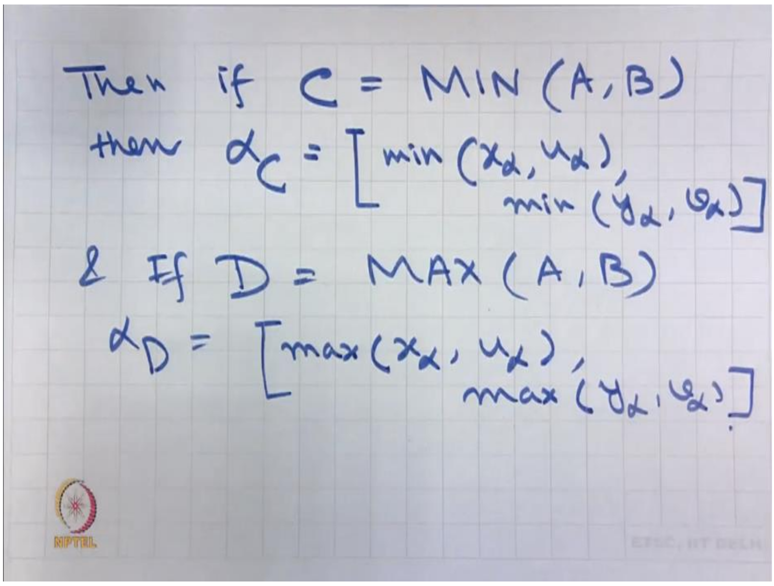
(Refer Slide Time: 47:08)



Let A and B be two fuzzy TFNs or triangular fuzzy numbers.

Let $^{\alpha}A = [x_\alpha, y_\alpha]$ and $^{\alpha}B = [u_\alpha, v_\alpha]$

(Refer Slide Time: 48:06)



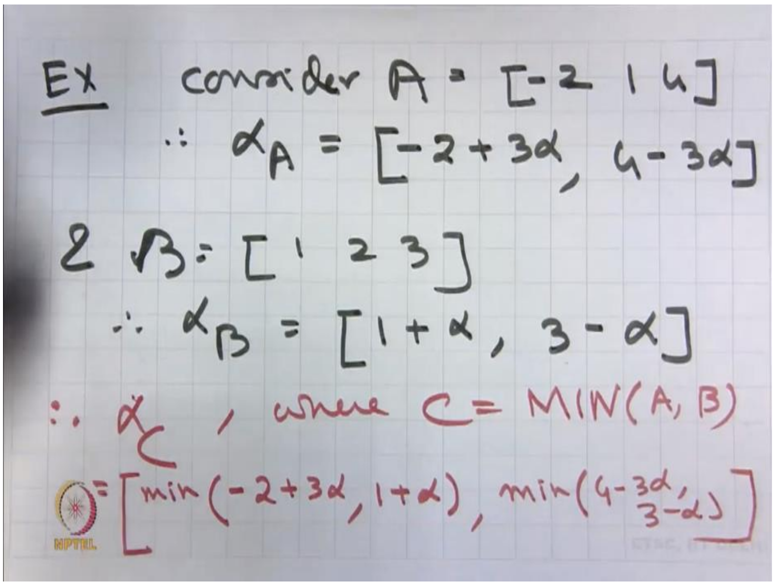
Then, by definition, if $C = MIN(A, B)$ then,

$^{\alpha}C = [\min(x_\alpha, u_\alpha), \min(y_\alpha, v_\alpha)]$

And if $D = MAX(A, B)$

$^{\alpha}D = [\max(x_\alpha, u_\alpha), \max(y_\alpha, v_\alpha)]$

(Refer Slide Time: 49:15)



For example, consider $A = [-2\ \ 1\ \ \ 4]$

Therefore, ${}^{\alpha}A = [-2 + 3\alpha,\ 4 - 3\alpha]$

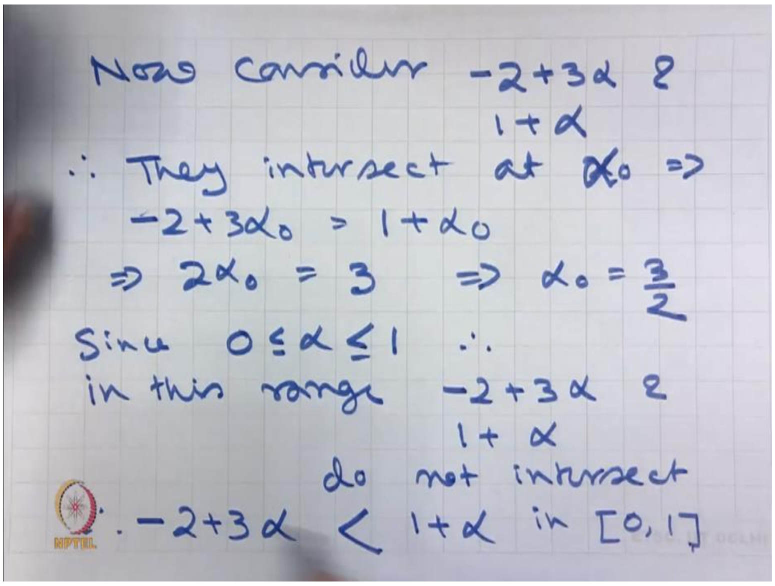
And $B = [1\ \ \ 2\ \ 3]$

Therefore, ${}^{\alpha}B = [1 + \alpha, 3 - \alpha]$

$\therefore\ {}^{\alpha}C$, where $C = MIN(A, B)$

$${}^{\alpha}C = [\min(-2 + 3\alpha, 1 + \alpha), \min(4 - 3\alpha, 3 - \alpha)]$$

(Refer Slide Time: 50:52)



Now, consider $-2 + 3\alpha$ and $1 + \alpha$.

Therefore, they intersect at α_0

$\Rightarrow -2 + 3\alpha_0 = 1 + \alpha_0$

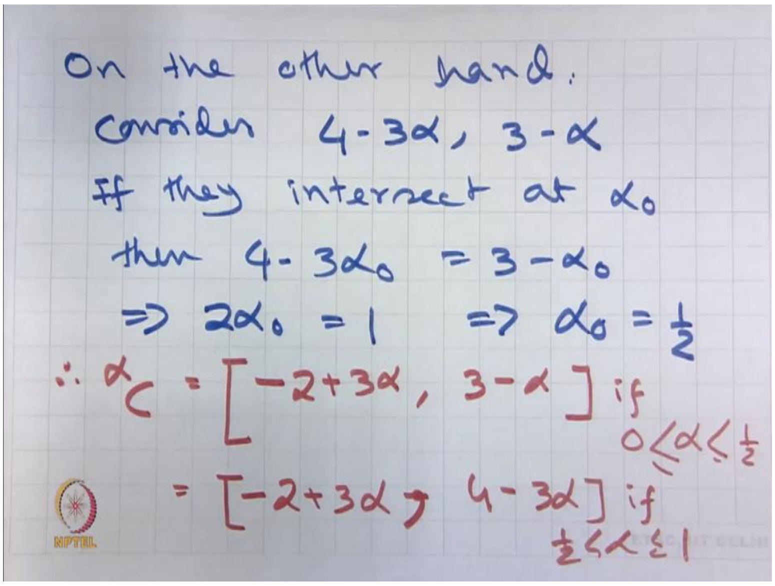
$\Rightarrow 2\alpha_0 = 3$

$\Rightarrow \alpha_0 = \frac{3}{2}$

Since $0 \leq \alpha \leq 1$ therefore, in this range, $-2 + 3\alpha$ and $1 + \alpha$ do not intersect.

Therefore, $-2 + 3\alpha < 1 + \alpha$ in $[0, 1]$

(Refer Slide Time: 52:14)



On the other hand, consider $4 - 3\alpha, 3 - \alpha$.

If they intersect at α_0, then $4 - 3\alpha_0 = 3 - \alpha_0$

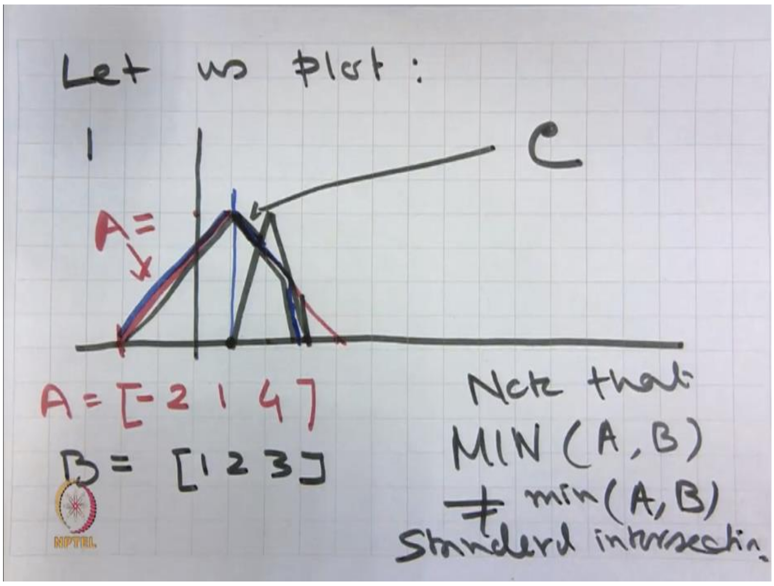
$\Rightarrow 2\alpha_0 = 1$

$\Rightarrow \alpha_0 = \frac{1}{2}$

Therefore,

$$
\begin{aligned}
{}^{\alpha}C &= [-2 + 3\alpha, 3 - \alpha] \text{ if } 0 \leq \alpha \leq \tfrac{1}{2} \\
&= [-2 + 3\alpha, 4 - 3\alpha] \text{ if } \tfrac{1}{2} < \alpha \leq 1
\end{aligned}
$$

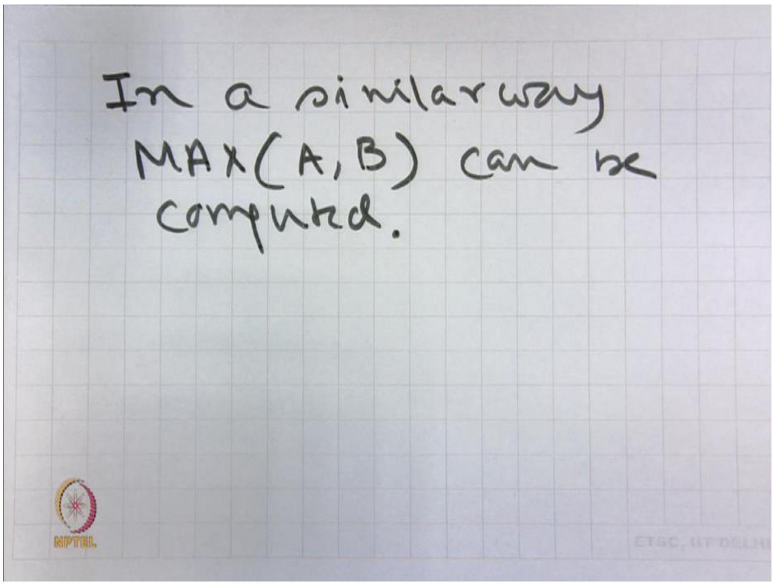
(Refer Slide Time: 53:50)



So, let me plot it.

Note that $MIN(A,\ B) \neq \min(A, B)$, which is the standard intersection. In a similar way, one can compute the $MAX(A, B)$.

(Refer Slide Time: 56:11)



I leave it for you as an exercise. Okay student, I stop here now. In the last few lectures, we have done different problems of fuzzy arithmetic. From the next class, I will start more about fuzzy sets. In particular, I will look at certain properties and also we will look at extension principle and decomposition theorem, etcetera, in the next few classes. Thank you.