Integral Transform and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute Of Information Technology

Lecture - 72
Discrete Haar, Shanon and Debauchies Wavelet- Part 03

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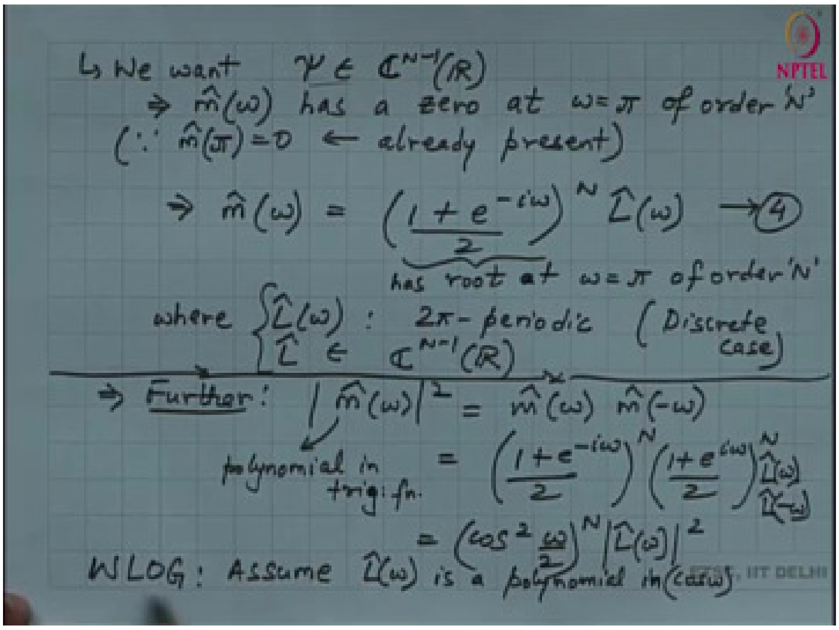

So, what we want is that this function ψ which is a wavelet is (N-1) times differentiable in the complex plane. So, which means in the transformed plane, my associated generating function $\hat{m}(w)$ has a zero $w = \pi$ which is of order of order N. Well, where is this coming from? Notice that $\hat{m}(\pi) = 0$. So, at least we have a root of my associated generating function at π, which is of order 1. So, if we want the original function to be (N-1) times continuously differentiable, then its corresponding value in the transformed value of the associated generating function must have at least one root which is of order N, one greater than the requirement. So, I have that this is already present. So, since there is one root at π already present if I were to have impose danother requirement that this root is of order N then I can get my original function to be(N-1) times continuously differentiable. So, which means that I can describe my associated generating function as follows:

$$\hat{m}(w) = \left(\frac{1 + e^{-iw}}{2}\right)^N \hat{L}(w)$$

So, by introducing this factor I have made sure that this associated generating function has a root of order N at omega equals pi. So, then now so let me call this IV. So, this is a very important expression. So, which means that if I know what is this function $\hat{L}(w)$ I can plug it into this relation to get the associated generating function and from the associated generating function, I can create the scaling function ϕ and hence the wavelet ψ. So, all I need is this function$\hat{L}(w)$ in the transformed plane. So, notice that $\hat{L}(w)$ is 2π periodic. Why? Because this is for the discrete case of the Daubechies wavelet, I am working in a 2 pi periodic domain. So, I need that my function $\hat{L}(w)$ is 2 pi periodic and must be (N-1) times differentiable over the complex plane. So, if that is the case I have made sure that my original function which will be the inverse transform of this associated generating polynomial function times some other function, the original function in the physical plane will be also (N-1) times continuously differentiable. So, I need these two conditions. So, let us look at what is the absolute value of this associated generating function.

So, let us work in terms of the square of the absolute value. Since this is the complex function in general I can write this as

$$
\begin{aligned}
|\hat{m}(w)|^2 &= \hat{m}(w)\hat{m}(-w) \\
&= \left(\frac{1+e^{-iw}}{2}\right)^N \left(\frac{1+e^{iw}}{2}\right)^N \hat{L}(w)\hat{L}(-w) \\
&= (\cos^2 w/2)^N \; |\hat{L}(w)|^2
\end{aligned}
$$

So, what we can assume? We can assume without loss of generality, I can assume that my $\hat{L}(w)$ is a polynomial in $\cos(w)$. We could assume that it is a polynomial in $\sin(w)$, but then sin and cos are related by a phase shift, so it that does not really matter. So, let us assume that $\hat{L}(w)$ is a polynomial in $\cos(w)$.

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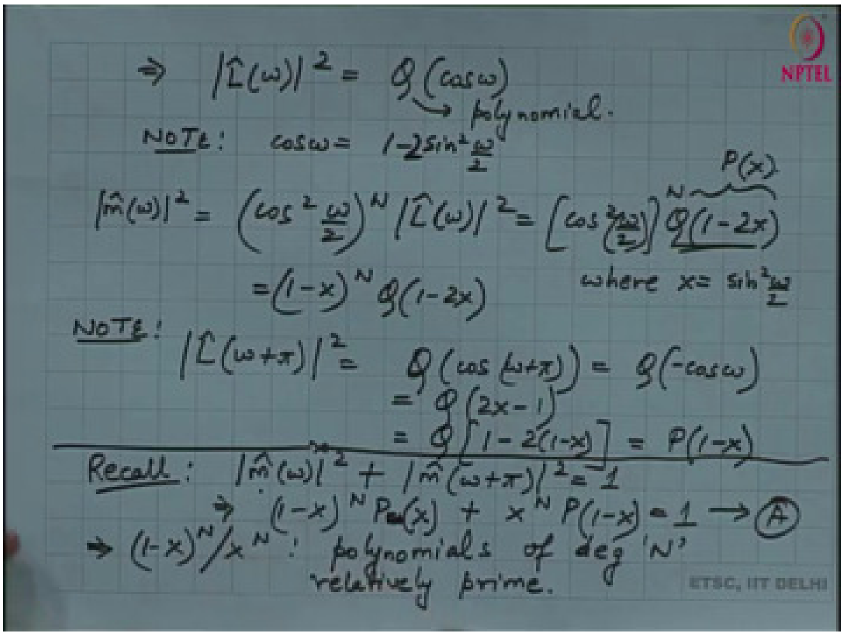

So, then let us look at so, then what I have assumed is the following, I have assumed that $\hat{L}(w) = Q(\cos(w))$, where Q is a polynomial. So, then now notice that I can rewrite my $\cos(w) = 1 - 2\sin^2 w/2$. So, which means

$$
\begin{aligned}
|\hat{m}(w)|^2 = (\cos^2 w/2)^N \; |\hat{L}(w)|^2 &= \left[\cos^2(w/2)\right] Q(1-2x) \\
&= (1-x)^N \; Q(1-2x)
\end{aligned}
$$

where my $x = \sin^2 \; w/2$. Note that

$$
\begin{aligned}
|\hat{L}(w+\pi)|^2 &= Q(\cos(w+\pi)) = Q(-\cos w) \\
&= Q(2x-1) = Q(1-2(1-x)) = P(1-x)
\end{aligned}
$$

So, then recall the following property of my associated generating function. Recall that

$$|\hat{m}(w)|^2 + |\hat{m}(w+\pi)|^2 = 1$$
$$=> (1-x)^N P(x) + x^N P(1-x) = 1$$

So, let me call this my expression A. Notice that this is completely a polynomial equation where P(x) is a polynomial and multiplied by polynomial factors. So, I see that $(1-x)^N/x^N$, these are all polynomials of degree N. Now, let me know also further these are polynomials and they are relatively prime. So, now if we were to find it turns out that there is a unique solution to this equation given by A.

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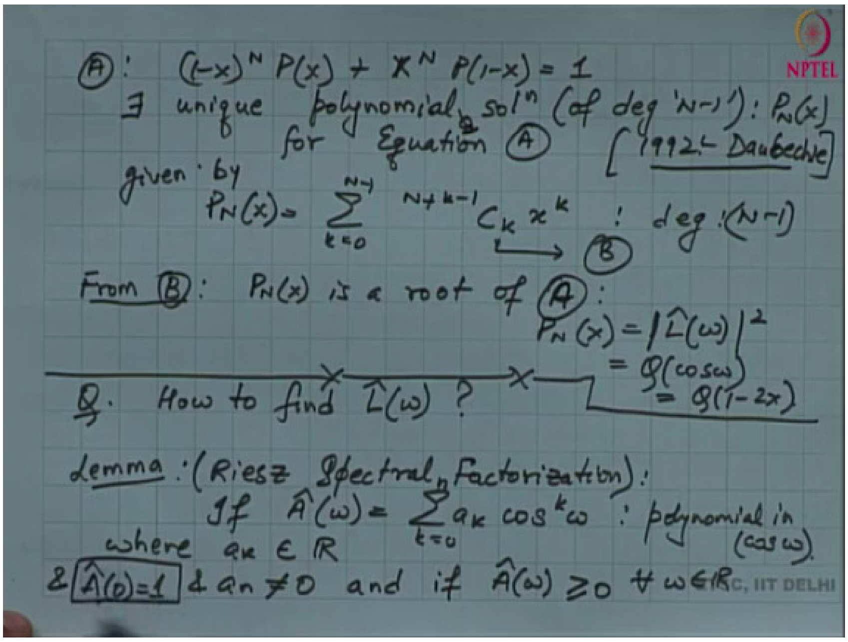

So, if I were to write this equation A:

$$(1-x)^N P(x) + x^N P(1-x) = 1$$

it turns out that there is a unique polynomial solution of degree N minus 1. So, the polynomial solution is denoted by $P_N(x)$ for equation given by A here. So, there is a unique polynomial solution of degree N minus 1 for equation A, all my students and readers are directed to this following paper by Daubechie to see how this unique solution arises, So, please look at the paper published by Daubechie in 1992, to see that a unique solution to exists and this unique solution is given by the following expression:

$$P_N(x) = \sum_{k=0}^{N-1} {}^{N+k-1}C_k x^k$$

So, this particular result is published in this particular paper the famous paper by Daubechie herself. So, then I see that the degree of this polynomial is N minus 1. Let me call this, expression B. Now, so from B what I see is that $P_N(x)$ is a root for this particular polynomial given by A:

$$P_N(x) = |\hat{L}(w)|^2 = Q(\cos w) = Q(1-2x)$$

So all I am left with now is how to find out $\hat{L}(w)$ from this function $P_N(x)$. So, I know that my $P_N(x)$ is also Q my polynomial in cos omega or this is also polynomial in 1 minus 2 x. So, all I am left is how to construct this L hat omega because from there I am going to construct my generating function and hence the scaling function or the Daubechie wavelet. So, the question that remains in this discussion is how to find how to find $\hat{L}(w)$, so how to find the transformed value of this polynomial? Now we do not have to do much because there is a very nice result in the form of a lemma. So, let me just highlight that result in the form of this Riesz spectral factorization lemma, which tells that if I am given a function, let me denote that function $\hat{A}(w)$, can be represented as a polynomial of cosines. So, A hat omega is a polynomial in terms of cosine. So, if I am given $\hat{A}(w)$ is a polynomial, where all my coefficients $a_k' s$ are real numbers and further I have that $a_n \neq 0$, then and further I am given that $\hat{A}(w) \geq 0$ for all values of the argument which are real values and $\hat{A}(0) = 1$ So, I have all these conditions, so I have a polynomial in omega given that all its coefficients are real and the coefficients are non-zero and also that the polynomial is nonnegative and that the value at omega equal to 0 the polynomial takes the value 1.

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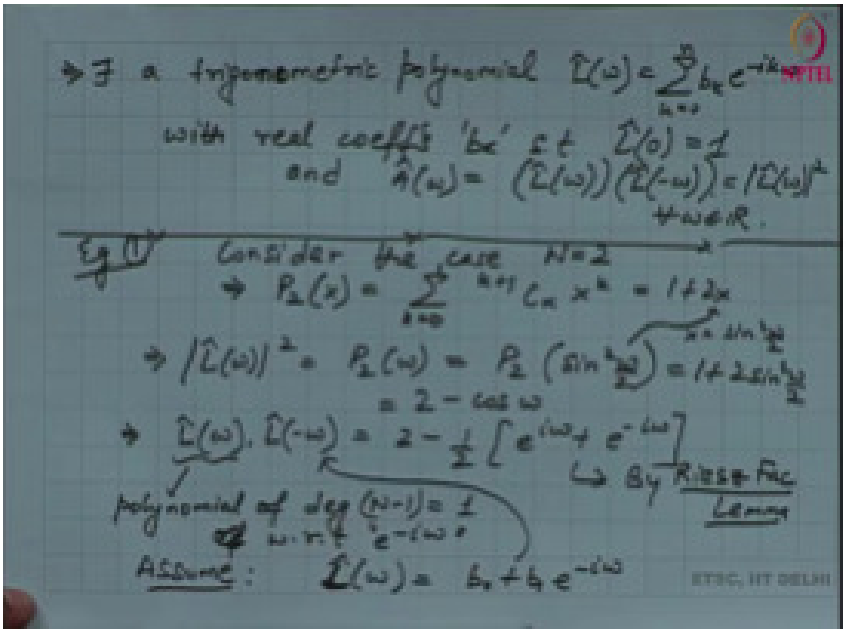

Then, what I get is it the Riesz representation lemma tells that, it tells that there exists a trigonometric polynomial

$$\hat{L}(w) = \sum_{k=0}^{n} b_k e^{-ikw}$$

with real coefficients $'b_k'$ s.t. $\hat{L}(0) = 1$ and

$$\hat{A}(w) = (\hat{L}(w))\ (\hat{L}(-w)) = |\hat{L}(w)|^2 \quad \text{for all} \quad w \in \mathbb{R}$$

It tells me that if you give me a polynomial in cosine with all those conditions that I have specified the lemma says that I can always factorize that polynomial to find my $\hat{L}(w)$ which will be used to generate the associated generating function. So, let us look at an example on how to construct the Daubechie wavelet. So let me consider let me consider the case when N is 2. So, I am going to consider the case where I have vanishing moments up to order 2. So, which means my polynomial P_2 which is a unique solution to my equation:

$$P_2(x) = sum_{k=0}^{1} {}^{k+1}C_k x^k = 1 + 2x; \quad \text{where} \quad x = \sin^2 w/2$$
$$=> |\hat{L}(w)|^2 = P_2(w) = P_2(\sin^2 w/2) = 1 + 2\sin^2 w/2 = 2 - \cos w$$
$$=> \hat{L}(w)\hat{L}(-w) = 2 - 1/2(e^{iw} + e^{-iw})$$

So, let us now that notice that my $P_2(w)$ is a polynomial of order 1, with respect to this factor e^{iw} which means that my L hat of omega is also a polynomial of degree N minus 1 in this case this will be equal to 1 with respect to e^{iw}. So, with respect to this factor my L hat of omega is a polynomial of degree 1. So let me assume $\hat{L}(w) = b_0 + b_1 e^{-iw}$

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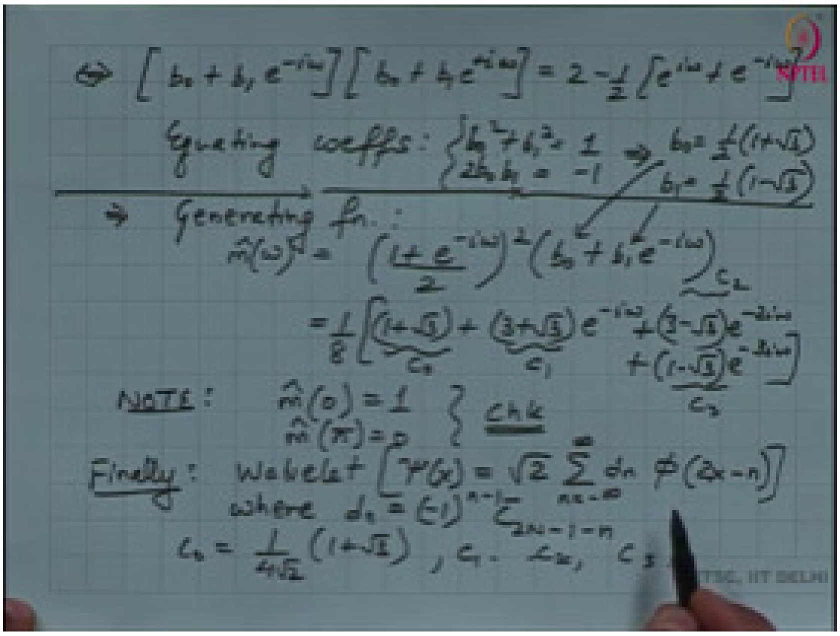

So, let us now use this expression in this equation to see what happens. I see that the following things happen, so

$$\hat{L}(w) = \left[b_0 + b_1 e^{-iw}\right]\left[b_0 + b_1 e^{iw}\right] = 2 - \frac{1}{2}\left[e^{iw} + e^{-iw}\right]$$

So, all I have to do is to find the coefficients b_0 and b_1. So, equating my coefficients I get the following that

$$b_0^2 + b_1^2 = 1$$
$$2b_0 b_1 = -1$$
$$=> b_0 = \frac{1}{2}(1+\sqrt{3}), \quad b_1 = \frac{1}{2}(1-\sqrt{3})$$

So, with this I have found my $\hat{L}(w)$. So, which means my generating function is as follows:

$$\hat{m}(w) = \left(\frac{1+e^{-iw}}{2}\right)^2 (b_0 + b_1 e^{-iw})$$
$$= \frac{1}{8}\left[(1+\sqrt{3}) + (3+\sqrt{3})e^{-iw} + (3-\sqrt{3})e^{-2iw} + (1-\sqrt{3})e^{-3iw}\right]$$

So, then students can also note and check that this generating function satisfies all the properties that we had to begin with namely $\hat{m}(0) = 1, \hat{m}(\pi) = 0$. So, students are asked to check that indeed these two are satisfied. So, which means finally, I have to find out what is the wavelet. So, finally, my wavelet which is the Daubechie wavelet can be constructed as psi is given by this following summation

$$\psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} dn\phi(2x - n)$$

$$\text{where,} \quad dn = (-1)^{n-1} \bar{C}_{2N-1-n}$$

So, all I have to do is to find the coefficients d ns from this c ns and plug it into this particular expression for to get my wavelet ok. So, in particular if I were to if I were to solve. So,

$$c_0 = \frac{1}{4\sqrt{2}}(1 + \sqrt{3}), c_1, c_2, c_3 \cdots$$

So, notice that my discussion on the wavelet for with 2 orders of vanishing moments is complete this in particular is called the Daubechie wavelet this is the Daubechie wavelet D2.

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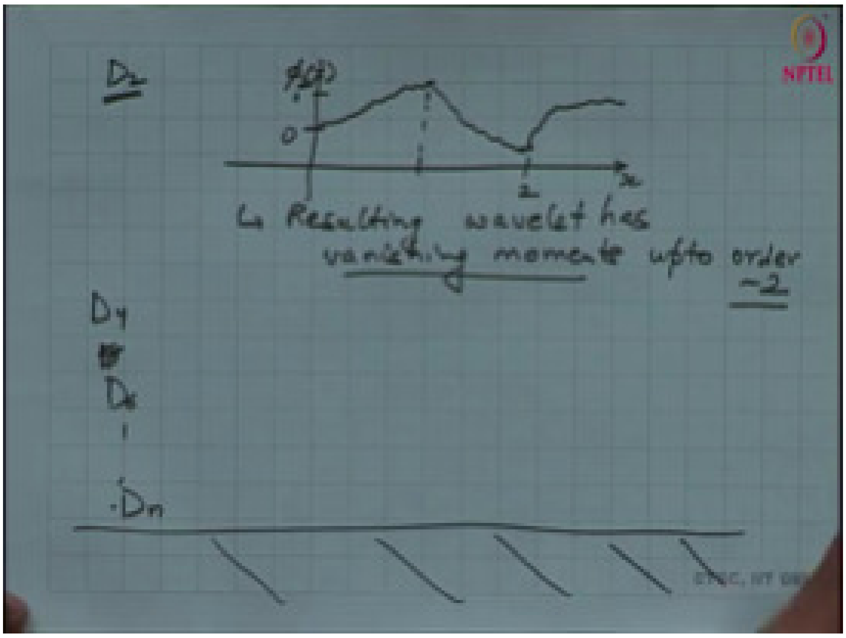

So, let us look at the particular scaling function let me show you the scaling function of D2 versus x. So, it looks as in figure. So, D2 has the following graph, it is some sort of an increasing function say it is an increasing function up till 1 and then it decreases up till 2 and then it again starts to increase from 2 onwards. So, I see that this is the scaling function, I see that the resulting wavelet has vanishing moments up to up to order 2 . So, that is my D2 and then we can similarly construct D4, D6 or any Dn and so on. So, all we need is what is the requirement and based on the requirement we say that the moments vanish and from there we construct these scaling functions. So, that that completes my discussion on wavelet transforms which is possible to the scope of this course, the students are also requested to see many interesting applications of wavelets which arises here I have just highlighted some properties of this wavelets in particular cases. So, in this course we have looked at several different transforms. I hope this course is going to help be helpful to a lot of students in engineering and sciences and with this I conclude my discussion.

Thank you very much, thank you for listening.