Integral Transforms and Their Applications

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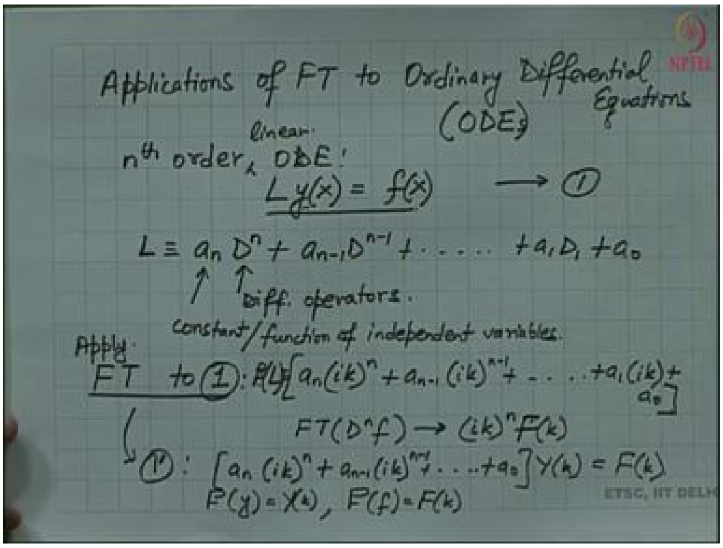
Lecture – 3

Applications of Fourier Transforms Part - 01

The last 2-3 lectures we have developed sufficient theory to look at some complex examples arising from engineering and sciences that can be solved analytically mostly analytically or numerically otherwise as well via the use of Fourier transform.



So, I will start with my applications of Fourier transforms to ordinary differential equations. I from now on I am going to refer ordinary differential equations as ODEs. So, consider the n th order ODE. So, consider this n th order ODE, I can write my n th order ODE as follows:

$$Ly(x) = f(x) \tag{1}$$

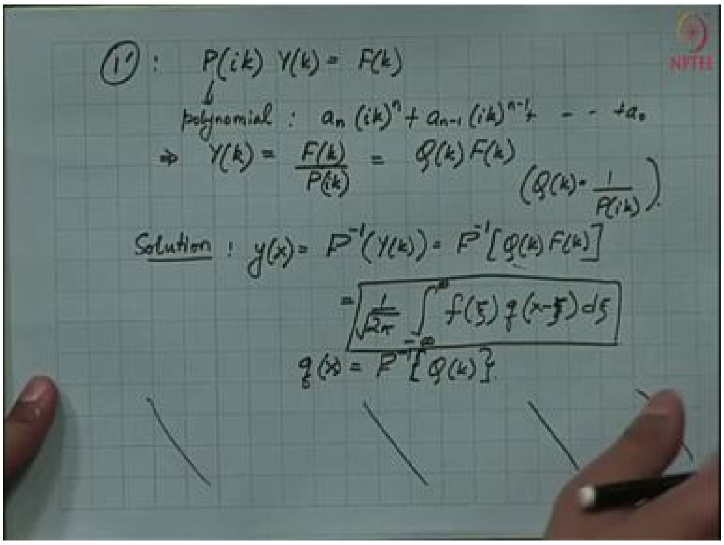
So, it is an operator L which is acting on y and L of y is equal to f of x right let me denote it as 1 where my operator L is denoted by this following expression.

$$L \equiv a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D_1 + a_0$$

So, these D's are my differential operators ok. So, I am talking about nth order linear ODE to begin with. So, which means let us keep these coefficients as let us say constant or functions of independent variables. So, in that case if I were to apply the Fourier transform. So, if I apply Fourier transform to 1; so, one implying this equation that I have outlined here, I get the following algebraic equation.

$$f(L) = \big[a_n(ik)^n + a_{n-1}(ik)^{n-1} + \ldots + a_1(ik) + a_0$$

So, this derivation was done in an earlier lectures where ,i k to the power n times Fourier transform of small f and this expression was already deduced in one of my previous lectures.



So, my application of Fourier transform to 1 gives me the following expression.

$$FT\,(D^n f) \rightarrow (ik)^n F(k)$$

So, I get the following expression,

$$\left[a_n(ik)^n + a_{n-1}(ik)^{n-1} \cdots + a_0\right] Y(k) = F(k)$$

let us say the Fourier transform,

$$F(y) = Y(k), F(f) = F(k)$$

So, I get Fourier transform of Y is equal to Fourier transform of F ok. So, in general; so what I get is; so, 1 prime can be written as follows.

$$P(ik)Y(k) = F(k)$$

So, it is a product of two functions. So, where this is a polynomial; it is a polynomial which looks as like follows.

$$a_n(ik)^n + a_{n-1}(ik)^{n-1} + \ldots + a_n$$

So, it is a polynomial of order n and the variable is k or i can always invert this polynomial to get my solution in the Fourier plane.

$$Y(k) = \frac{F(k)}{P(k)} = Q(k)F(k)$$

where my function Q(k):

$$Q(k) = \frac{1}{P(ik)}$$

So, then my solution to the ODE; the solution to the ODE is given by:

$$y(x) = F^{-1}(Y(k)) = F^{-1}[Q(k)F(k)]$$

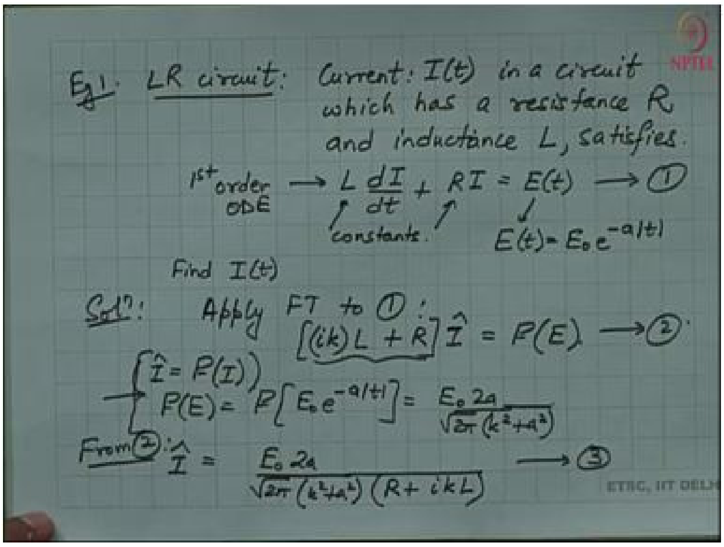
So, then that will lead to the expression. So, we see inside the argument of this inverse transform F have a product of two Fourier transform which means that in the physical plane this must be the convolution of the corresponding two function. So, when I take the inverse I get the following expression,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)q(x-\xi)d\zeta$$

So, the final answer can be simplified from this convolution result.

$$g(x) = F^{-1}[Q(k)]$$

So, let us look at some examples.



So, in electrical engineering one of the common examples is to study the LR circuit. So, let us say I have the current the current in a circuit denoted by I(t) which has resistance R and inductance L. So, the current satisfies the following ODE which is given by :

$$L\frac{dI}{dt} + RI = E(t) \tag{1}$$

So, I am going to take my L and my R as constants. to begin with ok. So, let us take for the time being let us take my forcing function E(t) to be or the following simple form.

$$E(t) = E_0 e^{-a|t|}$$

So, to find the solution notice that this is an 1st order; 1st order ODE right. So, once we apply the Fourier transform to this 1st order ODE I get the following:

$$[(ik)L + R]\hat{I} = F(E) \tag{2}$$

So, this is equal to the Fourier transform of E; the Fourier transform of E. So, notice that the Fourier transform of E is of course, the Fourier transform of this function

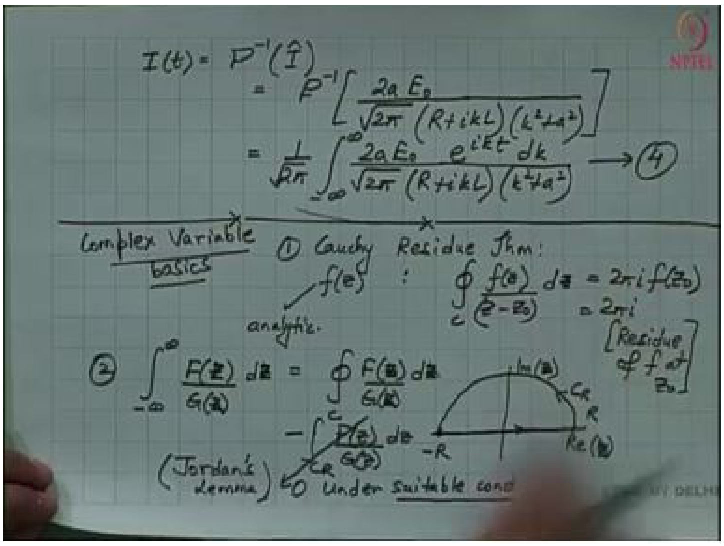
$$\hat{I} = F(I)$$

$$F(E) = F\left[E_0 e^{-a|t|}\right] = \frac{E_0 2a}{\sqrt{2\pi}\,(k^2+a^2)}$$

So, using these known results from 2 I get:

$$\hat{I} = \frac{E_0 2a}{\sqrt{2\pi}\,(k^2+a^2)\,(R+ikL)} \tag{3}$$

so let us call this as 3. So, now, from this I can find the inverse of the Fourier transform.



So, the solution I(t) is given by:

$$I(t) = F^{-1}(\hat{I})$$

$$= F^{-1}\left[\frac{2aE_0}{\sqrt{2\pi}(R+ikL)\,(k^2+a^2)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2aE_0 e^{ikt}dk}{\sqrt{2\pi}(R+ikl)\,(k^2+a^2)} \tag{4}$$

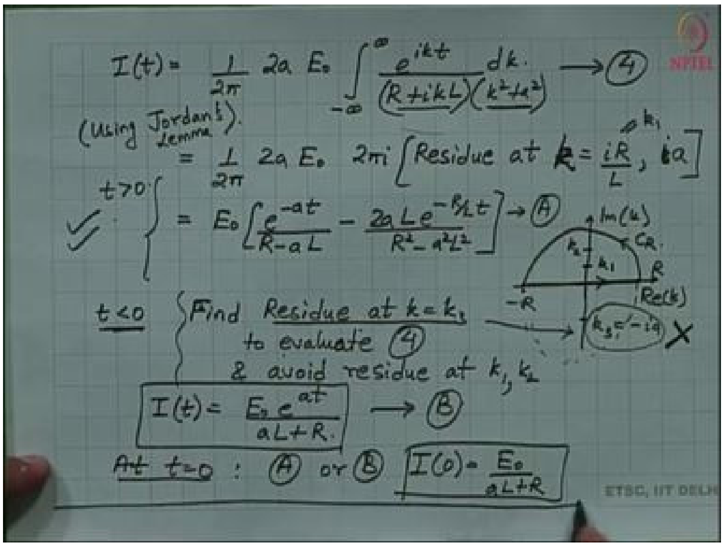
Now, to find the solution we have to evaluate this integral. Now to evaluate this integral we need some basics I need to revise some basics of complex variables. So, let me just revise some complex variables basics. So, the two things that are important to find the solution to this integral one of them is the Cauchy Residue theorem. So, what is Cauchy Residue theorem it tells me the following. It tells me that if I have a function f(z) and f(z) is an analytic function in the real in on the real domain this could boiled down to being differentiable, but in the complex plane analytic is a little bit more than differentiability. So, f x being analytic then it tells me that 1.Cauchy Residue Theorem: So, this particular integral has a singularity at $z = z_0$ and the singularity is of the 1st order also called as the pole. So, if I have the integration of the following function then over the complex plane this is equal to

$$f(z) : \oint_c \frac{f(z)}{(z-z_0)}dz = 2\pi i f\,(z_0) = 2\pi i$$

Then the next result that is important for us is the fact is that we want to integrate integrals of this form:

$$\int_{-\infty}^{\infty}\frac{F(z)}{G(z)}dz = \oint_c \frac{f(z)}{G(z)}dz - \oint_{C_R}\frac{F(z)}{G(z)}dz$$

So, notice that this could this integral is of this form where over the complex plane. So, let me draw this. So, I am talking about a plane where I am I have the variables k being complex. The contour integral is we have to draw a contour. So, let us draw us let us say a semi circle and the choice of the semi circle is due to the fact that one of the sides of this contour is our integral from in the limit R tending to infinity. So, we complete this curve to have a closed contour. So, c; so let me denote the circular part of this contour by c R. So, what I have is that well just to keep continuity. So, let me say that my variable of integration is z without loss of generality. So, any ways z is a complex variable and. So, was k. So, let me say this is z and this is also z. So, in this range I have that this integral which is over the real over the real z or real axis is equal to the contour integral minus the integral over the curved part right and further it can be shown that this integral goes to 0 under suitable conditions condition which is given by the. So, called Jordans lemma. Now, I am not going to go in details of Jordans lemma people can search up the basics of complex variables and Jordans lemma is quite available now one important corollary out of this Jordons lemma is that specially if I have that F over G is such that the order of F suppose F and G both are polynomials and the order of F the order of F is two orders of magnitude less than the order of G then this integral can be shown to be equal to 0 in the limit R tending to infinity right.



So, Jordans lemma is much more than that, but we are going to use the fact that our F(z) is two orders of magnitudes smaller than G and both are polynomials. So, coming back to our integral here.

$$I(t) = \frac{1}{2\pi} 2aE_0 \int_{-\infty}^{\infty} \frac{e^{ikt}dk}{(R + ikL)(k^2 + a^2)} \tag{4}$$

So, let us say this is my 4. So, coming back to 4:

$$I(t) = \frac{1}{2\pi} 2aE_0 \quad 2\pi i \left[\text{ Residue at } k = \frac{iR}{L}, \text{ ia } \right]$$

So, on notice that if I were to take the absolute value of this integrand then on the numerator I am going to get one and the order of the polynomial in the denominator will be 3. So, some conditions; so, one of the conditions that the order of the numerator should be the polynomial order of the numerator should be less than 2 or more than that of the denominator is quite satisfied. So, what I have is that this is also equal to I am now going to use. So, using my Jordans lemma result this integral boils down to :

$$= E_0 \left[\frac{e^{-at}}{R - aL} - \frac{2aLe^{-\frac{R}{L}t}}{R^2 - a^2L^2} \right] \qquad (A)$$

So, I am going to take the residue at well in our case z is k. So, $k = \frac{iR}{L}$ again let me draw the diagram here. So, I have real k imaginary k here and the contour i am talking about is the following. So, this is the integral let us say this is the integral that I have.So, I do not have to count this root at all because it is not inside this contour. So, finally, evaluating the residue I get the above following result:

$$I(t) = \frac{E_o e^{at}}{aL + R} \qquad (B)$$

Now, this was the case when; this was the case when t was positive. Well if t is time then t of course, needs to be positive, but suppose t is a variable and independent variable we could have t negative so, in that case we have to calculate. So, let us say this is our integral 4. So, you have to find the residue at k=k3 which is given this value to evaluate the integral 4 and avoid residue at k1 and k2 because in that case the contour would be on the negative half of the plane or the covering the negative imaginary axis when t is negative. So, all I have to do was to will be to evaluate the residue at k=k3 for t=0 and let me write down the answer in that case my (I)t my current becomes :

$$I(0) = \frac{E_0}{aL + R}$$

So, at t=0 you can check that for even for A or B my I of 0 comes out to be the following constant whether you want to evaluate it from A or you want to evaluate it from B; it will come out to be the same ok so, that completes my problem here. So, let us precede to look at some more complicated examples.