Integral Transform and Their Applications
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Lecture - 69
Introduction to Wavelet Transform Part 3

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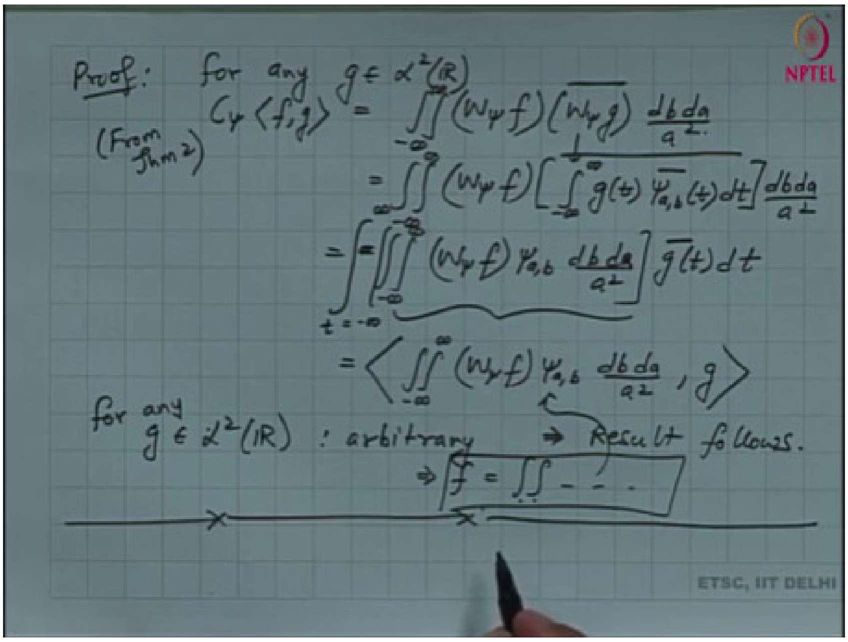

So far I have seen how to construct wavelets, I have looked at some particular examples of wavelets, how to construct wavelets from some simple wavelets and further I have also shown you how to construct the inverse of the wavelet transform. So, I am going to now look at some properties of these transforms.So, let us look at these properties.

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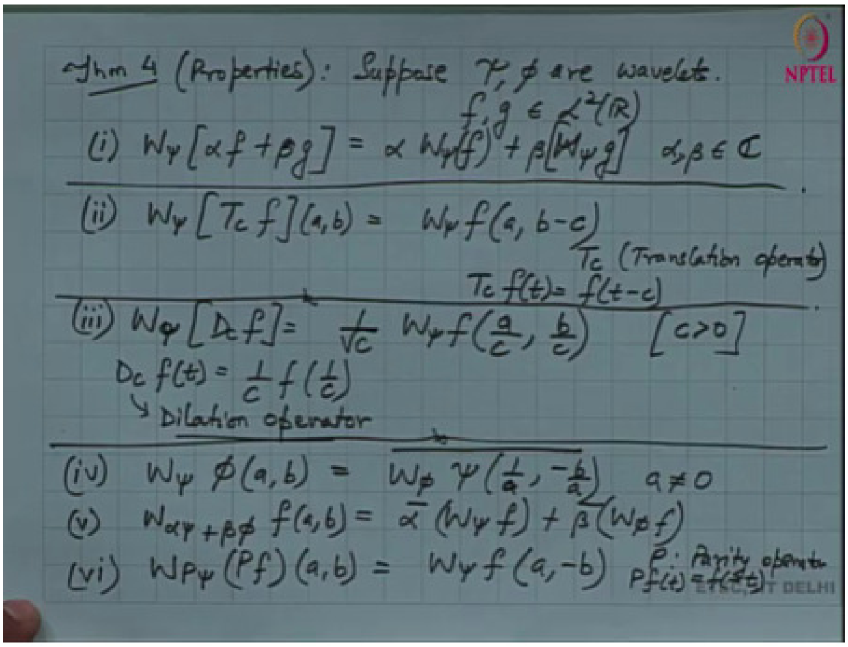

So, I am going to consider these properties and state them as a theorem. So, again most of these properties can be very easily shown and that will be left as an exercise to the students. So, suppose I am given ψ and ϕ are 2 wavelets and I am given 2 functions f and g which are square integrable functions.So, then the following property holds:

$$
\begin{aligned}
(i):&\quad W_\psi\big[\alpha f+\beta g\big]=\alpha W_\psi(f)+\beta W_\psi(g);\qquad \alpha,\beta\in\mathbb{C}\\
(ii):&\quad W_\psi\big[T_c f\big](a,b)=W_\psi f(a,b-c);\qquad T_c:\text{Translation property, i.e. } T_c f(t)=f(t-c)\\
(iii):&\quad W_\psi\big[D_c f\big]=\frac{1}{\sqrt{c}}W_\psi f(a/c,b/c);\qquad c>0,\ D_c f(t)=\frac{1}{c}f\left(\frac{1}{c}\right)\\
(iv):&\quad W_\psi\phi(a,b)=W_\phi\psi(1/a,-b/a);\qquad a\neq 0\\
(v):&\quad W_{\alpha\psi+\beta\phi}\ f(a,b)=\bar{\alpha}(W_\psi\ f)+\bar{\beta}(W_\phi\ f)\\
(vi):&\quad W_{P\psi}(Pf)(a,b)=W_\psi f(a,-b);\qquad P:\text{Parity function i.e. }\ Pf(t)=f(-t)
\end{aligned}
$$

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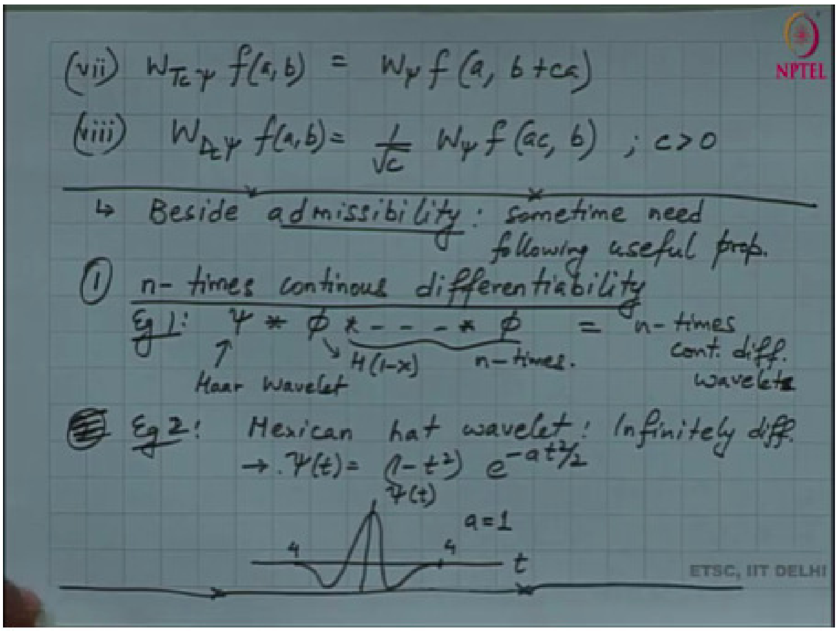

So, moving on there is further results properties of wavelets that are useful. Another result shows that:

$$
\begin{aligned}
(vii): &\quad W_{T_c\psi}\ f(a,b) = W_\psi\ f(a, b+ca) \\
(viii): &\quad W_{D_c\psi}\ f(a,b) = \frac{1}{\sqrt{c}} W_\psi\ f(ac,b); \qquad c>0
\end{aligned}
$$

So, we see that all these properties are quite useful properties for wavelet transforms to have. Now, this question is what are some of the useful properties that we must have in order to have nice wavelets. So, one thing is that we have seen that admissibility condition that particular integration of the transform of the wavelet should be finite. Are there any more conditions that we need to look at to describe these wavelets properly? So, let us look at that.

So, I am going to show you that besides admissibility, there are some other factors which are necessary or useful sometimes to describe wavelets. So, the first type of properties that will be quite useful if have the wavelet which is n times continuously differentiable. I saw that if the wavelet has discontinuity, then the decay of the wavelet in the transform plane is very slow and that is an undesirable feature of the wavelet. So, in ideality I would like to have as many times differentiability as I want. So, let us say n times continuously differentiable wavelet.

So, let us take an example of Haar wavelet. For my Haar wavelet, if I were to convolve with my heavyside function $H(1-x)$ and I convolve it n times then the resultant is an n times continuously differentiable wavelet. I know that the convolution is a wavelet by the result that I have proved earlier. So, that is one way of constructing n times continuously differentiable wavelet by introducing n times the convolution.

Again, to show you some other properties which are useful let me introduce one more wavelet which is quite useful. It is also known as the Mexican hat wavelet. So, this wavelet is particularly useful in describing n times continuously differentiable functions. So, the in fact, this particular wavelet is infinitely differentiable. This has the property of being infinitely differentiable. So, the wavelet is as follows:

$$\psi(t) = \psi(t) = (1-t^2)e^{-at^2/2}$$

If I were to draw this wavelet I see the figure. This is for $a = 1$. We see that the wavelet looks like the Mexican hat that we are familiar with. So, hence the name the Mexican hat wavelet.

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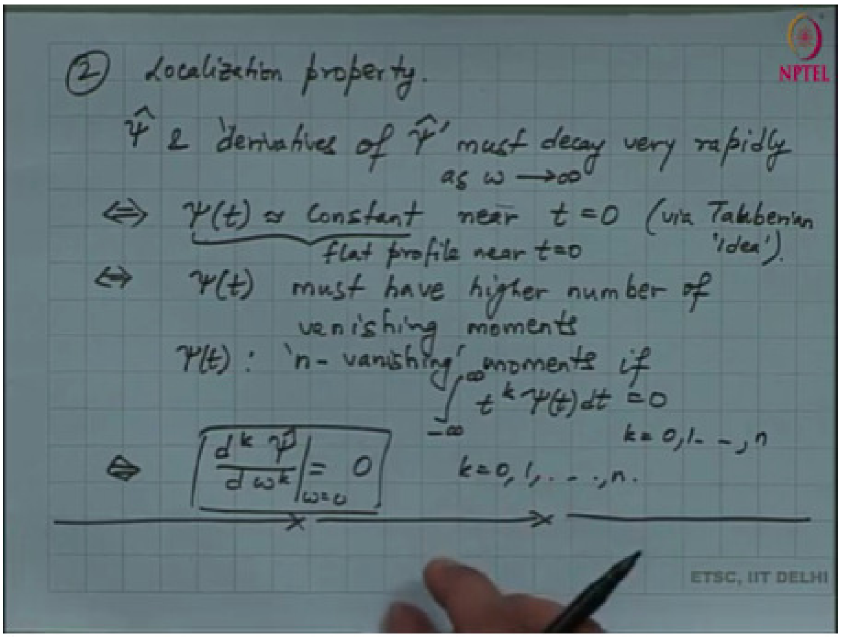

So, let us move ahead. So, let us look at the second important feature that is the localization property of the wavelet. So, what this means? We saw that the wavelet should decay as we take larger and larger values of the transformed variable. So, which means that I must have that ψ and derivatives of ψ must decay very rapidly as my ω goes to infinity. I saw that for Haar wavelet the decay is very slow. So, that is not a very useful feature to have or this also means that I need that $\hat{\psi}$ and derivatives of $\hat{\psi}$ must decay in the asymptotic limit. So, this is also equivalent to the fact that my $\psi(t)$, that is the wavelet in the physical plane, must be a constant near the value $t = 0$.

Now, this is similar to our Tauberian result (if people recall) that according to Taubers result, that was done in Laplace transform, to look at the solution at infinity we need to look at the solution in the physical plane at 0. So, which means that if the solution, at the physical plane is near constant at the value $t = 0$, the solution in the transform plane will rapidly decay as the value in the transform variable goes to infinity. So, this is a similar to your Tauberian idea that was introduced earlier. So, which means that this will be true if I have the following conditions which is satisfied, which means that ψ must have and higher vanishing moments. Let us say $\psi(t)$ has n vanishing moments. That is possible if I have:

$$\int_{-\infty}^{\infty} t^k \psi(t) dt = 0; \quad k = 0, 1, 2 \cdots, n$$

So, if for all values of k this particular integral vanishes, I say that ψ has n vanishing moments. So, the higher the number of vanishing moments the nicer or the more flatter the value of this wavelet is. So, this is the flat profile near $t = 0$, which means the faster the transformed wavelet decays with respect to ω going to infinity. So, this also means having n vanishing moments means that the following derivatives of $\hat{\psi}$ with respect to ω near $\omega = 0$ goes to 0. So, these are for all values of k from 1 to n. So, that is my condition. So, then I have seen a lot of ideas about continuous wavelets. So, let me now start describing the discrete form of these continuous wavelets. So, let me introduce the discrete wavelets.

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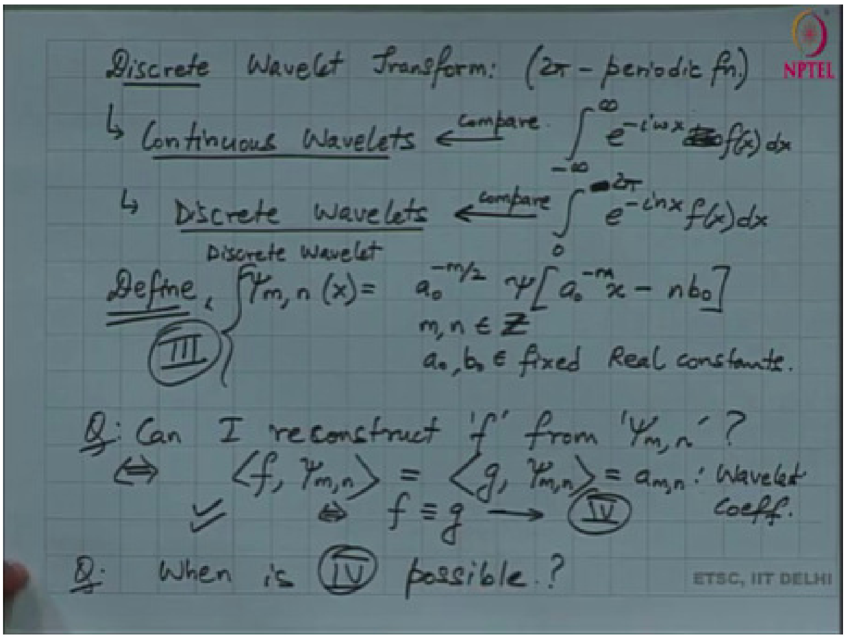

So, let me introduce the discrete wavelet transform. Now as the name discrete suggests these transforms will be quite useful for functions which are periodic let us say 2π periodic. So, in that situation I am going to use my discrete version of the continuous wavelets. So, let us start the discussion. So, let me again go back to my discussion on continuous wavelets. Please recall that for describing continuous wavelets I was comparing those wavelets with respect to Fourier transform. So, I was comparing with respect to this following integral:

$$\int_{-\infty}^{\infty} e^{-iwx} f(x)dx$$

So, to find continuous wavelets, I was regularly using my Fourier transforms. Now, if I were to use and describe my discrete wavelets, I have to resort to Fourier series or the integrals of this form which are 2π periodic. So,

$$\int_{0}^{2\pi} e^{-inx} f(x)dx$$

So, then let me just start by definition of the discrete wavelets. So, let me define discrete wavelet as follows. So,

$$\psi_{m,n}(x) = a_0^{-m/2}\ \psi\big[a_0^{-m}x - nb_0\big]; \quad m, n \in \mathbb{Z}, \quad a_0, b_0 \in \text{fixed real constants}$$

Let me call this definition by III. So, what I have done is instead of defining ψ_x, I have introduced to two new constants. So, here m and n are integers and a_0 and b_0 are fixed real constants. So, once I fixed two constants, I can describe my discrete wavelets using the continuous definition of the continuous wavelets here. So, then what I see is this the definition is fine, but this question to be asked that can I reconstruct my function f from these wavelets $\psi_{m,n}$. So, if that is true, then this definition is going to make sense. So now, I have to see how can I reconstruct f from these discrete wavelets. So, this question can be equivalently posed as follows. This equivalently means that if I am given the following inner product $< f, \psi_{m,n} >$, and suppose that is also equal to this inner product $< g, \psi_{m,n} >$ then this must be equivalent to saying that f is identical to g. Let me call this this expression as IV. So, so these are my wavelet coefficients in describing the wavelet series of the function f with respect to $\psi_{m,n}$. So, if I say that the wavelet coefficients are equal for 2 functions then then 2 functions are almost equal. So, if that is true if this is true, then the answer to my question above is also true. The question is when is IV possible. So, when can I have that the coefficients going equal implying for two functions implying that the functions are equal? Now the it turns out that this argument IV is possible if these coefficient supposed to be close implying the functions are close to each other and vice versa.

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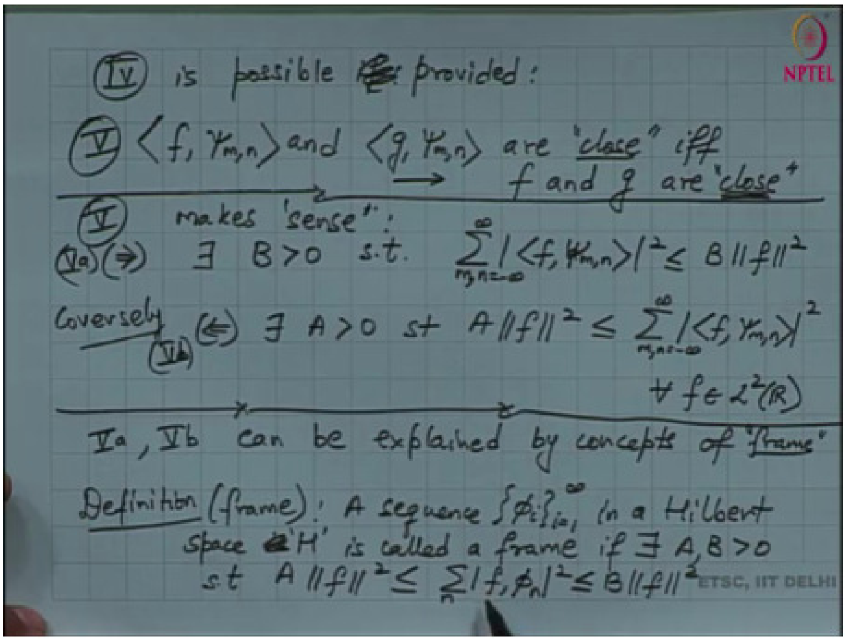

So, what I mean is the following. What I mean is that IV is possible if the following statement is true. If I am given that the inner product $< f, \psi_{m,n} >$ and the inner product $< g, \psi_{m,n} >$ are close are close if and only f and g are close.

Now, I have to describe what do I mean by this term close. Let me write statement Va. So, the statement Va makes sense if these two statement happens.

$$\exists\ B > 0 \ \text{ s.t. } \sum_{m,n=-\infty}^{\infty} |< f, \psi_{m,n} >|^2 < B\ ||f||^2$$

$$\text{conversely,} \ \ \exists\ A > 0 \ \text{ s.t. } \ A||f||^2 \leq \sum_{m,n=-\infty}^{\infty} |< f, \psi_{m,n} >|^2 \ \text{ for all } \ f \in L^2(\mathbb{R})$$

Let me call this last statement as Vb. Then I know that for f and g close implies that the coefficients are close. Now using these two statements, let me come to the definitions. If Va and Vb holds then my discrete wavelet transform is well defined or I can use the definition that I have just stated in the previous slide. So, Va and Vb can be explained by the concept of the so-called frames. So, I am going to introduce the concept of frames. So, let us see what is this. So, let me introduce a definition. What it says is a sequence $\{\phi_i\}_{i=1}^{\infty}$ in a Hilbert space is denoted by 'H' is called a frame if there exists and A ,B positive such that

$$A||f||^2 \leq \sum_n |f, \phi_n|^2 \leq ||f||^2$$

So, if I have these following bounds on these coefficients, then I call this sequence as frames where my constants A ,B are called frame bounds.

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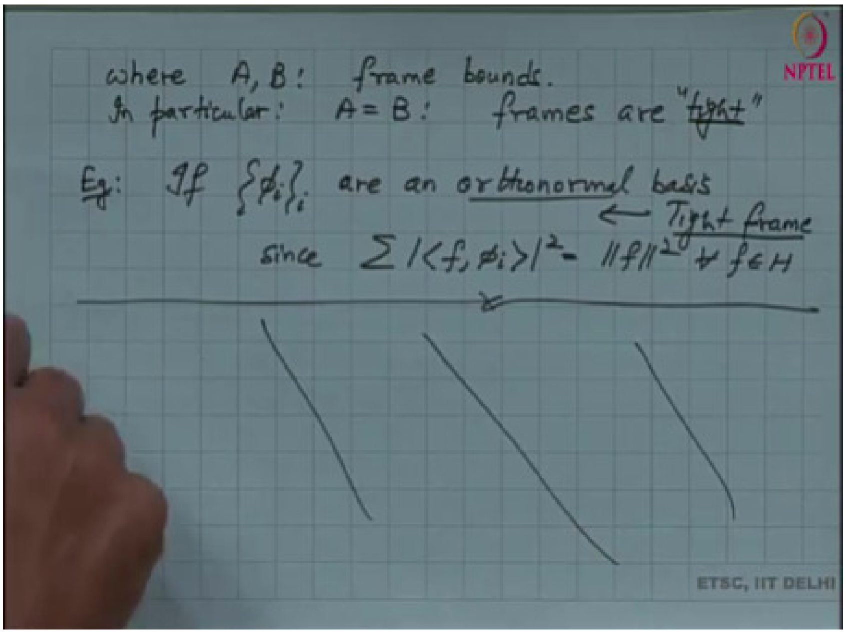

Now in particular I have a special case. In particular if I am given that A=B, then the frames are the so-called tight frames. So, which means that the inequality changes to an equality if A is equal to B.

Now, let me give you one example. So, suppose I am given the sequence of functions $\{\phi_i\}_i$ in this Hilbert space and they are orthonormal. So, these are an orthonormal basis function. Then in that case these $\{\phi_i\}_i$ are tight frame. Why? Because:

$$\sum | < f, \phi_i > |^2 = ||f||^2 \quad \text{for all} \quad f \in H$$

So, for any ϕ the sequence of orthonormal basis I see that the frames are tight or if I can select an orthonormal basis, I know that I can define my discrete wavelet transform. So in my next lecture I am going to give certain examples of these discrete wavelet transforms, namely the discrete Haar wavelet, the discrete Shannon wavelet and the discrete Debauchee wavelets for certain values of the parameter of the debauchee wavelets. So, thank you for listening.