Integral Transform and Their Applications
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Lecture - 68
Introduction to Wavelet Transform Part 2

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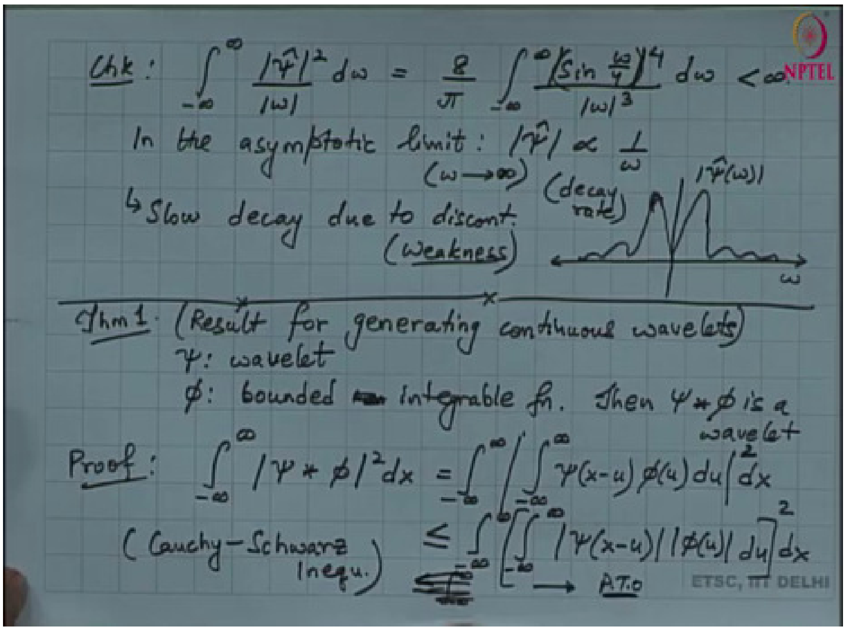

So, then let us look at some results now. So, the first result that I am going to construct how to construct continuous wavelet. So, this is a result for generating continuous wavelets. So, if I am given that ψ is a wavelet and is a bounded integrable functions then $\psi * \phi$ is a wavelet. Using this theorem I can construct wavelets of any possible kind that I want by appropriately convolving with a bounded integrable function.

So, if I start with a Haar wavelet I can find a continuous wavelet by suitably taking the convolution to suitable functions which are square integrable. So, then let us look at the proof of this result, so I am given let us start with the square integration of $\psi * \phi$. So this result is:

$$\int_{-\infty}^{\infty} |\psi * \phi|^2 dx = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \psi(x-u)\phi(u)du \right]^2 dx$$

Now, notice that I am going to replace my equality with an inequality by suitably changing the order of integration.

$$\leq \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |\psi(x-u)|\ |\phi(u)|du \right]^2 dx$$

So, I see that what I have used in this inequality is the Cauchy Schwarz relation. So, the inequality arises via Cauchy Schwartz inequality relation. So, then let me just further break this product down.

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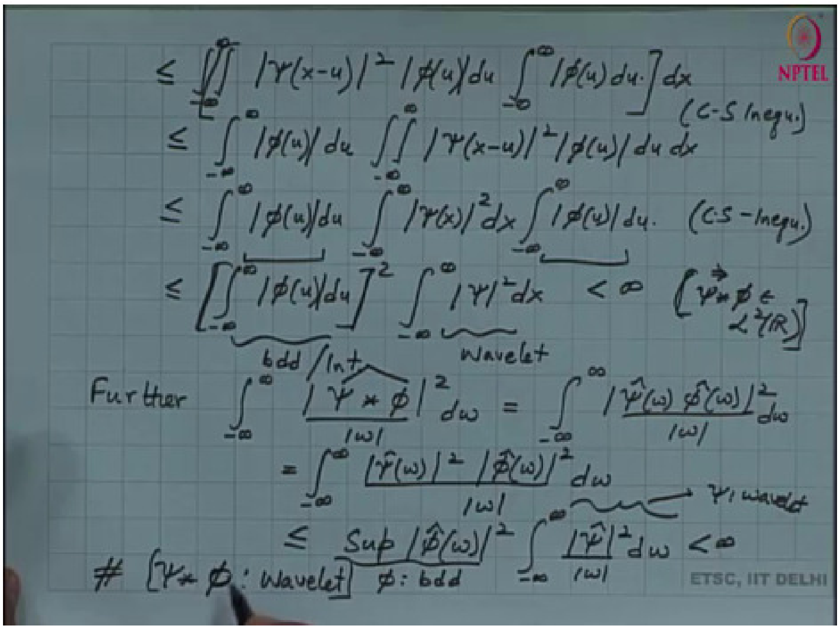

So, then I see that this inequality is further bounded above by the following inequality:

$$\leq \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|\psi(x-u)|^2\ |\phi(u)|du\int_{-\infty}^{\infty}|\phi(u)|du\right]dx \qquad \text{(C.S. Inequality)}$$

$$\leq \int_{-\infty}^{\infty}|\phi(u)|du\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\psi(x-u)|^2\ |\phi(u)|dudx$$

$$\leq \int_{-\infty}^{\infty}|\phi(u)|du\int_{-\infty}^{\infty}|\psi(x)|^2dx\int_{-\infty}^{\infty}|\phi(u)|du \qquad \text{(C.S. Inequality)}$$

And I can now combine this third integral with the first to see that this is also the integral of absolute value of ϕ with respect to u whole square. So,

$$\leq \left[\int_{-\infty}^{\infty}|\phi(u)|du\right]^2\int_{-\infty}^{\infty}|\psi(x)|^2dx$$

So, I see that this is finite why because I have that ϕ is bounded so that this particular integration is finite and ψ is my wavelet. Since ψ is wavelet, I know that ψ is a square integrable so this integration is finite. Further notice the following integration of the following transform.

$$\int_{-\infty}^{\infty}\frac{|\psi\hat{*}\phi|^2}{|\omega|}d\omega = \int_{-\infty}^{\infty}\frac{|\hat{\psi}(w)\hat{\phi}(w)|^2}{|\omega|}d\omega$$

$$= \int_{-\infty}^{\infty}\frac{|\hat{\psi}(w)|^2|\hat{\phi}(w)|^2}{|\omega|}d\omega$$

So, I have just use my standard definition of convolution of two functions to arrive at this particular integration. I see that this will be bounded above by the supremum value or the maximum value of phi. I know that ϕ is bounded. So, I can always write:

$$\leq \text{Sup}|\hat{\phi}(w)|^2\int_{-\infty}^{\infty}\frac{|\hat{\psi}|^2}{|w|} < \infty$$

I know that, this is also finite because ψ is a wavelet and supremum value is finite because ϕ is bounded and integrable. So, then what I have shown here is that the convolution of ψ with this bounded integrable function ϕ is a wavelet. So, that result allows us to generate wavelets starting from a very basic a very basic wavelet. So, let us look at the application of this particular result.

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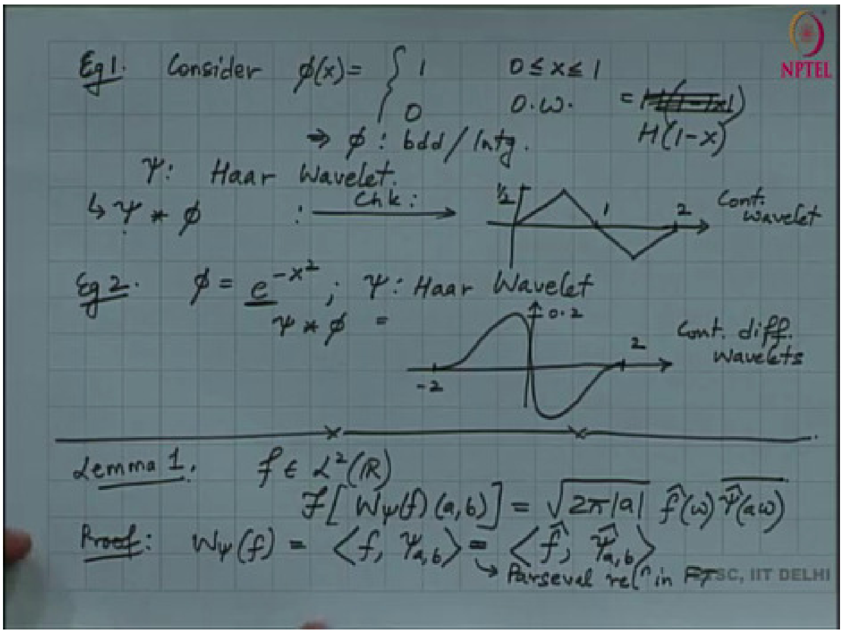

So, let us consider one example. Consider:

$$\phi(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

So, we can consider ϕ to be a Heaviside function $H(1-x)$. So if I were to use my Haar wavelets, let me call ψ to be my Haar wavelets. And then if I were to find what is the convolution of this Haar wavelet with this particular function we can see that in this case ϕ is bounded and integrable.So, I see that ψ convolve with ϕ and students can check that the convolution of these two function brings me a wavelet which is continuous and which looks as in figure. So, I have that this is from half to 1 to 2. So, I see that, this is my convolution of the Haar wavelet with the following Heaviside function and I get a continuous wavelet by the definition of the convolution of the wavelet with a bounded integrable function. In fact, we can see that if I continue to convolve the Haar wavelet with this ϕ let us say n times, I will make sure that I get at least (n-1) derivatives of the resulting wavelet to be continuous. So, here convolution with one application of this Heaviside function has led to the function being continuous or c_0. As we continue to convolve I can increase my smoothness in this particular wavelet.

So, then let us look at another example. Let me say that:

$$\phi = e^{-x^2}; \quad \psi : \text{Haar wavelet}$$

So, then in that case ψ convolve with ϕ. If students can check that after the convolution of the Haar wavelet with this smooth function, I get a resulting wavelet which is smooth and looks as in figure. So, this wavelet is continuously differentiable wavelet. So, it has derivatives of all orders available because of the convolution with this continuous and continuously differentiable bounded integrable functions. So, then let us look at some other results.

So, then before moving ahead, I have the small following small result which I denoted by a lemma, it tells that if I am given a function which is square integrable. So, suppose I am given a function square integrable then the Fourier transform of the wavelet transform is:

$$\mathscr{F}\left[W_{\psi}(f)(a,b)\right] = \sqrt{2\pi|a|}\hat{f}(w)\bar{\hat{\psi}}(aw)$$

So, this is a short result that can be proved very quickly. So, let us look at the proof of this lemma. So, what it says is let us look at what is the wavelet transform of this particular square integrable function. So, the wavelet transform of this function is defined as the following:

$$W_{\psi}(f) =< f, \psi_{a,b} >=< \hat{f}, \hat{\psi}_{a,b} >$$

This last relation comes due to the Parseval's relation in Fourier transform that was done in my Fourier transforms lecture.

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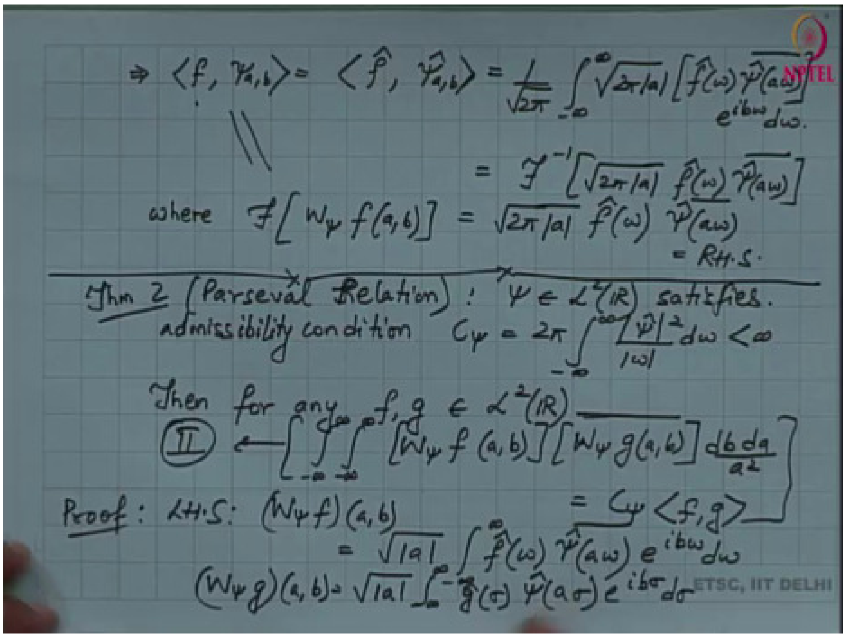

So, notice that if I were to take this inner product, I see that:

$$< f, \psi_{a,b} >=< \hat{f}, \hat{\psi}_{a,b} >= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi|a|}\ \left[\hat{f}(w)\bar{\hat{\psi}}(a,b)\right]e^{ibw}dw$$

So, then I see that this particular inner product is very nicely describing the inverse transform of this particular quantity inside the integral. Namely this is the inverse transform of the quantity:

$$\mathscr{F}^{-1}\left[\sqrt{2\pi|a|}\ \hat{f}(w)\bar{\hat{\psi}}(a,b)\right],$$

$$\text{where,}\quad \mathscr{F}\left[W_{\psi}f(a,b)\right] = \sqrt{2\pi|a|}\ \hat{f}(w)\bar{\hat{\psi}}(a,b) = \text{R.H.S.}$$

So, moving ahead let us look at another relation that will be useful in describing wavelet transform. So, I am going to denote that relation by the so called Parseval theorem. So, this is the Parseval relation for wavelet transform. So, what it says is suppose I am given ψ to be square integrable and of course, it is a wavelet. So, it satisfies my wavelet condition or my admissibility condition:

$$C_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}|^2}{|w|} dw < \infty$$

So, as long as this is finite my ψ is a wavelet. Suppose my ψ is a square integrable function and it is a wavelet. So, in that case for any f and g which is square integrable, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[W_\psi \ f(a,b)\right] \left[\overline{W_\psi \ g(a,b)}\right] \frac{db \ db}{a^2}$$
$$= C_\psi < f, g >$$

Let me call this result as II here. So, in describing this relation I am going to use the lemma that I have used above. So, let us start the proof of the result by starting from the left hand side. I see the following:

$$(W_\psi f)(a,b) = \sqrt{|a|} \int_{-\infty}^{\infty} \hat{f}(w)\hat{\psi}(aw)e^{ibw} dw$$
$$(W_\psi g)(a,b) = \sqrt{|a|} \int_{-\infty}^{\infty} \hat{g}(\sigma)\hat{\psi}(a\sigma)e^{ib\sigma} d\sigma$$

Let me just substitute the expressions for the wavelet transform of f with g and take the conjugate of the second quantity and plug it back into this integral.

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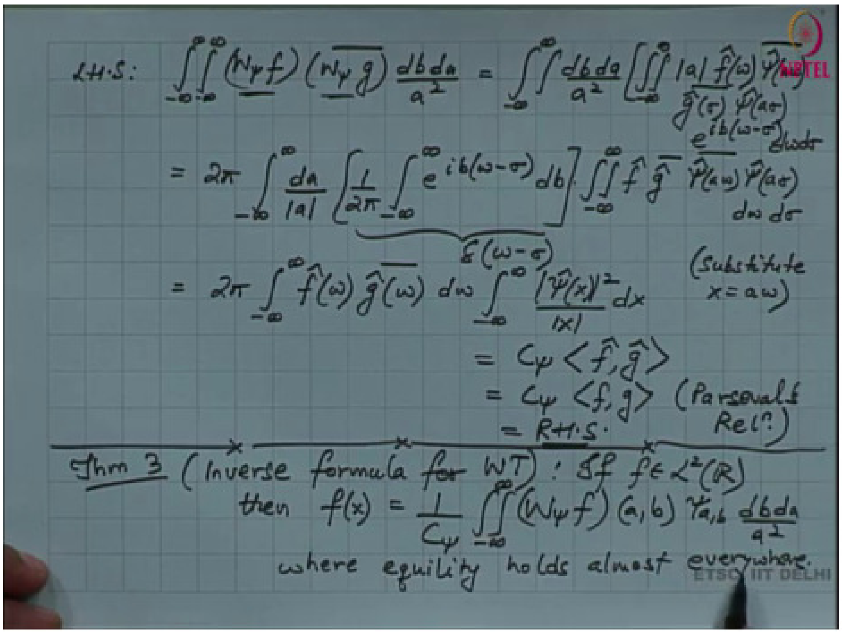

I see that my left hand side is as follows:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)\overline{(W_\psi g)} \frac{db \ db}{a^2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{db \ db}{a^2} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |a| \hat{f}(w)\hat{\psi}(aw)\overline{\hat{g}(\sigma)}\hat{\psi}(a\sigma)e^{ib(w-\sigma)dwd\sigma} \right]$$
$$= 2\pi \int_{-\infty}^{\infty} \frac{da}{|a|} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ib(w-\sigma)} db \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}\bar{\hat{g}}\overline{\hat{\psi}(aw)}\hat{\psi}(a\sigma) dw d\sigma$$

So, then I see that this particular integral inside this is nothing but the delta function of $(w - \sigma)$, that is by the standard definition of delta function which means that I can safely replace my w with σ and then this double integral will be reduced to a single integral.

$$
\begin{aligned}
&= 2\pi \int_{-\infty}^{\infty} \hat{f}(w)\bar{\hat{g}}(w)dw \int_{-\infty}^{\infty} \frac{|\hat{\psi}(x)|^2}{|x|}dx \\
&= C_\psi < \hat{f}, \hat{g} > \\
&= C_\psi < f, g > \qquad\qquad \text{(Parseval's rel.)} \\
&= R.H.S.
\end{aligned}
$$

So, then the next result gives me the idea on how to find the inverse of the wavelet transform. So, let me introduce the result in the form of a third theorem, that is related to the inverse formula for wavelet transform and how to calculate the inverse of the wavelet transform?So, if I am given that f is square integrable then,

$$
f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a, b)\psi_{a,b} \frac{db\ da}{a^2}
$$

where equality holds almost everywhere. Notice that the last statement that I have used where equality holds almost everywhere has been introduced because I am going to show this particular inverse with regards to the inner product.

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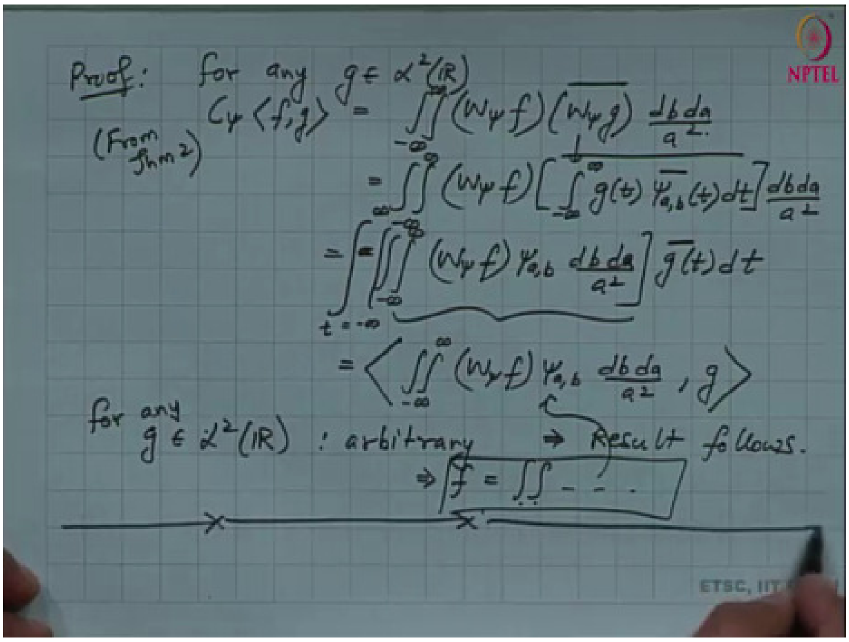

So, let me start showing you the proof. The proof is quite straightforward. I have that supposed I am given any square integrable function $C_\psi(f, g)$. So, this is:

$$
\begin{aligned}
C_\psi(f,g) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)\overline{(W_\psi g)}\frac{db\ da}{a^2} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)\overline{\left[\int_{-\infty}^{\infty} g(t)\bar{\psi}_{a,b}(t)dt\right]}\frac{db\ da}{a^2} \\
&= \int_{t=-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)\psi_{a,b}\frac{db\ da}{a^2}\right]\bar{g}(t)dt \\
&= \left\langle \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)\psi_{a,b}\frac{db\ da}{a^2}, g\right\rangle
\end{aligned}
$$

So, the choice of g was arbitrary, which means that my result follows. So, I see that f is equal to this double integral, which is described inside this inner product. So, the result follows. So, let us continue and describe some properties of wavelets.