Integral Transform and Their Applications
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Lecture - 67
Introduction to Wavelet Transform Part 1

Good afternoon everyone. So, in today?s and the next lecture I am going to introduce yet another new transform that is the Wavelet Transform. Now the Introduction of Wavelets interestingly was not done by any mathematician, but engineers and this is one prime example of a transform that came into existence not because of the theoretical requirement, but because of the practical applications and practical needs of solving certain problems in domains of engineering and science. So, emphasizing the importance of wavelets, I just want to say that wavelets have revolutionized many areas of science and engineering starting from biomedical medicine, signal processing and and many more. So, let me start the discussion on wavelet transform.

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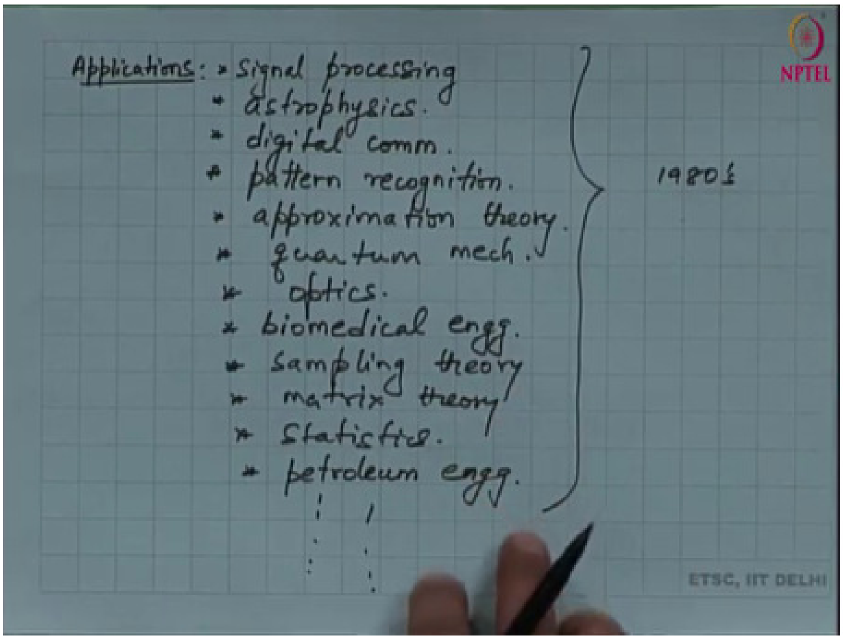

So, before that again I just want to briefly emphasize the applications of wavelets. So, the applications of wavelets are manyfolds starting from areas in signal processing, then from signal processing, to astrophysics, to digital communications, to image processing or pattern recognition, in approximation theory, in quantum mechanics, in optics. So you can see that there is a list of areas where wavelets are widely used. Also we have biomedical engineering, sampling theory or statistics, sampling theory, matrix theory, petroleum engineering. Specially the use of wavelets are used in petroleum engineering to record the segmentation of well logs. So we see the wavelets are widespread in almost any areas, that in science and engineering that we all can think of and let me just highlight that the start of all these the applications that the wavelets started as late as in 1980s and moving on to 90s and 2000s. So, let me just start our discussion by just briefly mentioning what are these wavelets?

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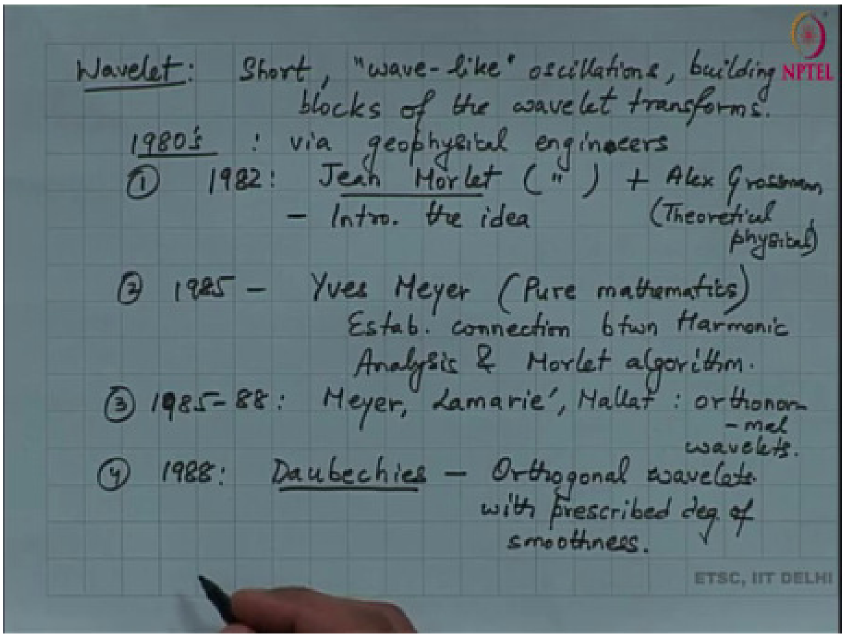

So, wavelets is a short wave like pulse. These are short wave like oscillations or pulses. So, these are the building blocks of the transform. So, when we construct the wavelet transforms, it is all about reasonable integration using these wavelets. So, let me continue by showing you some of the development in this area and then we are going to discuss during the course of a lecture, some of these recent developments in the form of results and theorems. So, starting of wavelets was in 80s and that is again via some engineers, some geophysical engineers. So, interestingly it was not brought about by any mathematicians, but engineers.

So, the first person in this list of record is in 1982 by Jean Morlet. Morlet was a geophysical engineer and his idea along with Alex Grossmann ,who was a theoretical physicist ,was the start of this area. So, the area started with the initial ideas put forward by these two scientists. So, they introduced the idea of wavelets to describe complex engineering processes. So, then the next major development was in 1985, where a pure mathematician by the name of Yves Meyer. So, Yves Meyer saw the relation between wavelets and harmonic analysis. So, this scientist was able to identify the relation between how this wavelet influences certain areas of harmonic analysis. So, established connection between harmonic analysis and the Morlet algorithm, that was introduced by Jean Morlet earlier. So, then there were several other developments which I am not going to go into great details, but I am going to highlight some of the major milestones between 1985 to 1988 and there were certain work by these three people Meyer, Lamarie and Mallat. These are all French scientists, they introduced the concept of orthonormal wavelets.

We will also see in our discussion as to how to construct orthonormal wavelets. And then another big achievement was in 88 by a Belgian mathematician by the name of Daubechies. So, Daubechie introduced the so-called Daubechie wavelets which are orthogonal wavelets of any given prescribed order of continuity or order of differentiability.

So, these were special orthogonal wavelets with prescribed degree of smoothness. Later on in my second lecture, I will highlight some of the wavelets that were developed by Daubechies namely the orthonormal wavelets of the second and the fourth kind. So, these were some of the major milestones, let me just highlight briefly some of the other milestones. Although lot of work was done in the early 80s to late 80s, but it was seen that despite all these developments, it is very difficult to construct symmetric and orthogonal wavelets. Some work was done in the early 90s on that area.

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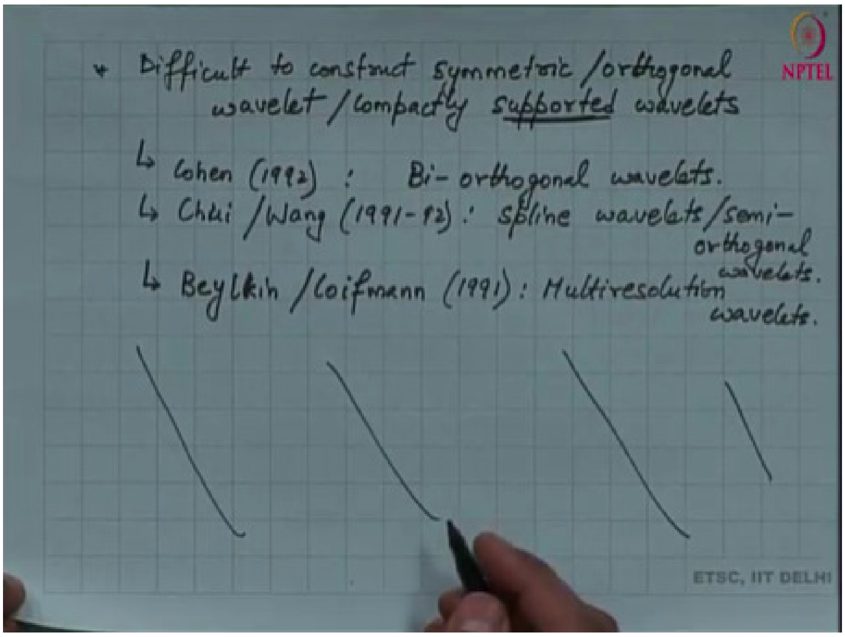

So, it was seen that it is difficult to construct symmetric orthogonal wavelets, orthogonal compactly supported wavelets. So, when I am going to use the word support, then by support, I mean a sub domain where the wavelets are defined. So, when I say compactly supported this means that the wavelets are defined in a particular sub domain of the real or the complex plane. So, then Cohen in 1992, he did some initial work on orthogonal wavelets and then Chui and Wang also introduced so. That was all done in the early 90s and these two scientists introduced the spline wavelets which were semi orthogonal wavelets. So, spline wavelets are semi orthogonal wavelets. And then another notable mention was by Beylkin, a Russian scientist and Coifmann. So in 1991, they introduced the multi resolution wavelets. So, we are also going to study one particular case of multi-resolution wavelets namely the Daubechie wavelets. I am just going to highlight some of the development, the early development followed by some of the major results and also later on I am going to introduce the Daubechies wavelets. So, moving on let me just start with the definition of the wavelet and also introduce the concept of wavelets.

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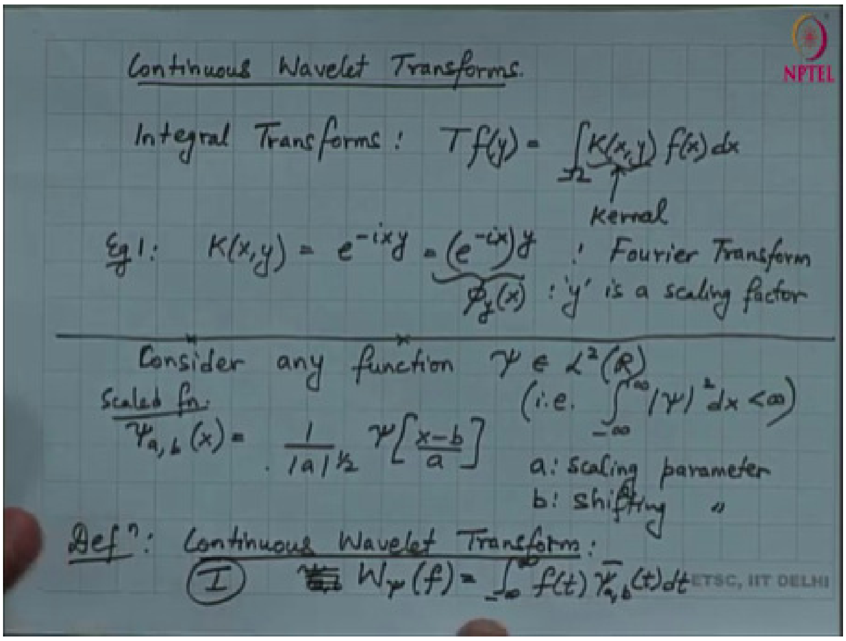

So, I am going to start with the continuous wavelets and the discrete wavelet will follow sometime later in this lecture. So far we have seen different types of integral transforms, I have seen that integral transforms are to be defined by the following operator

$$Tf(y) = \int_{\Omega} K(x,y)f(x)dx$$

$K(x,y)$, we call this particular function as my Kernel of the integration domain. So, then well we have seen different types of Kernels in our discussions of different transforms, in particular if suppose my Kernel is the following trigonometric function:

$$K(x,y) = e^{-ixy} = (e^{-ix})^y$$

So, I know that in this is the case of my Fourier transform. So, this is already been discussed in great detail. So, let me call this particular function as $\phi_y(x)$. So, what it means is that as if y is like a scaling factor to this function e^{-ix}. So, we see that for this particular transform, we have this following Kernel. Now, in the case of wavelets we are given the complete independence of describing the Kernel which means that depending on specific types of wavelets that we choose or Kernel is going to vary on a case by case basis. So, let me start with a general function which is square integrable and I am going to describe how to construct wavelet transform. So, consider any function ψ such that ψ is square integrable over R. So, what it means is that the following integration of ψ^2 is following integration $\int_{-\infty}^{\infty} \psi dx$ is finite. So, ψ^2 is integrable. So, then let us consider a scaled version of this function. So, scaled function:

$$\psi_{a,b}(x) = \frac{1}{|a|^{1/2}} \; \psi\left[\frac{x-b}{a}\right]$$

where, I have that a is my scaling parameter and b is my shifting parameter. So, I have scaled and I have shifted my function by the respective constants a and b. So, then under suitable conditions which I am going to describe slightly later, I can always define my continuous wavelet transform as follows. Here, the wavelet transform I denoted by this capital W. So,

$$W_{\psi}(f) = \int_{-\infty}^{\infty} f(t)\bar{\psi}_{a,b}(t)dt$$

So, if I am given a suitable function which is square integrable and plus some other condition, I can always define my continuous wavelet transform by this definition I. Note that I have taken a conjugate of this function ψ. In general I can be a complex function and we can take conjugate.

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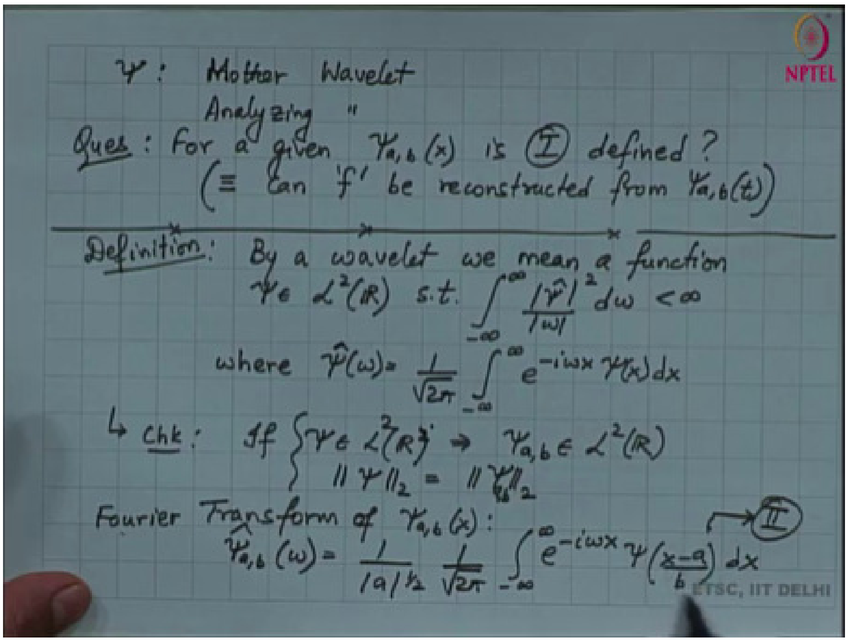

So, then, let before moving ahead this ψ is known as the mother wavelet or it is also called the analyzing wavelets. So, psi is also called the mother wavelet or the analyzing wavelet. Now, in this question the first question that arose in this definition is what are the suitable values or what are the suitable functions ψ that can be used to describe my continuous wavelet transform. So, which means for a given ψ, is I defined?

Now, or an equivalence question to be asked is, can my function f be reconstructed from my wavelet functions $\psi_{a,b}$. So, namely can the inverse be suitably found so, that I get back my original function? So, to answer that question I have to introduce another definition. So, let me introduce a definition of the wavelets. It turns out that not all functions are going to be considered in my definition of the wavelet transform, it is only these particular functions that I described now will be considered. So, for a wavelet we mean a function ψ which is all square integrable(i.e. $\psi \in L^2(R)$) such that:

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}|^2}{|w|} dw < \infty$$

where this hat denotes the Fourier transform of this mother wavelet. So, by wavelet I mean all those square integrable function such that this particular integration is finite, where:

$$\hat{\psi}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} \psi(x) dx$$

So, that is my Fourier transform of the wavelet function. So, if I am given that the wavelet function is square integrable, it also means that the scaled and the shifted version of the function is also square integrable. So, that can be easily checked. So, what I mean to say is that the L^2 norm of this function is also equal to the L^2 norm of the scaled version of this function. So, these two norms are equal in the L^2 sense in the square integrable sense. So, notice that my Fourier transform of $\psi_{a,b}(x)$ can be found like:

$$\hat{\psi}_{a,b}(w) = \frac{1}{|a|^{1/2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} \psi\bigg((x-a)/b\bigg) dx$$

We see that the Fourier transform of the wavelet can be evaluated by this integral of the scaled wavelet. Let me call this expression by II. So, if I were to evaluate this expression. So, let me substitute another variable t by this scaled and shifted variable.

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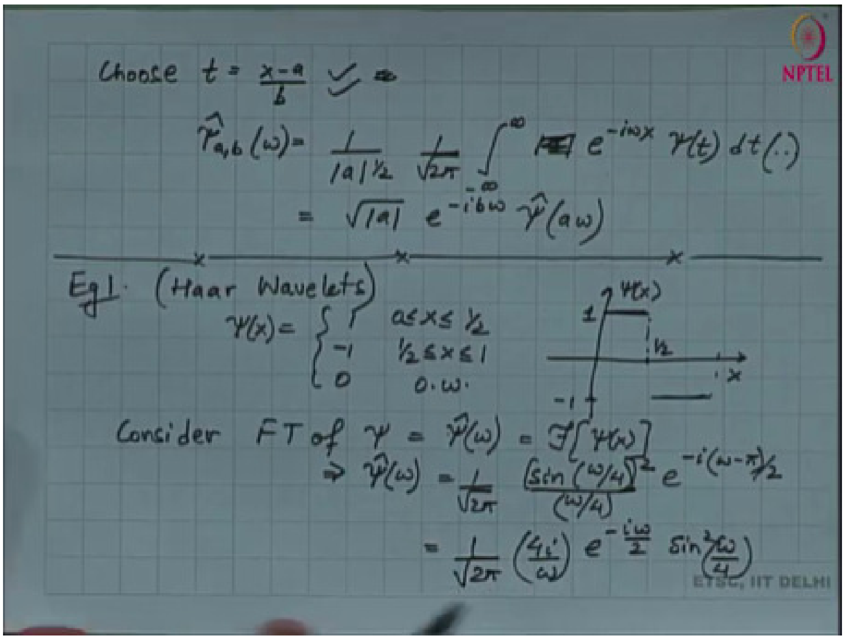

So, choose my new variable $t = \frac{x-a}{b}$, I see that my integration is as follows:

$$\hat{\psi}_{a,b}(w) = \frac{1}{|a|^{1/2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} \psi(t) dt(\cdots)$$
$$= \sqrt{|a|} e^{-ibw} \hat{\psi}(aw)$$

So, we are going to use all these results when we show some specific examples of wavelets. So, let me start with the first example. So, the first example that I want to show is the famous Haar wavelets. Haar wavelets are step functions which means the wavelets are described by the following values:

$$\psi(x) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ -1 & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

So, these are my wavelet functions values and if you were to draw this function, I see that there is a step function and the step is at $x = 1/2$. At 1/2 there is a sudden jump from value 1 to -1. So, that is my Haar wavelet. Now let us see what is the Fourier transform of this Haar wavelet. So, the Fourier transform of ψ is:

$$\hat{\psi}(w) = \mathscr{F}\big[\psi(x)\big]$$
$$=> \hat{\psi}(w) = \frac{1}{\sqrt{2\pi}} \frac{\big[\sin(w/4)\big]^2}{(w/4)} e^{-i(w-\pi)/2}$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{4i}{w}\right) e^{-iw/2} \sin^2(w/4)$$

So, then let us now check what is the square integration of $\hat{\psi}^2/w$ to see whether this is indeed a wavelet or not.

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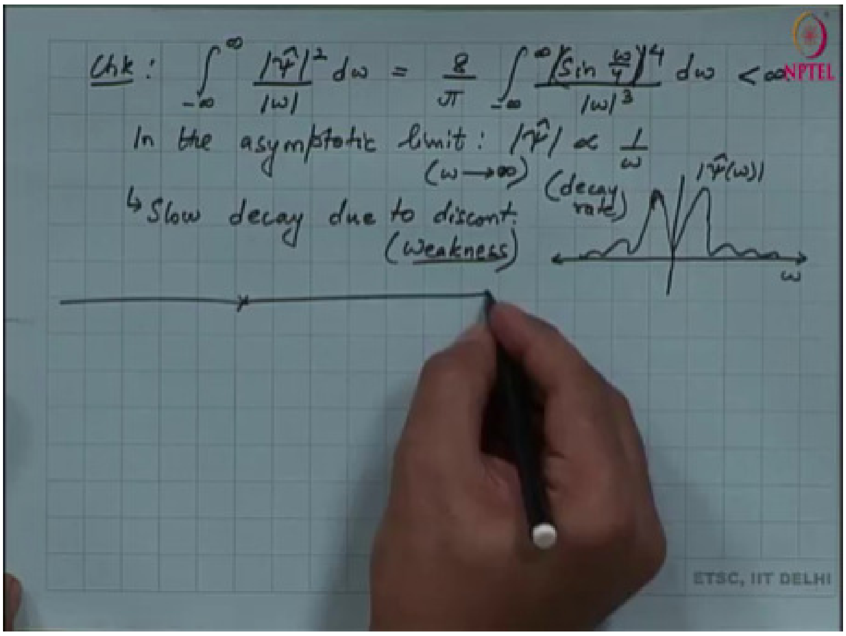

So, students are asked to check this following integral.

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}|^2}{|w|} dw = \frac{8}{\pi} \int_{-\infty}^{\infty} \frac{|\sin\ w/4|^4}{w^3} dw < \infty$$

It is easy to see that the denominator is bounded for larger and larger values of omega becomes bigger and bigger. So, this integration is finite. So, I see that the in the asymptotic limit my $\hat{\psi}$ is proportional to $1/\omega$. We see that in this limit when ω goes to infinity for large values of ω, I see that $\hat{\psi}$ is proportional to $1/\omega$. Now, let me just show you how this transform looks like. The absolute value of the transform of ψ, the Haar wavelet, versus the transform parameter. I see that the wavelet has a first-order discontinuity at 0.
Now, notice that the decay of this wavelet in the transform plane is of the order of $1/\omega$. So, that the decay rate that I mentioned is of the order $1/\omega$. So, in wavelet theory this is a very slow decay. Now, we want that the decay of the wavelet for large values of this transform should be very fast. So, in that sense the decay is very slow for the Haar wavelet and the reason is that there is a discontinuity in the original wavelet transform. So, there is a discontinuity and due to discontinuity, I see that the decay rate is relatively slow in the transform plane and that is considered a weakness in the transform. So, later on I will show you a special case of a transform that is the Daubechies transform, where I can remove this weakness by assuming as many derivatives to vanish to 0 as I want to construct. So, that is the degree of freedom that particular wavelet has.